The Pillbox Cavity Completely Worked Out

The fields in the pillbox cavity are (Wangler, page 28)

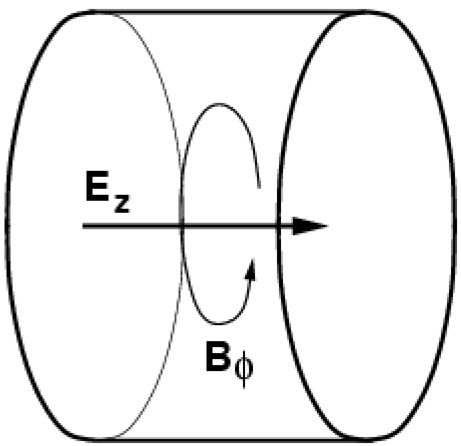
$$E_z = E_0 J_0(k_r r) \cos\omega t$$

$$B_\theta = -\frac{E_0}{c} J_1(k_r r) \sin\omega t$$

$$\omega_c = k_r c = \frac{2.405\, c}{R_c}$$

Length = L

Radius = R_c



The stored energy, power and unloaded quality factor are

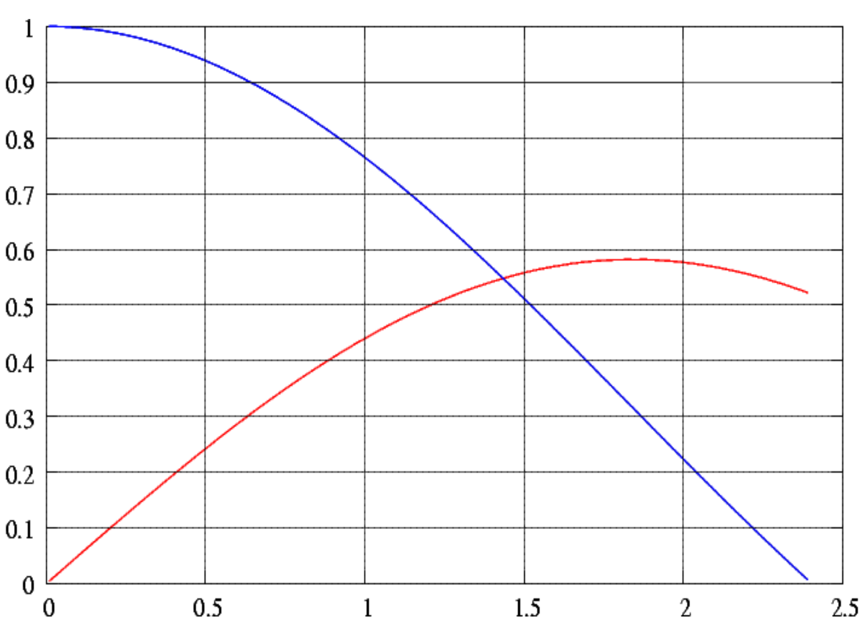
$$U = \frac{\pi \epsilon_0 L R_c^2}{2} E_0^2 J_1^2(2.405)$$

$$P = \pi R_c R_s E_0^2 \frac{\epsilon_0}{\mu_0} J_1^2(2.405)\,(L + R_c)$$

$$Q = \frac{\omega_c U}{P} = \frac{2.405 \sqrt{\mu_0/\epsilon_0}}{2 R_s} \frac{1}{1 + R_c/L}$$

$$R_s = \frac{1}{\delta \sigma}, \qquad \delta = \sqrt{\frac{\lambda}{\pi \sigma \mu_0 c}}$$

$$\frac{1}{\sigma_{copper}} = \rho_{copper} = 1.724 \times 10^{-8}\, \Omega - m$$



Pillbox: Power on Walls

Power on the outer wall at $r = R_c$

$$P_{outer} = \frac{R_s}{2} H_{wall}^2 \times Area = \pi R_s R_c \, E_0^2 \, \frac{\epsilon_0}{\mu_0} \, J_1^2(2.405) \, L$$

Power on each endwall (this is a little more difficult)

$$P_{end} = \frac{1}{\mu_0^2} \frac{R_s}{2} \int B_\theta^2 \, 2\pi \, r \, dr = \pi R_s E_0^2 \frac{\epsilon_0}{\mu_0} \int_0^{R_c} J_1^2 (2.405 \frac{r}{R_c}) r \, dr$$

The identities that allow the integral to be evaluated are

$$\int_0^P [J_n(ax)]^2 x \, dx \equiv \frac{P^2}{2} \left([J_n'(aP)]^2 + (1 - \frac{n^2}{a^2 P^2}) [J_n(aP)]^2 \right)$$

and

$$J_1'(a) = J_0(a) - \frac{1}{a} J_1(a)$$

Some terms cancel and one goes to zero.

$$P_{end} = \pi R_s E_0^2 \frac{\epsilon_0}{\mu_0} \frac{R_c^2}{2} J_1^2(2.405)$$

The total power on the surfaces is then

$$\begin{aligned} P_{total} = P_{wall} + 2 P_{end} &= \pi R_c R_s E_0^2 \frac{\epsilon_0}{\mu_0} J_1^2(2.405)\,(L + R_c) \\ &= \pi R_s (Z_0 E_0)^2 R_c (L + R_c) J_1^2(2.405) \end{aligned}$$

Note that this is the **rms** (thermal) power, and the fields are expressed as **peak** fields. We have converted from peak to rms power by the factor of ½ in the expression for power.

$$P_{rms} = \frac{R_s}{2} H^2 \times Area$$

Compare results with Superfish. A cavity with R_s = L = 0.1 meters is computed.

The default electric field of 1 MV/m on the axis is used.

```
title
.1 m radius, .1 m tall pillbox
run 1
fish
xmax 10.0
ymax 11.0
nseg 4
rseg 0 10 10 0
zseg 0 0 10 10
dz .1
boundary 1 0 1 1
freq 1150.0
power 2 3 4
beta 1.0
begin
end
```