Scattering Matrices

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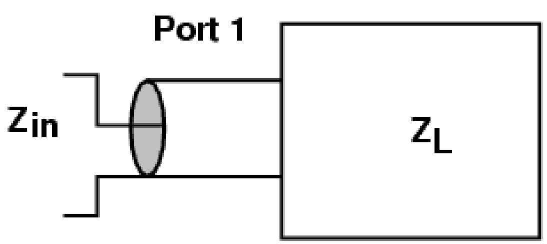
October 2007

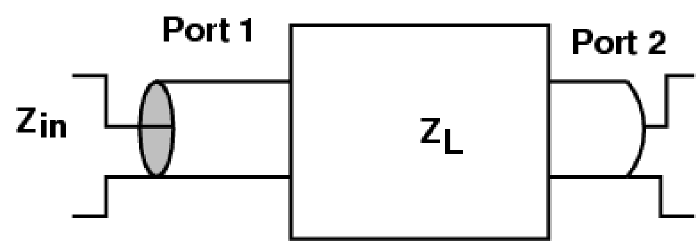
Scattering Matrices

At low frequencies (wavelengths long compared to the geometry of the device being measured), the circuit elements can be considered to be lumped (grouped into identifiable elements). At high frequencies, this is no longer possible or even desirable.

An RF circuit may be defined by its **ports**, the input and output connections. The **scattering** and **Z-matrices** define the properties of the circuit through its external ports.

We will consider circuits comprising one or two ports, accessible through coaxial cables with characteristic impedance Z_c.





A wave may enter a port and be reflected back from that port, or may be transmitted through the DUT and exit from the other port.

Signals Present at the Ports

We adopt the symbols V^+ and V^- for the voltage of a wave of angular frequency ω.
(We will leave out the time dependence in the following equations.)

Toward a port $$V^+ = V_0 \, e^{-j\beta z}$$

Away from a port $$V^- = V_0 \, e^{+j\beta z}$$

β is a wavenumber with units of inverse length:

$$\beta = \frac{\omega}{v_p} = \frac{2\pi f}{v_p} = 2\pi \frac{c}{v_p} \frac{1}{\lambda} = \frac{2\pi}{\lambda}\sqrt{\epsilon_r}$$

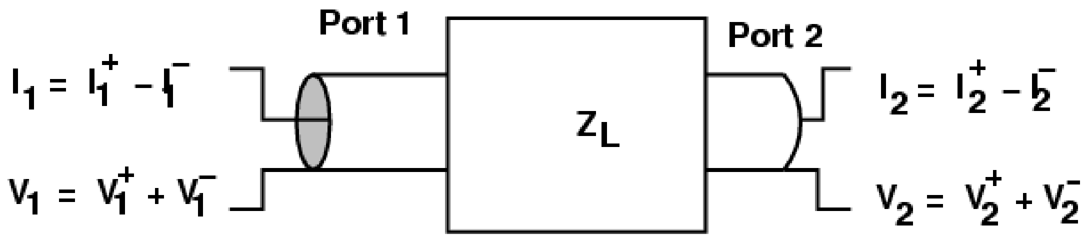
where v_p is the propagation velocity along the cable and ϵ_r is the relative permeability of the dielectric in the coaxial cable.

For RG-58A/U cable, $\epsilon_r = 2.29$, so the propagation velocity $v_p/c = 0.66$.

One wavelength of a transmission line is then $$L = \frac{c}{f\sqrt{\epsilon_r}} = \frac{\lambda_{free-space}}{\sqrt{\epsilon_r}}$$

Impedance Matrix Z

For a two-port circuit, the forward and reflected voltages and currents can be specified for each of the two ports.



V^+, I^+ are directed *in to* the port
V^-, I^- are directed *away from* the port

The voltage V is the *sum* of the ingoing and outgoing waves.
The current I is the *difference* of the ingoing and outgoing waves.

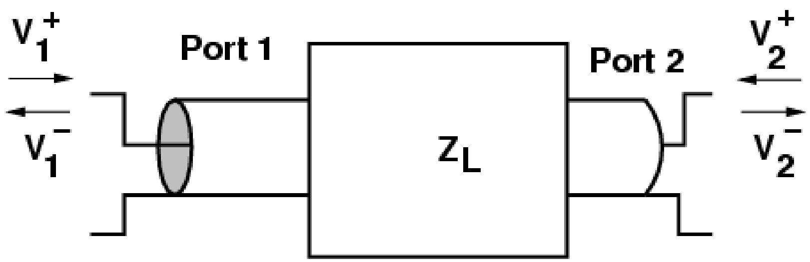
The Z-matrix relates the current and voltages at the two ports:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

This formulation requires the measurement of voltages *and* currents at the ports.

Scattering Matrix S

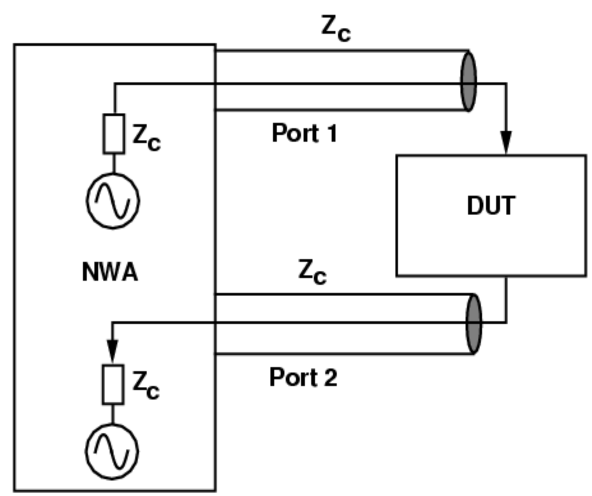
The scattering matrix is defined only in terms of voltages, easily measured with a network analyzer. (The network analyzer, with its directional couplers, can differentiate between a forward and reverse wave as it measures the voltage of each wave.)



$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

Note that currents are not involved directly, but inferred from the impedance of the network analyzer, the cables and the terminations, normally 50 ohms.

The NWA provides reverse termination of all ports, so signals are not reflected from them.



Significance of the S-Matrix Elements

The NWA supplies a known voltage to each port in turn. All ports are back-terminated in Z_c, the characteristic impedance of the NWA ports and cables to the DUT.

If a voltage V_1^+ is applied from to port 1 of the DUT, and no voltage to port 2, the voltage of the reflected waves from the ports is measured: V_1^- and V_2^-.

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ 0 \end{bmatrix}$$

or

$$S_{11} = \frac{V_1^-}{V_1^+}, \qquad S_{21} = \frac{V_2^-}{V_1^+}$$

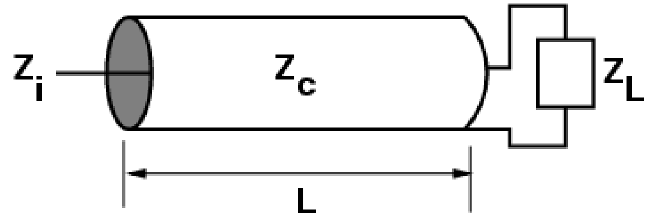
S_{11} is the voltage *reflected* from the input port = Γ_0 = the **reflection coefficient**

S_{21} is the voltage *transmitted* from port 1 to port 2.

For a lossless network, $|S_{11}| = |S_{22}|$

For homework, show that $S_{12} = \sqrt{1 - |S_{11}|^2}$

Find the Load Impedance from the Reflection Coefficient



From the wave equation, a wave propagating down the transmission line with characteristic impedance Z_c is

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$$
$$I(z) = I^+ e^{-j\beta z} - I^- e^{+j\beta z}$$

In the coaxial cable, the relationship between V and I is $I = \frac{V}{Z_c}$

At the load Z_L, $\frac{V(Z_L)}{I(Z_L)} = \frac{V^+ + V^-}{I^+ - I^-} = Z_L = Z_c \frac{V^+ + V^-}{V^+ - V^-}$

But $S_{11} = \Gamma_0 = \frac{V^-}{V^+}$, so $Z_L = Z_c \frac{1+\Gamma_0}{1-\Gamma_0}$

And at a distance L from the load, the impedance Zi, looking into the transmission line is

$$Z_i = Z_c \left[\frac{Z_L + j Z_c \tan \beta L}{Z_c + j Z_L \tan \beta L} \right], \qquad \beta = \frac{2\pi}{\lambda'}, \qquad \lambda' = \frac{\lambda_{free-space}}{\sqrt{\epsilon_r}}$$