Key Concepts - 1

William A. Barletta
Director, United States Particle Accelerator School
Dept. of Physics, MIT

Why do we need high energy beams

- Resolution of "Matter" Microscopes
  - ➞ Wavelength of Particles (γ, e, p, ...) (de Broglie, 1923)

$$\lambda = h / p \quad = 1.2 \ \text{fm} / \boldsymbol{p} \ [\ \text{GeV}/c]$$

  - ➞ Higher momentum => shorter wavelength => better the resolution

- Energy to Matter
  - ➞ Higher energy produces heavier particles

$$E = mc^2 = \frac{m_o c^2}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_o c^2$$

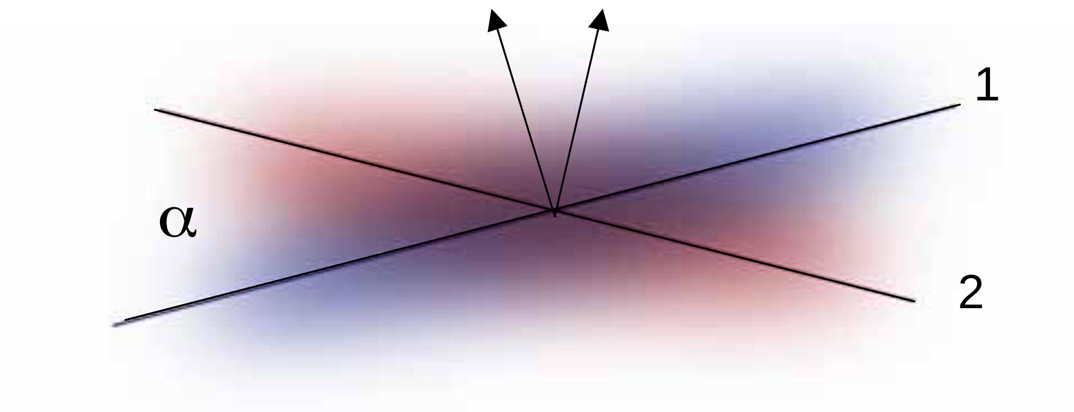
- Penetrate more deeply into matter

Figures of merit

High Energy Physics
Figure of Merit 2: Number of events

$$Events = Cross\text{-}section \times \langle Collision\ Rate \rangle \times Time$$

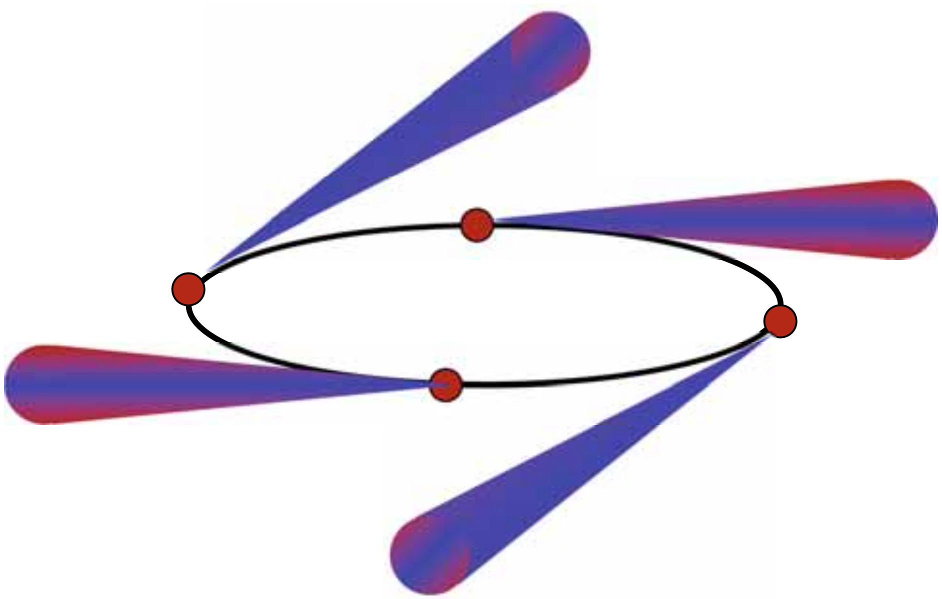
Beam energy: sets scale of physics accessible



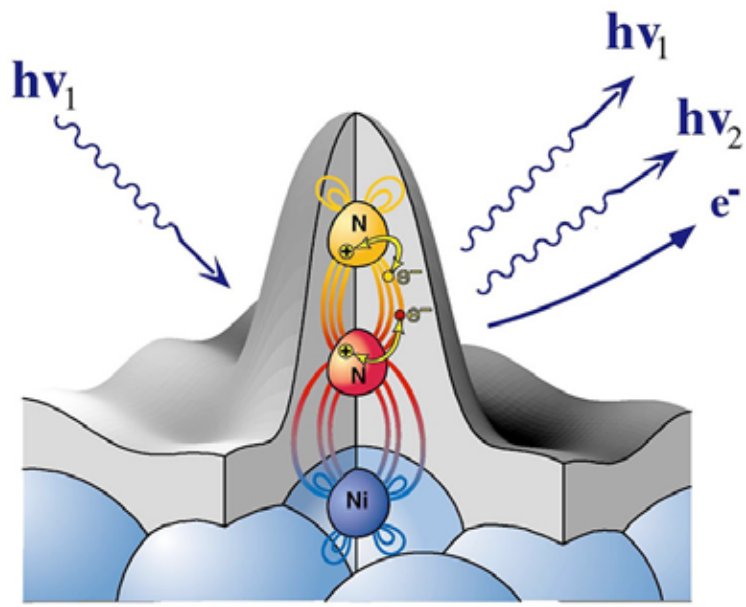
$$\text{Luminosity} = \frac{N_1 \times N_2 \times frequency}{Overlap\ Area} = \frac{N_1 \times N_2 \times f}{4\pi\sigma_x\sigma_y} \times Correction\ factors$$

We want large charge/bunch, high collision frequency & small spot size

Synchrotron light source



- Science with X-rays
  - Microscopy
  - Spectroscopy



FOM: Brilliance v. λ

B = ph/s/mm²/mrad²/0.1%BW

Special relativity

Thus we have the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad , \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \quad , \quad z' = z$$

Or in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z' \end{pmatrix}$$

Proper time & length

- We define the proper time, τ, as the duration measured in the rest frame

- The length of an object in its rest frame is L_o

- As seen by an observer moving at v, the duration, T, is

$$T = \frac{\tau}{\sqrt{1 - v^2/c^2}} \equiv \gamma\tau > \tau$$

And the length, L, is

$$L = L_o/\gamma$$

Velocity, energy and momentum

- For a particle with 3-velocity ***v***, the 4-velocity is

$$u^{\alpha} = (\gamma c, \gamma \mathbf{v}) = \frac{dx^{\alpha}}{d\tau}$$

- The total energy, E, of a particle is its rest mass, m_o, plus kinetic energy, T (what is cited as the energy of the beam)

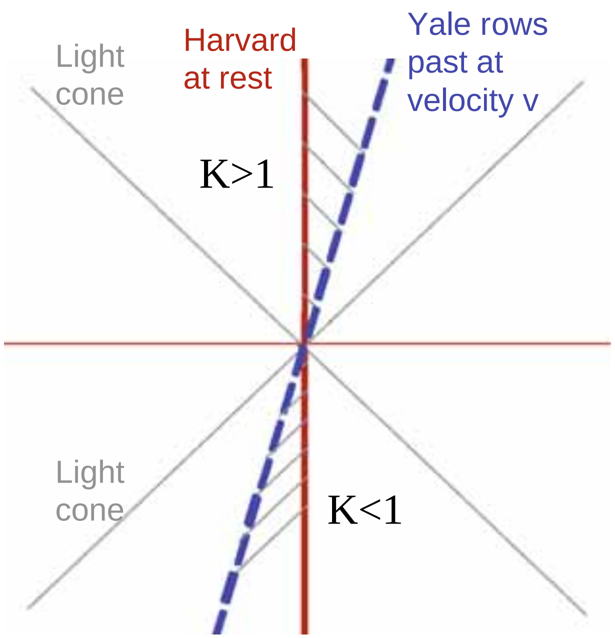
$$E = m_o c^2 + T = \gamma m_o c^2 \quad \text{and} \quad E^2 = p^2 c^2 + m_o^2 c^4$$

- The 4-momentum, p^{μ}, is

$$p^{\mu} = (c\gamma m_0, \gamma m_0 \mathbf{v})$$

$$p^2 = m_o{}^2 c^2$$

Doppler shift of frequency



Distinguish between coordinate transformations and observations

- Yale sets his signal to flash at a constant interval, $\Delta t'$
- Harvard sees the interval foreshortened by K(v) as Yale approaches
- Harvard see the interval stretched by K(-v) as Yale moves away

$$K(v) = \left(\frac{1+v}{1-v}\right)^{1/2} \approx 2\gamma$$

Particle collisions

- Two particles have equal rest mass m_0.

Laboratory Frame (LF): one particle at rest, total energy is E_{lab}.

$$\mathbf{P_1} = (E_1/c, \mathbf{p_1}) \qquad \mathbf{P_2} = (m_0 c, \mathbf{0})$$

Centre of Momentum Frame (CMF): Velocities are equal & opposite, total energy is E_{cm}.

$$\mathbf{P_1} = (E_{\rm cm}/(2c), \mathbf{p}) \qquad \mathbf{P_2} = (E_{\rm cm}/(2c), -\mathbf{p})$$

Exercise: Relate E to E_{cm}

$$E_{cm} = \sqrt{2mc^2 E_{lab}}$$

A simple problem - bending radius

- Compute the bending radius, R, of a non-relativistic particle particle in a uniform magnetic field, B.
  - Charge = q
  - Energy = $mv^2/2$

$$F_{Lorentz} = q\frac{v}{c}B = F_{centripital} = \frac{mv^2}{\rho}$$

$$\Rightarrow \rho = \frac{mvc}{qB} = \frac{pc}{qB}$$

$$\rho(\mathrm{m}) = 3.34 \left(\frac{p}{1\ \mathrm{GeV/c}}\right)\left(\frac{1}{q}\right)\left(\frac{1\ \mathrm{T}}{B}\right)$$

Lorentz transformations of E.M. fields

$$E'_{z'} = E_z$$

$$E'_{x'} = \gamma\left(E_x - vB_y\right)$$

$$E'_{y'} = \gamma\left(E_y + vB_x\right)$$

$$B'_{z'} = B_z$$

$$B'_{x'} = \gamma\left(B_x + \frac{v}{c^2}E_y\right)$$

$$B'_{y'} = \gamma\left(B_y - \frac{v}{c^2}E_x\right)$$

$$\Rightarrow \mathbf{B}'_{\perp} = \gamma\frac{\mathbf{v}}{c^2}\times\mathbf{E}$$

Fields are invariant along the direction of motion, z

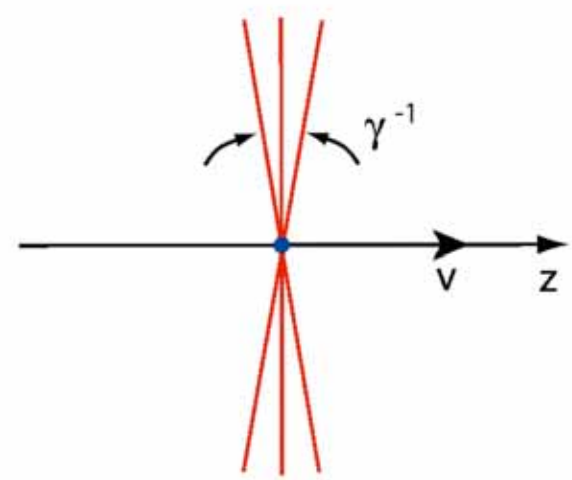
The E field gets swept into a thin cone

- We have $E_x = \gamma E'_x$, $E_y = \gamma E'_y$, and $E_x = E'_z$
- Transforming r´ gives $r' = \sqrt{x^2 + y^2 + \gamma^2(z - vt)^2} \equiv \gamma R$
- Draw **r** is from the current position of the particle to the observation point, r = (x, y, z − vt)
- Then a little algebra gives us

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\frac{q\mathbf{r}}{\gamma^2 R^3}$$

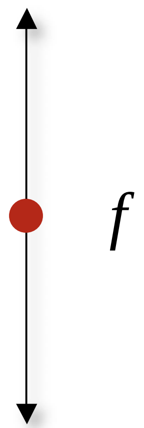
- The charge also generates a B-field

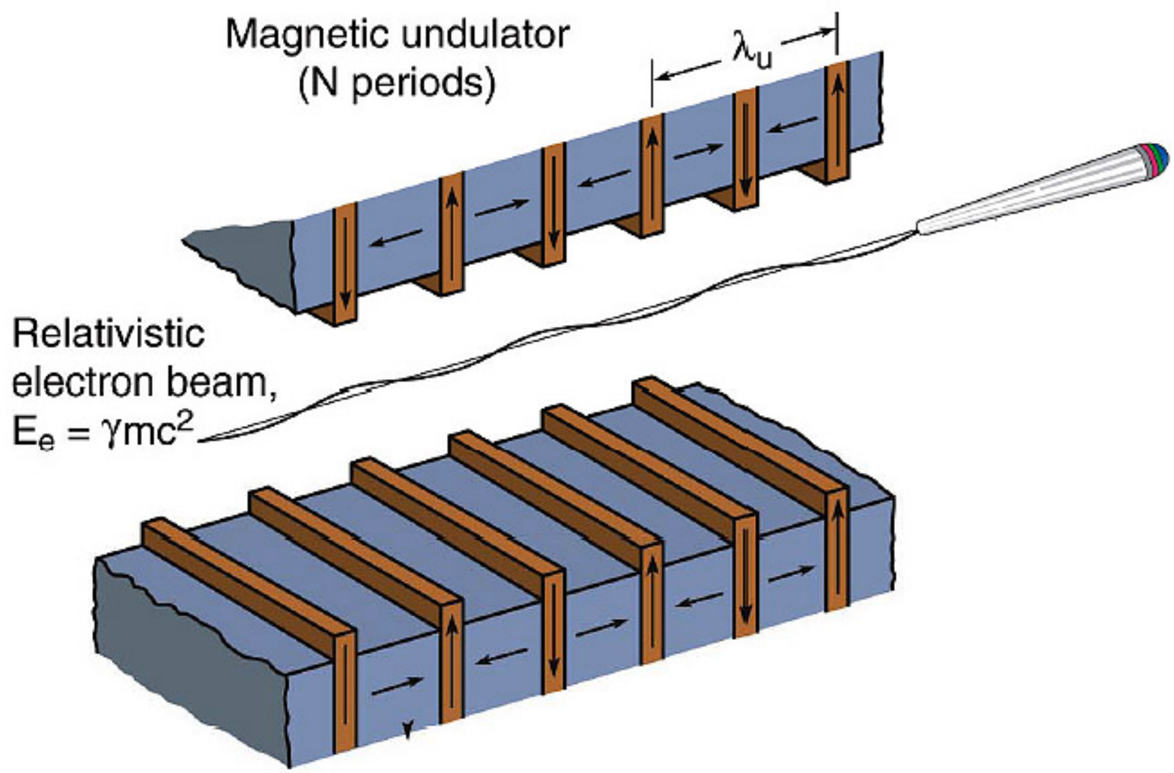
$$\mathbf{B} = \frac{1}{c^2}\mathbf{v}\times\mathbf{E}$$



Undulator radiation: What is λ_{rad}?

An electron in the lab oscillating at frequency, *f*, emits dipole radiation of frequency *f*





What about the relativistic electron?

Electromagnetism

Newton's law

- We all know

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}$$

- The 4-vector form is

$$F^{\mu} = \left(\gamma c \frac{dm}{dt}, \gamma \frac{d\mathbf{p}}{dt} \right) = \frac{dp^{\mu}}{d\tau}$$

- Differentiate $p^2 = m_o{}^2 c^2$ with respect to τ

$$p_{\mu} \frac{dp^{\mu}}{d\tau} = p_{\mu} F^{\mu} = \frac{d(mc^2)}{dt} - \mathbf{F}\, 0\, \mathbf{v} = 0$$

- The work is the rate of changing mc^2

Harmonic oscillator

- Motion in the presence of a linear restoring force

$$F = -kx$$

$$\ddot{x} + \frac{k}{m} x = 0$$

$$x = A \sin \omega_o t \quad \text{where} \quad \omega_o = \sqrt{k/m}$$

- It is worth noting that the simple harmonic oscillator is a linearized example of the pendulum equation

$$\ddot{x} + \omega_o^2 \sin(x) \approx \ddot{x} + \omega_o^2 (x - x^3/6) = 0$$

that governs the free electron laser instability

Electric displacement & magnetic field

In vacuum,

- The electric displacement is $\mathbf{D} = \varepsilon_o \mathbf{E}$,

- The magnetic field is $\mathbf{H} = \mathbf{B}/\mu_o$

Where

$$\varepsilon_o = 8.85 \times 10^{-12} \text{ farad/m} \quad \& \quad \mu_o = 4\,\pi \times 10^{-7} \text{ henry/m.}$$

Maxwell's equations (1)

- Electric charge density ρ is source of the electric field, **E** (Gauss's law)

$$\nabla \bullet \mathbf{E} = \rho$$

- Electric current density **J** = ρ**u** is source of the magnetic induction field **B** (Ampere's law)

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_0 \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

If we want big magnetic fields, we need large current supplies

Maxwell's equations (2)

- Field lines of **B** are closed; i.e., no magnetic monopoles.

$$\nabla \bullet \mathbf{B} = 0$$

- Electromotive force around a closed circuit is proportional to rate of change of **B** through the circuit (Faraday's law).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Maxwell's equations: integral form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_0} \quad => \quad \oint_S \vec{E} \bullet d\vec{a} = \frac{Q_{enclosed}}{\varepsilon_0} \quad \text{Gauss' Law}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad => \quad \oint_C \vec{E} \bullet d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t} \bullet d\vec{a} \quad \text{Faraday's Law}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad =>$$

Displacement current

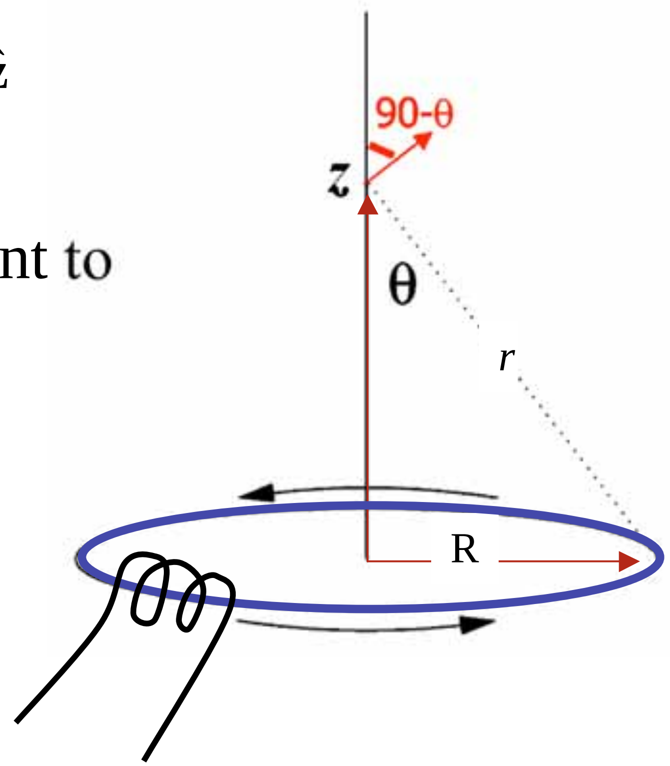
$$\oint_C \vec{B} \bullet d\vec{l} = \mu_0 I_{enclosed} + \mu_0 \varepsilon_0 \oint_S \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a} \quad \text{Ampere's Law}$$

We computed the B-field from current loop with I = constant

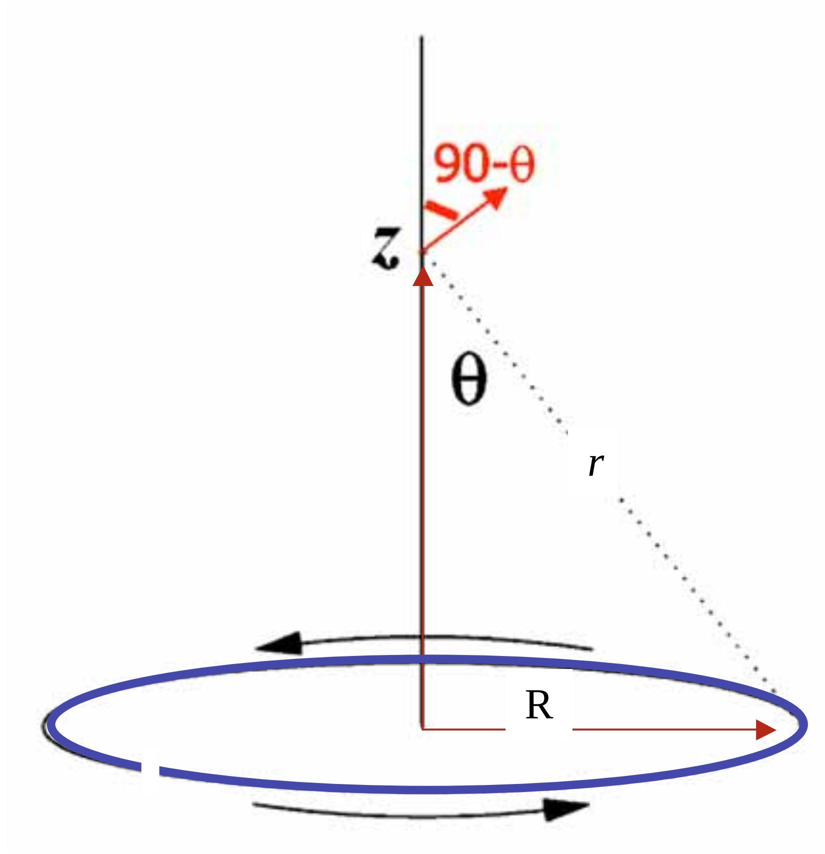
- By the Biot-Savart law we found that on the z-axis

$$\mathbf{B} = \frac{I}{cr^2} R \sin\theta \int_0^{2\pi} d\varphi \, \hat{\mathbf{z}} = \frac{2\pi I R^2}{c\left(R^2 + z^2\right)^{3/2}} \hat{\mathbf{z}}$$

- What happens if we drive the current to have a time variation?



Question to ponder: What is the field from this situation?



We expect this situation to lead to radiation

Boundary conditions for a perfect conductor, $\sigma = \infty$

1. If electric field lines terminate on a surface, they do so normal to the surface
   a) any tangential component would quickly be neutralized by lateral motion of charge within the surface.
   b) The E-field must be normal to a conducting surface
2. Magnetic field lines avoid surfaces
   a) otherwise they would terminate, since the magnetic field is zero within the conductor
      i. The normal component of B must be continuous across the boundary for $\sigma \neq \infty$

Properties of beams

Brightness of a beam source

- A figure of merit for the performance of a beam source is the brightness

$$B = \frac{\text{Beam current}}{\text{Beam area} \cdot \text{Beam Divergence}} = \frac{\text{Emissivity (J)}}{\sqrt{\text{Temperature/mass}}}$$

$$= \frac{J_e}{\left(\sqrt{\frac{kT}{\gamma m_o c^2}}\right)^2} = \frac{J_e \gamma}{\left(kT / m_o c^2 \right)}$$

Typically the normalized brightness is quoted for $\gamma = 1$

Beams have directed energy

- The beam momentum refers to the average value of p_z of the particles

$$p_{beam} = \langle p_z \rangle$$

- The beam energy refers to the mean value of

$$E_{beam} = \left[\langle p_z \rangle^2 c^2 + m^2 c^4 \right]^{1/2}$$

- For highly relativistic beams $pc \gg mc^2$, therefore

$$E_{beam} = \langle p_z \rangle c$$

Beams have internal (self-forces)

- Space charge forces
  - Like charges repel
  - Like currents attract
- For a long thin beam

$$E_{sp}(V/cm) = \frac{60\ I_{beam}(A)}{R_{beam}(cm)}$$

$$B_{\theta}(gauss) = \frac{I_{beam}(A)}{5\ R_{beam}(cm)}$$

Envelope equation: Last steps

- Angular momentum conservation implies

$$P_\vartheta = \gamma L + \gamma \omega_c \frac{R^2}{c} = \text{ constant}$$

- The energy & virial equations combine to yield

$$\ddot{R} + \frac{\dot{\gamma}}{\gamma}\dot{R} + \frac{U}{R} + \frac{\omega_c^2 R}{4} - \frac{E^2}{\gamma^2 R^3} = \frac{1}{\gamma^2 R^3}\int_{t_o}^{t} dt' \left(\frac{2\gamma R^2}{m} \varepsilon' \right)$$

where

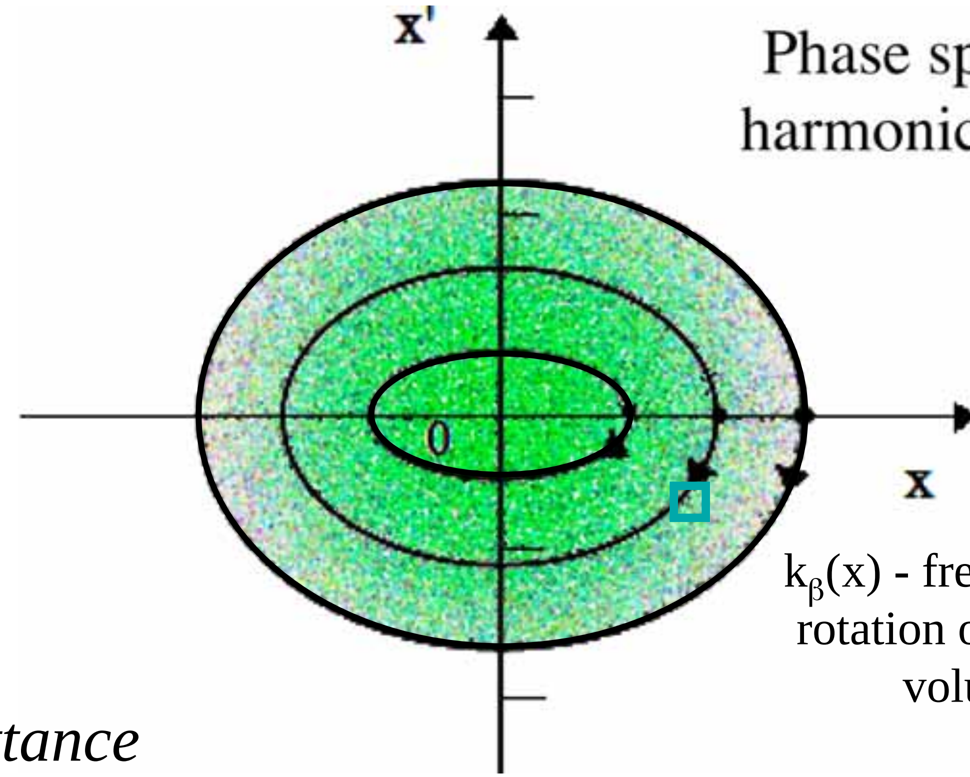
$$U = \left\langle \omega_\beta^2 r^2 \right\rangle = I / I_{Alfven}$$

and

$$E^2 = \gamma^2 R^2 \left(V^2 - (\dot{R})^2 \right) + P_\vartheta^2$$

Emittance describes the area in phase space of the ensemble of beam particles

Emittance - Phase space volume of beam



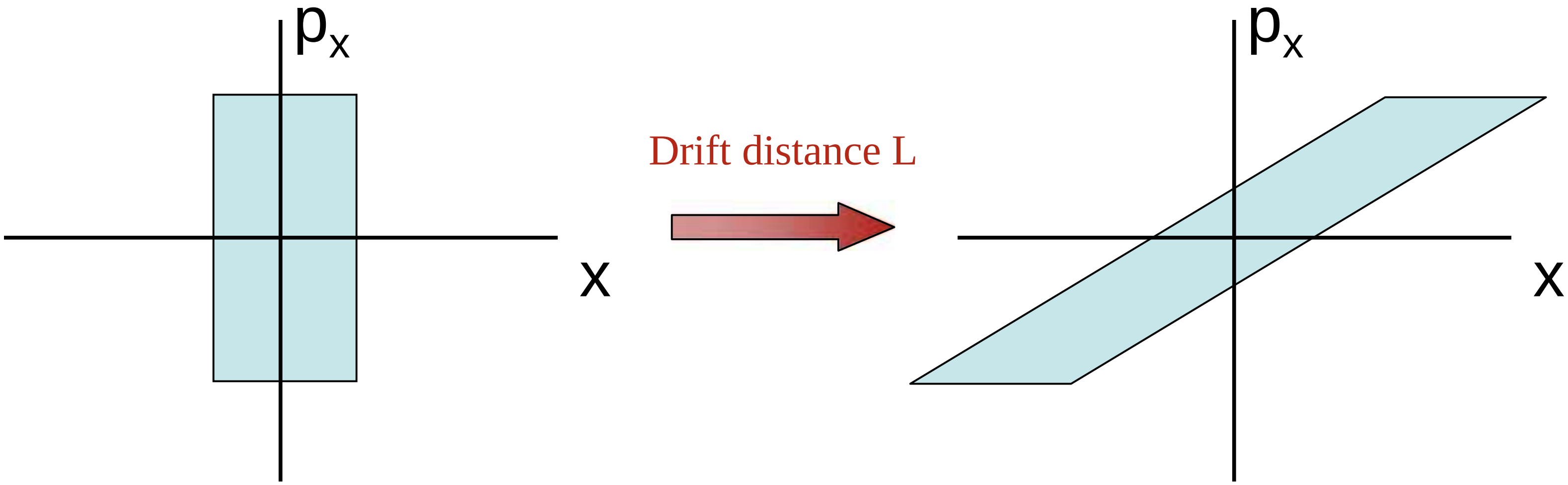
Phase space of an harmonic oscillator

$k_\beta(x)$ - frequency of rotation of a phase volume

RMS emittance

$$\varepsilon^2 \equiv R^2 (V^2 - (R')^2)/c^2$$

Force-free expansion of a beam



Notice: The phase space area is conserved

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \Rightarrow \begin{matrix} x = x_0 + Lx_0' \\ x' = x_0' \end{matrix}$$

Matrix representation of a drift

- From the diagram we can write by inspection

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} \Rightarrow \begin{matrix} x = x_0 + Lx_0' \\ x' = x_0' \end{matrix}$$

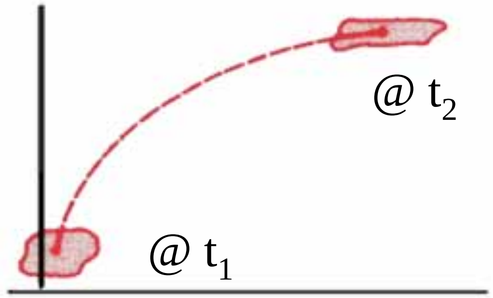
$$\langle x^2 \rangle = \langle (x_0 + Lx_0')^2 \rangle = \langle x_0^2 \rangle + L^2 \langle x_0'^2 \rangle + 2L \langle x_0 x_0' \rangle$$

$$\langle x'^2 \rangle = \langle x_0'^2 \rangle$$

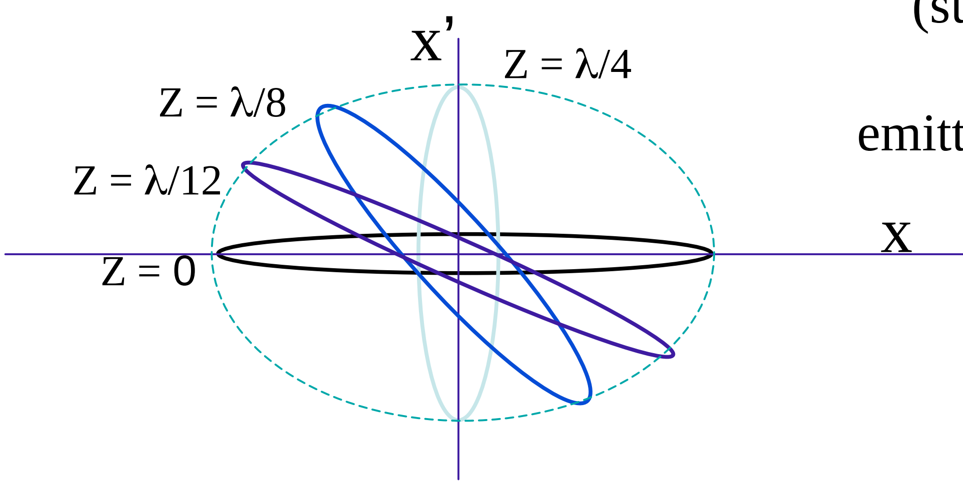
$$\langle xx' \rangle = \langle (x_0 + Lx_0') x_0' \rangle = L \langle x_0'^2 \rangle + \langle x_0 x_0' \rangle$$

- Now write these last equations in terms of β_T, γ_T and α_T

Why is emittance an important concept



1) Liouville: Under conservative forces phase space evolves like an incompressible fluid ==>

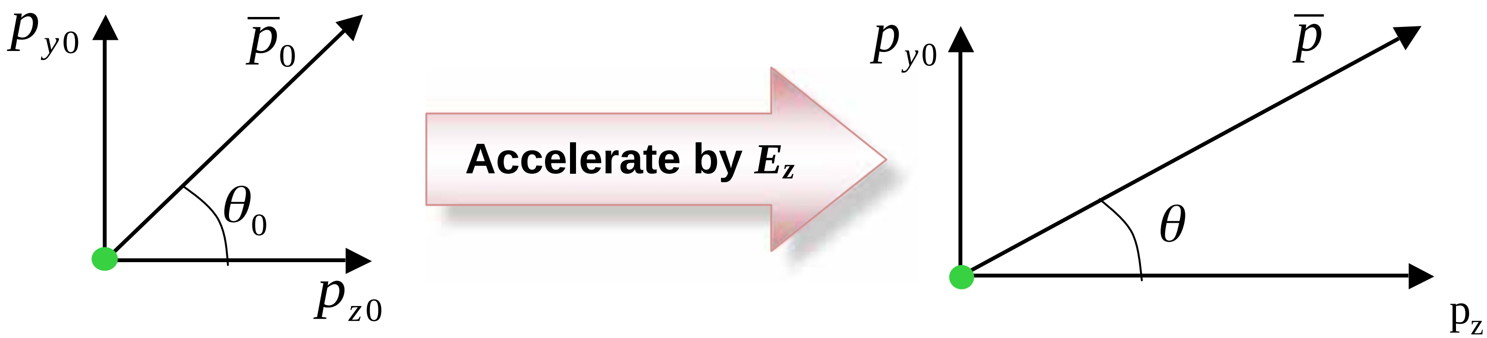


2) Under linear forces macroscopic (such as focusing magnets) &
γ =constant
emittance is an invariant of motion

3) Under acceleration
$\gamma\varepsilon = \varepsilon_n$
is an adiabatic invariant

Emittance during acceleration

- When the beam is accelerated, β & γ change
  - → x and x' are no longer canonical
  - → Liouville theorem does not apply & emittance is not invariant



$$p_z = \sqrt{\frac{T^2 + 2Tm_0c^2}{T_0^2 + 2T_0m_0c^2}}\, p_{z0}$$

$$T \equiv kinetic\ energy$$

From: Sannibale USPAS lectures

Then…

$$y_0' = \tan\theta_0 = \frac{p_{y0}}{p_{z0}} = \frac{p_{y0}}{\beta_0 \gamma_0 m_0 c} \qquad y' = \tan\theta = \frac{p_y}{p_z} = \frac{p_{y0}}{\beta \gamma m_0 c} \qquad \frac{y'}{y_0'} = \frac{\beta_0 \gamma_0}{\beta \gamma}$$

In this case $\frac{\varepsilon_y}{\varepsilon_{y0}} = \frac{y'}{y_0'}$ ==> $\boxed{\beta \gamma \varepsilon_y = \beta_0 \gamma_0 \varepsilon_{y0}}$

- Therefore, the quantity $\beta \gamma \varepsilon$ is invariant during acceleration.
- Define a conserved *normalized emittance*

$$\boxed{\varepsilon_{n\,i} = \beta \gamma \varepsilon_i \qquad i = x, y}$$

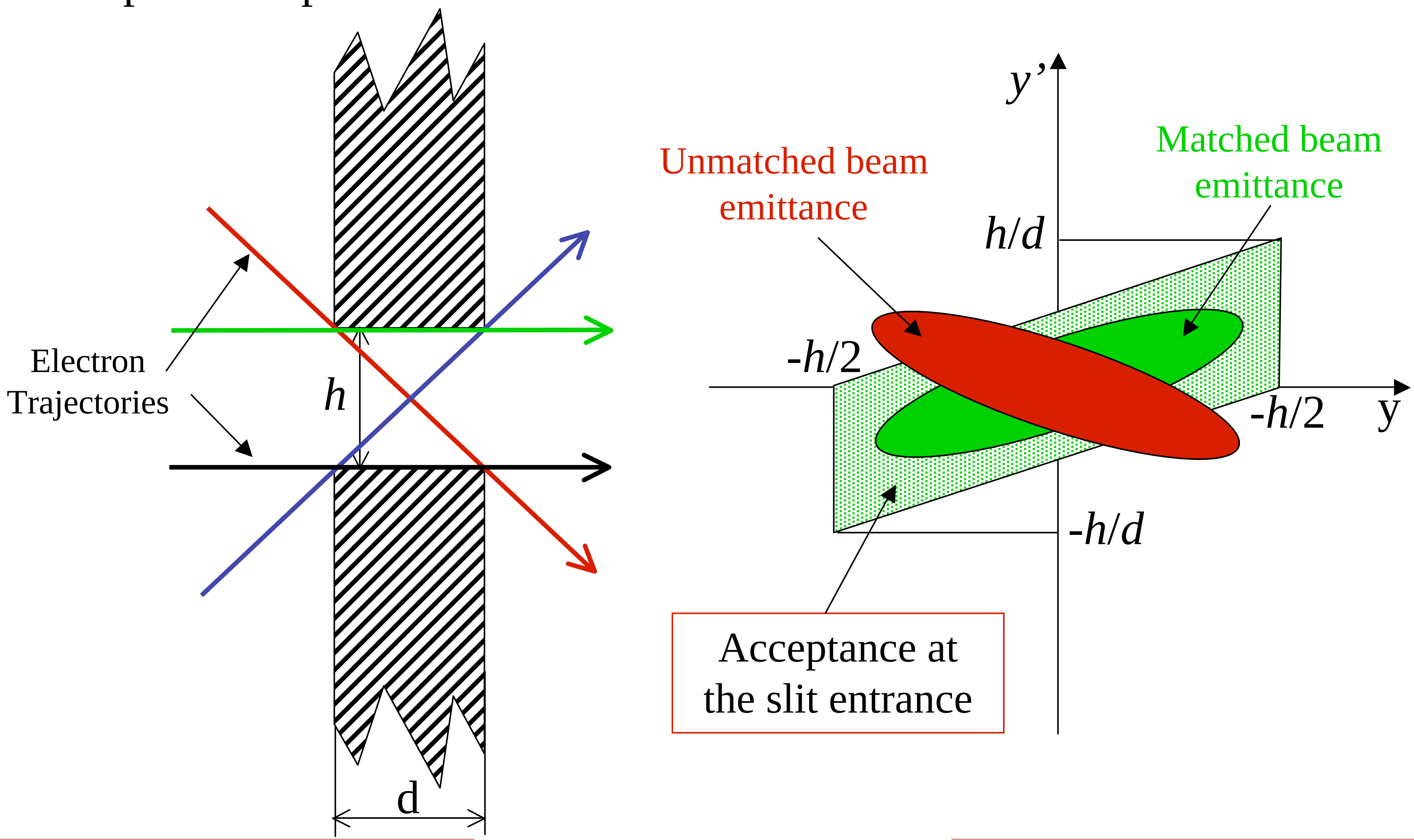
Acceleration couples the longitudinal plane with the transverse planes

The 6D emittance is still conserved but the transverse ones are not

From: Sannibale USPAS lectures

The Concept of Acceptance

Example: Acceptance of a slit



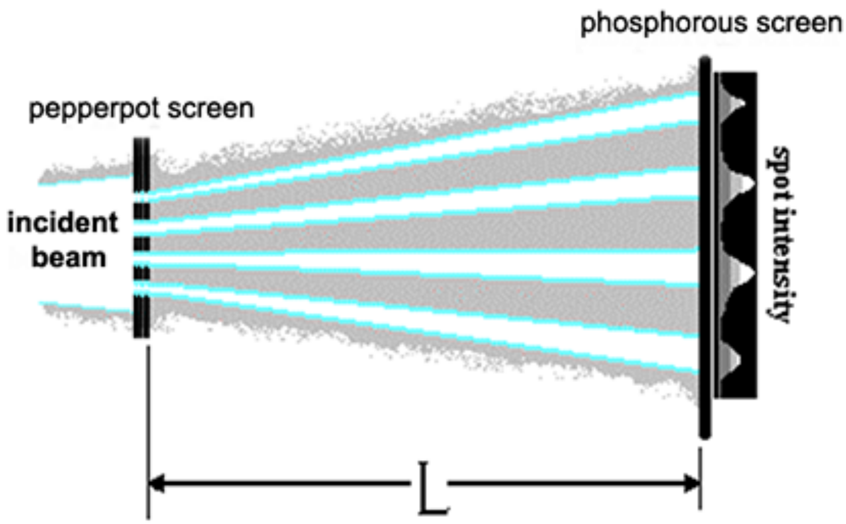
From: Sannibale USPAS lectures

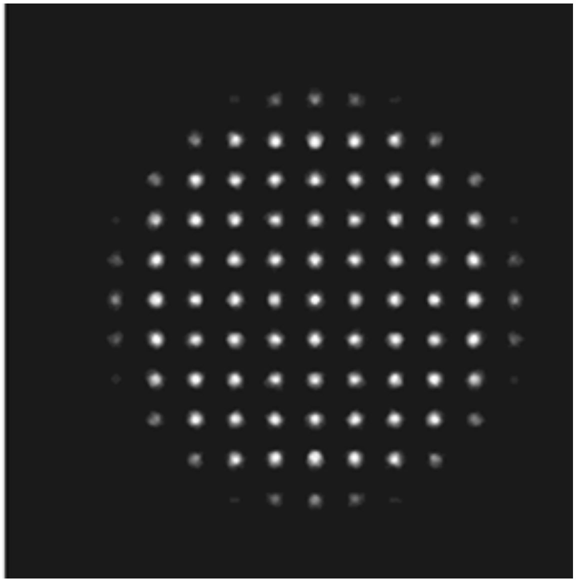
Measuring the emittance of the beam

$$\varepsilon^2 = R^2 (V^2 - (R')^2) / c^2$$

- RMS emittance
  - ➙ Determine rms values of velocity & spatial distribution
- Ideally determine distribution functions & compute rms values
- Destructive and non-destructive diagnostics

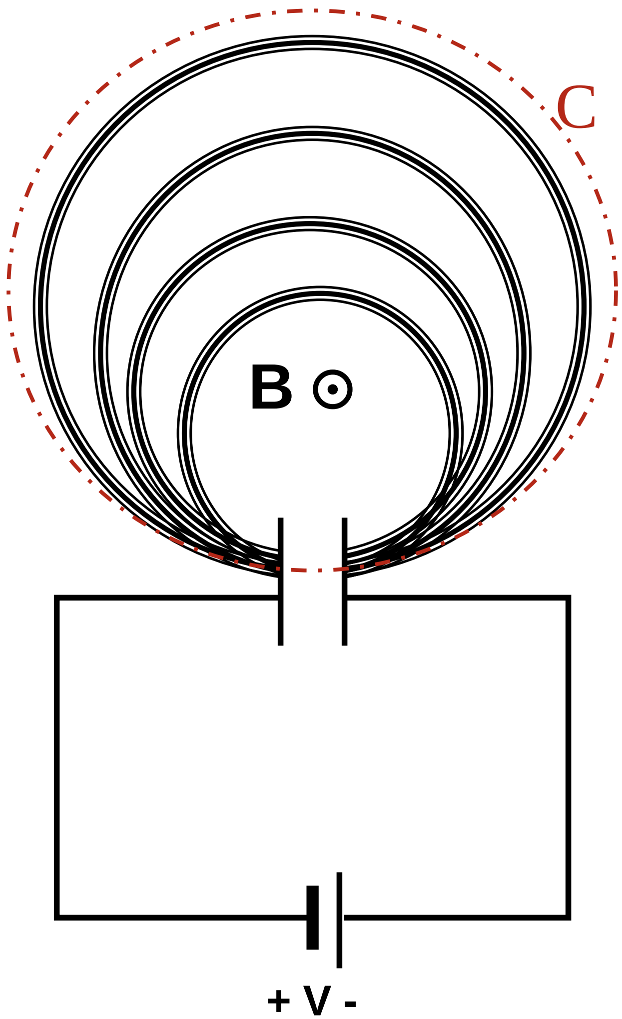
Example of pepperpot diagnostic





- Size of image ==> R
- Spread in overall image ==> R´
- Spread in beamlets ==> V
- Intensity of beamlets ==> current density

Maxwell forbids this!



$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

or in integral form

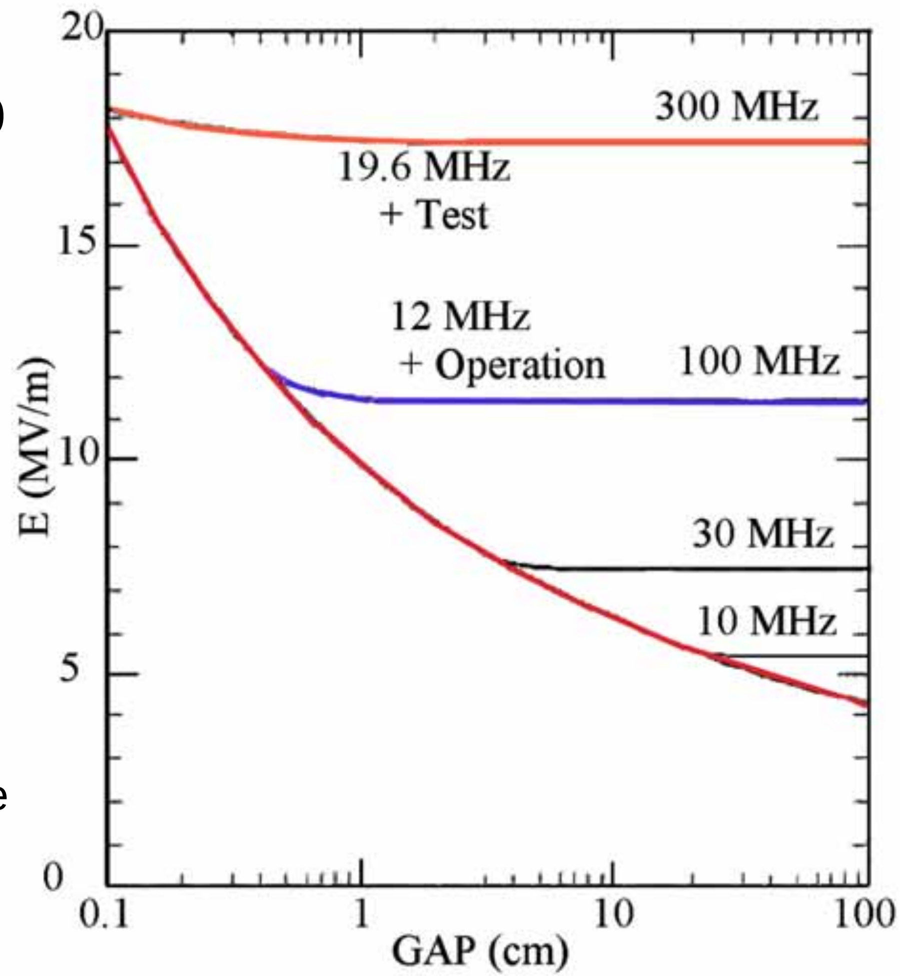
$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{n}\, da$$

∴ There is no acceleration without time-varying magnetic flux

Non-resonant accelerators

Characteristics of DC accelerators

- Voltage limited by electrical breakdown (~10 kV/cm)
  - High voltage
    - ==> Large size (25 m for 25 MV)
  - Exposed high voltage terminal
    - ==> Safety envelope
- High impedance structures
  - Low beam currents
- Generates continuous beams



Sparking electric field limits in the Kilpatrick model, including electrode gap dependence

Synchronism in the Microtron

$$\frac{1}{r_{orbit}} = \frac{eB}{pc} = \frac{eB}{mc^2\beta\gamma}$$

$$\tau_{rev} = \frac{2\pi r_{orbit}}{v} = \frac{2\pi r_{orbit}}{\beta c} = \frac{2\pi\, mc}{e}\frac{\gamma}{B}$$

Synchronism condition: $\Delta\tau_{rev} = N/f_{rf}$

$$\Delta\tau = \frac{N}{f_{rf}} = \frac{2\pi\, mc}{e}\frac{\Delta\gamma}{B} = \frac{\Delta\gamma}{f_{rf}}$$

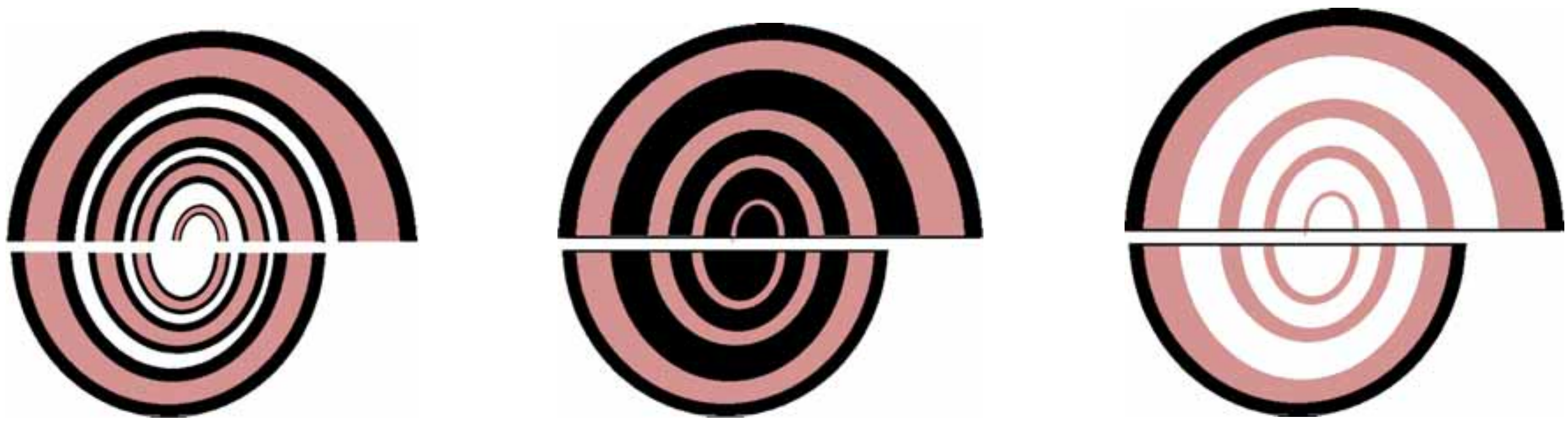
If N = 1 for the first turn @ $\gamma \sim 1$

Or $\Delta\gamma = 1 \Rightarrow E_{rf} = mc^2$

Possible for electrons but not for ions

But long as $\gamma \approx 1$, $\tau_{rev} \approx$ constant! Let's curl up the Wiederoe linac

Bend the drift tubes | Connect equipotentials | Eliminate excess Cu

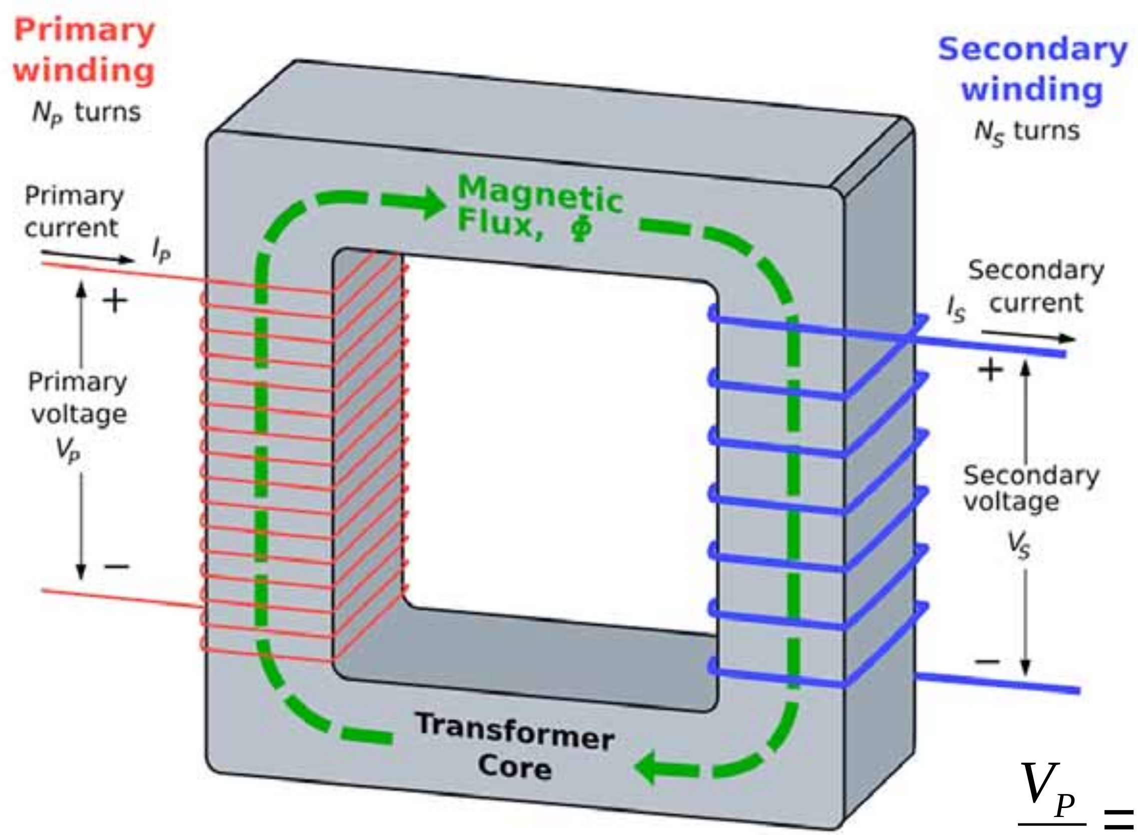


Supply magnetic field to bend beam

$$\tau_{rev} = \frac{1}{f_{rf}} = \frac{2\pi\, mc}{eZ_{ion}} \frac{\gamma}{B} \approx \frac{2\pi\, mc}{eZ_{ion} B} = const.$$

Transformers are highly efficient and can drive large currents

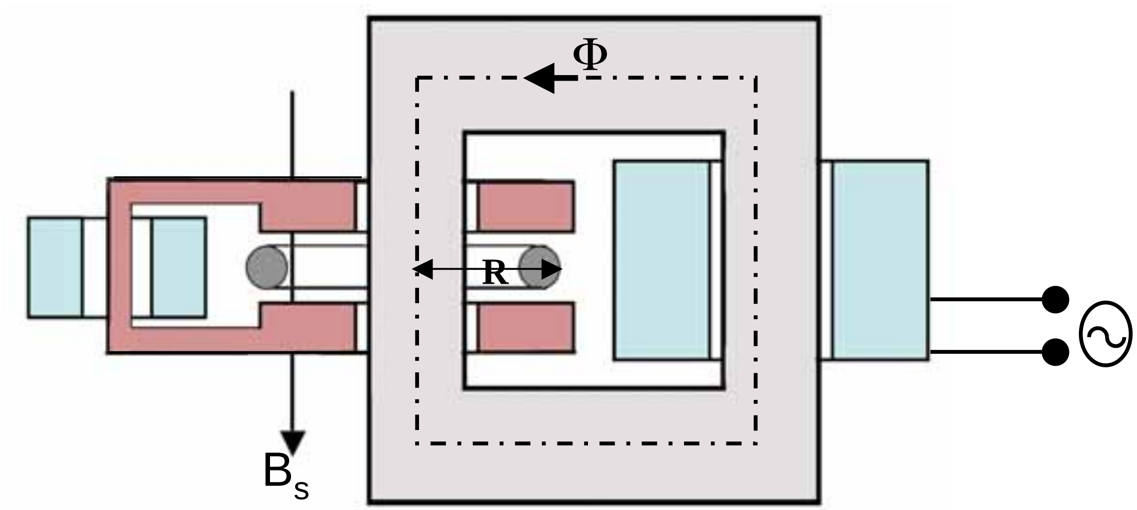
Large units can transfer > 99% of input power to the output



$$\frac{V_P}{V_S} = \frac{N_P}{N_S} = \frac{I_S}{I_P}$$

Image of step-down transformer from Wikipedia

Recall the ray transformer realized as the Betatron (D. Kerst, 1940)



The beam acts as a 1-turn secondary winding of the transformer

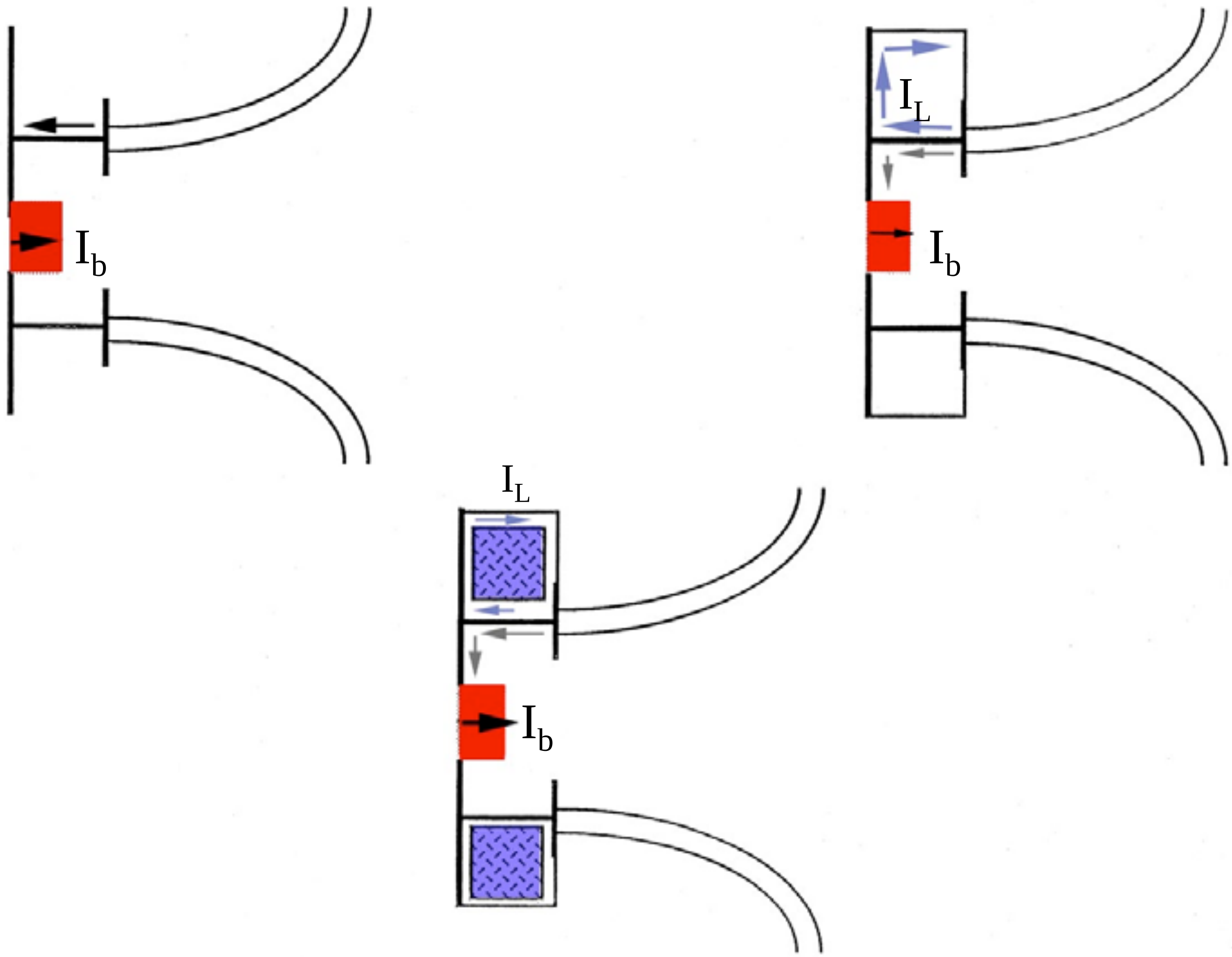
Magnetic field energy is transferred directly to the electrons

For the orbit size to remain invariant

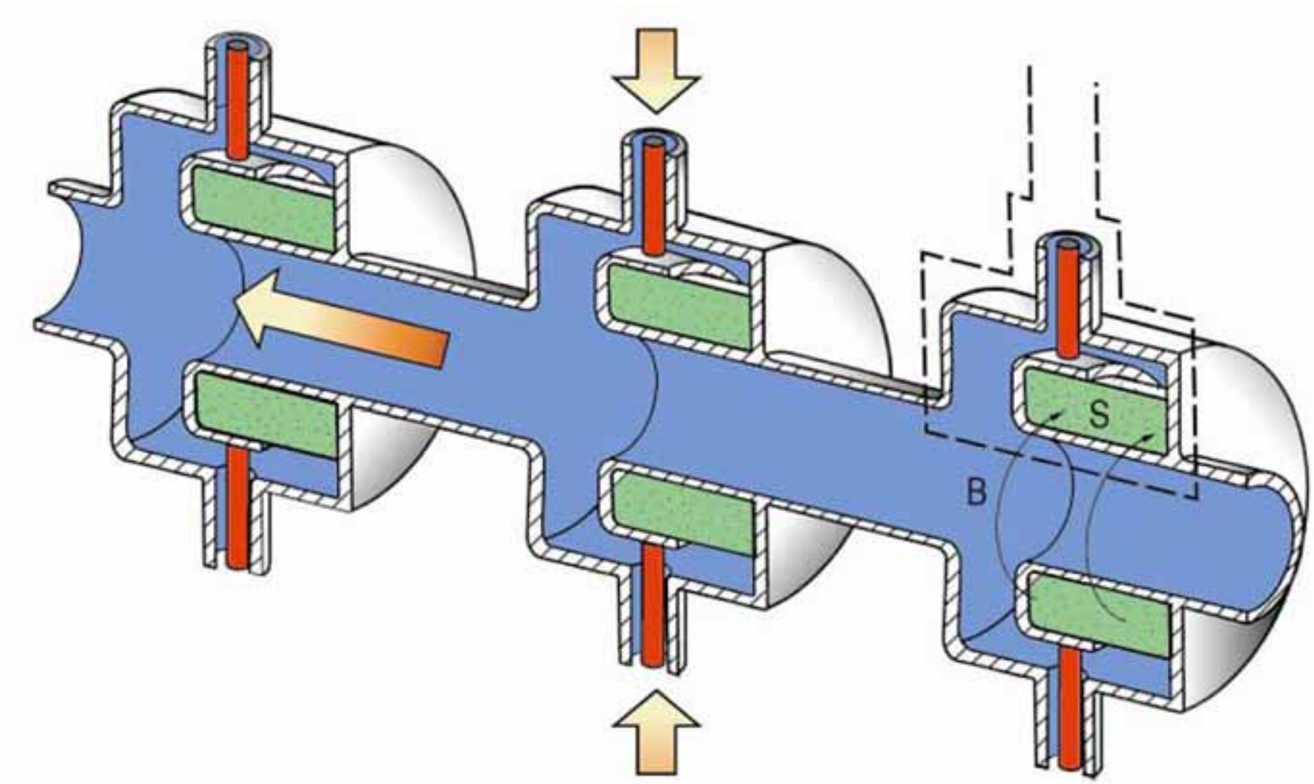
$$\dot{\Phi} = 2\pi R^2 \dot{B}_s$$

This was good for up to 300 MeV electrons. What about electrons or ions?

Principle of inductive isolation



The Linear Betatron: Linear Induction Accelerator



N. Christofilos

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{s}$$

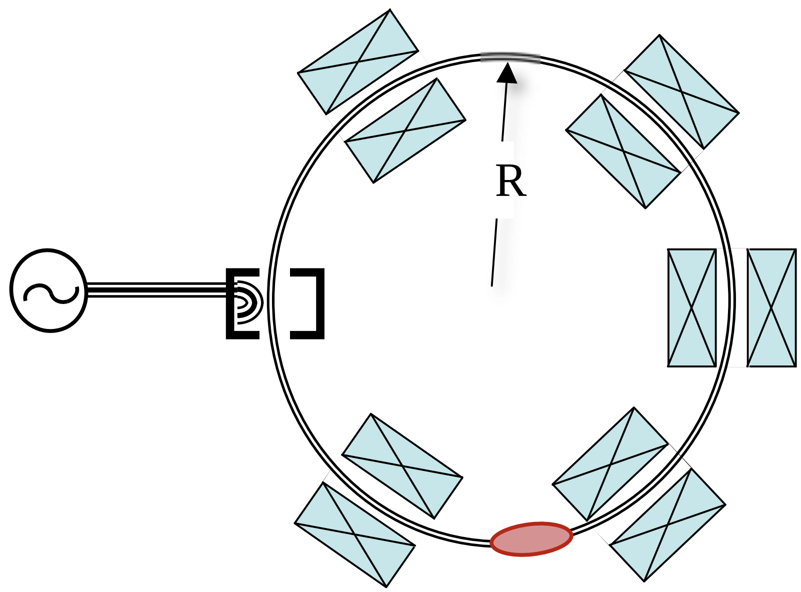
Synchrotrons & phase stability

The synchrotron introduces two new ideas: change B_{dipole} & change ω_{rf}

- For low energy ions, f_{rev} increases as E_{ion} increases
- ==> Increase ω_{rf} to maintain synchronism
- For any E_{ion} circumference must be an integral number of rf wavelengths

$$L = h\,\lambda_{rf}$$

- h is the harmonic number



$$L = 2\pi R$$

$$f_{rev} = 1/\tau = v/L$$

Ideal closed orbit in the synchrotron

- Beam particles will not have identical orbital positions & velocities
- In practice, they will have transverse oscillatory motion (betatron oscillations) set by radial restoring forces
- An ideal particle has zero amplitude motion on a closed orbit along the axis of the synchrotron

Energy gain -II

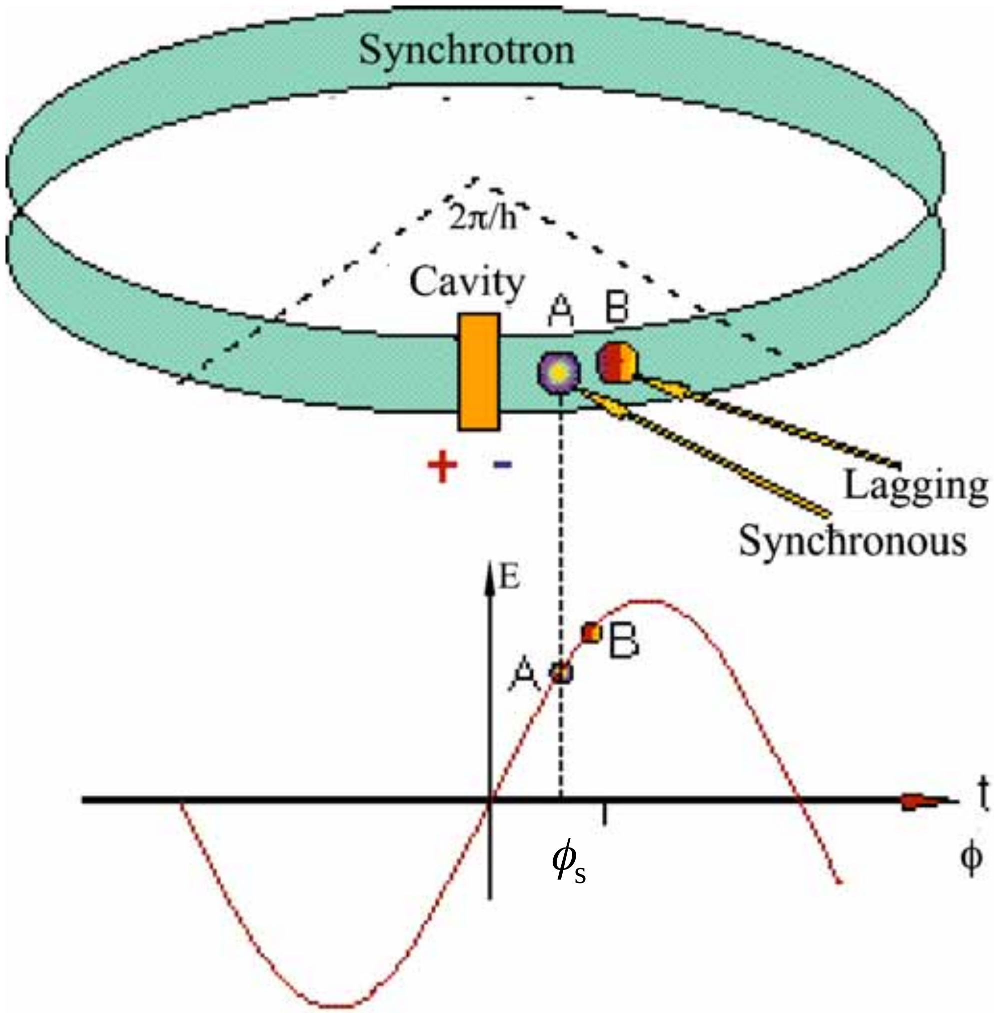
- The synchronism conditions for the synchronous particle
  - ➞ condition on rf- frequency,
  - ➞ relation between rf voltage & field ramp rate
- The rate of energy gain for the synchronous particle is

$$\frac{dE_s}{dt} = \frac{\beta_s c}{L} eV \sin\varphi_s = \frac{c}{h\lambda_{rf}} eV \sin\varphi_s$$

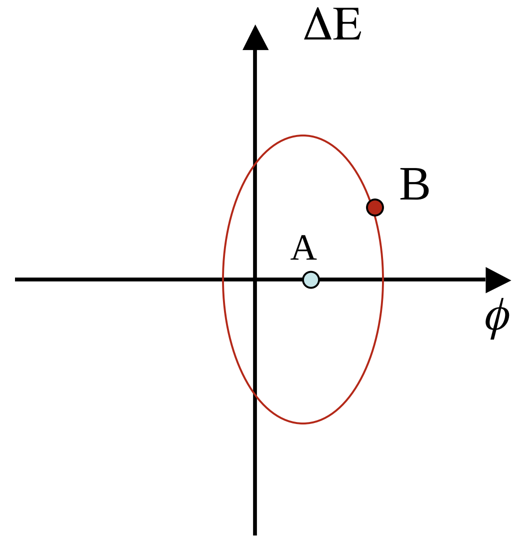
- Its rate of change of momentum is

$$\frac{dp_s}{dt} = eE_o \sin\varphi_s = \frac{eV}{L} \sin\varphi_s$$

What do we mean by phase?
Let's consider non-relativistic ions

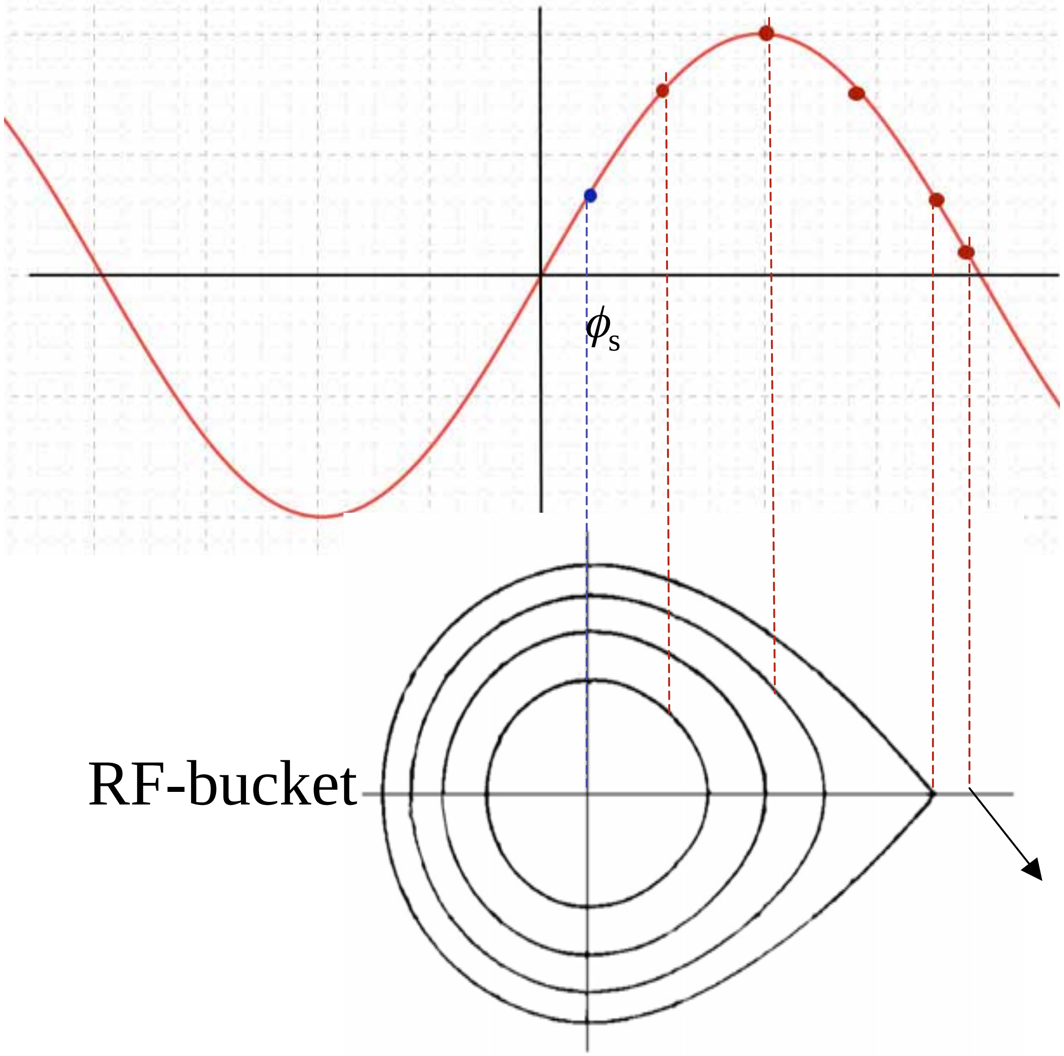


From E. J. N. Wilson CAS lecture



How does the ellipse change as B lags further behind A?

How does the ellipse change as B lags further behind A?



How does the size of the bucket change with ϕ_s ?

Two first order equations ==> one second order pendulum equation

$$\frac{d\varphi}{dn} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} \Delta E$$

and

$$\frac{\mathrm{d}\Delta \mathrm{E}}{dn} = eV(\sin\varphi - \sin\varphi_s)$$

yield

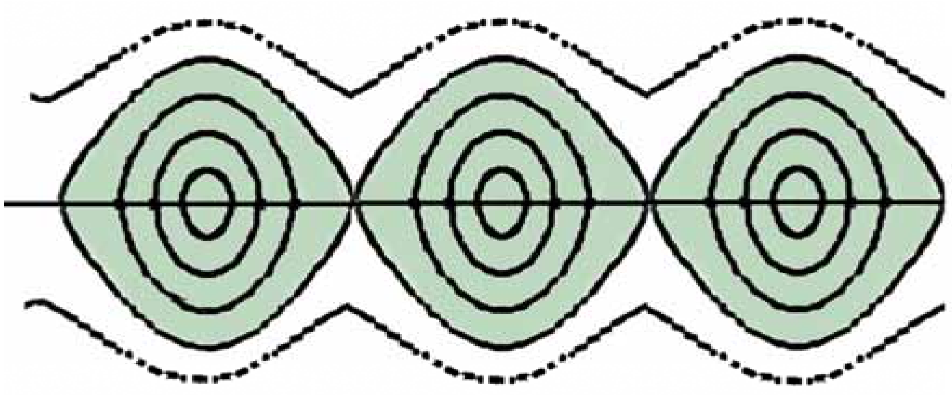
$$\frac{d^2\varphi}{dn^2} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\sin\varphi - \sin\varphi_s)$$

(Pendulum equation)

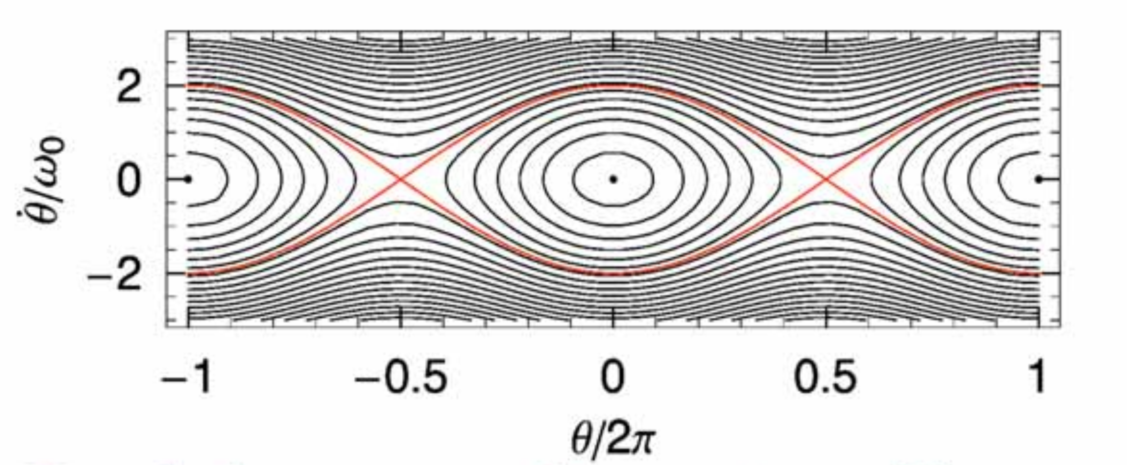
if

$$V = \text{constant and } \frac{\mathrm{dE_s}}{\mathrm{dn}} \text{ is sufficiently small}$$

For $\phi_\sigma = 0$ we have



We've seen this behavior for the pendulum



Now let's return to the question of frequency

For *small* phase differences, $\Delta\phi = \phi - \phi_s$, we can linearize our equations

$$\frac{d^2\varphi}{dn^2} = \frac{d^2\Delta\varphi}{dn^2} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\sin\varphi - \sin\varphi_s)$$

$$= \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV\left(\sin(\varphi_s + \Delta\varphi) - \sin\varphi_s\right)$$

$$\approx 4\pi^2 \left(\underbrace{\frac{\eta\omega_{rf}\tau}{4\pi^2\beta^2 E_s} eV\cos\varphi_s}_{-\nu_s^2} \right) \Delta\varphi$$

(harmonic oscillator in $\Delta\phi$)

$-\nu_s^2$ *Synchrotron tune*

$$\Omega_s = \frac{2\pi\nu_s}{\tau} = \sqrt{-\frac{\eta\omega_{rf}}{\tau\beta^2 E_s} eV\cos\varphi_s}$$ = synchrotron angular frequency

Choice of stable phase depends on η

$$\Omega_s = \sqrt{-\frac{\eta\omega_{rf}}{\tau\beta^2 E_s} eV\cos\varphi_s}$$

- Below transition ($\gamma < \gamma_t$),
  - → $\eta < 0$, therefore $\cos\phi_s$ must be > 0
- Above transition ($\gamma > \gamma_t$),
  - → $\eta > 0$, therefore $\cos\phi_s$ must be < 0
- At transition $\Omega_s = 0$; there is no phase stability
- Circular accelerators that must cross transition shift the synchronous phase at $\gamma > \gamma_t$
- Linacs have no path length difference, $\eta = 1/\gamma^2$; particles stay locked in phase and $\Omega_s = 0$

Bunch length

- In electron storage rings, statistical emission of synchrotron radiation photons generates gaussian bunches
- The over voltage *Q* is usually large
  - ➞ Bunch "lives" in the small oscillation region of the bucket.
  - ➞ Motion in the phase space is elliptical

$$\frac{\varphi^2}{\hat{\varphi}^2} + \delta^2 \left(\frac{h\omega_0 \eta_C}{\hat{\varphi}\Omega} \right)^2 = 1 \quad \Rightarrow \quad \hat{\varphi} = \frac{h\omega_0\eta_C}{\Omega}\hat{\delta} \Rightarrow \Delta s = \frac{c\eta_C}{\Omega}\frac{\Delta p}{p_0}$$

- For $\sigma_p/p_0 =$ *rms relative momentum spread*, the *rms bunch length* is

$$\sigma_{\Delta S} = \frac{c\eta_C}{\Omega}\frac{\sigma_p}{p_0} = \sqrt{\frac{c^3}{2\pi q}\frac{p_0\beta_0\eta_C}{h\, f_0^2\, \hat{V}\cos(\varphi_S)}}\;\frac{\sigma_p}{p_0}$$

Matching the beam on injection

- Beam injection from another rf-accelerator is typically "bucket-to-bucket"
  - rf systems of machines are phase-locked
  - bunches are transferred directly from the buckets of one machine into the buckets of the other

- This process is efficient for matched beams
  - Injected beam hits the middle of the receiving rf-bucket
  - Two machines are longitudinally matched.
    - They have the same aspect ratio of the longitudinal phase ellipse

Key concepts - 2

William A. Barletta
Director, United States Particle Accelerator School
Dept. of Physics, MIT

General Envelope Equation for Cylindrically Symmetric Beams

Can be generalized for sheet beams and beams with quadrupole focusing

Without scattering & in equilibrium

$$\ddot{R} + \frac{\dot{\gamma}}{\gamma}\dot{R} + \frac{U}{R} + \frac{\omega_c^2 R}{4} - \frac{E^2}{\gamma^2 R^3} = \frac{1}{\gamma^2 R^3}\int_{t_o}^{t} dt'\left(\frac{2\gamma R^2}{m}\varepsilon'\right)$$

(with $\ddot{R} = 0$, $\dot{\gamma} = 0$, $\varepsilon' = 0$)

$$\therefore \quad \frac{U}{R} + \frac{\tfrac{1}{4}\omega_c^2 R^2}{R} - \frac{E^2}{\gamma^2 R^3} = 0$$

Self-forces Focusing Emittance

More generally, $$\frac{U}{R} + \frac{\langle \omega_\beta^2 R^2 \rangle}{R} - \frac{E^2}{\gamma^2 R^3} = 0$$

RF Cavities

Basic principles and concepts

- Superposition
- Energy conservation
- Orthogonality (of cavity modes)
- Causality

Translate circuit model to a cavity model: Directly driven, re-entrant RF cavity

Outer region: Large, single turn Inductor

$$L = \frac{\mu_o \pi a^2}{2\pi(R+a)}$$

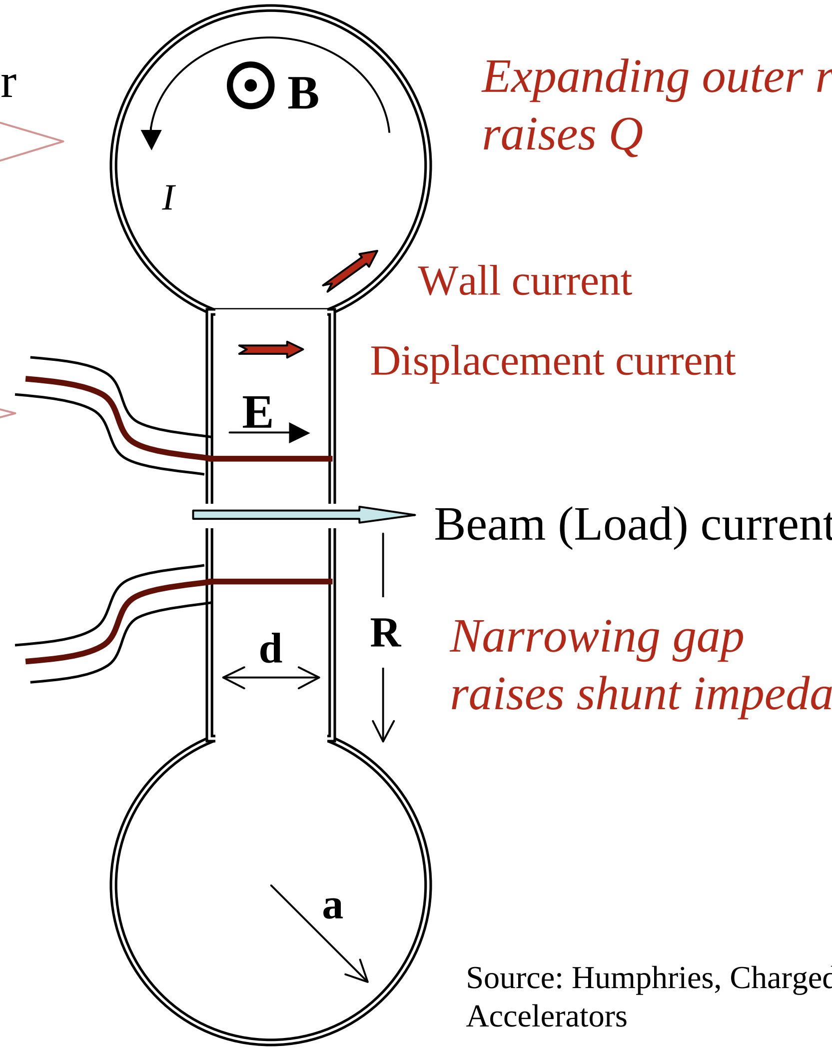
Central region: Large plate Capacitor

$$C = \varepsilon_o \frac{\pi R^2}{d}$$

$$\omega_o = 1/\sqrt{LC} = c\left[\frac{2((R+a)d}{\pi R^2 a^2}\right]^{1/2}$$

Q – set by resistance in outer region

$$Q = \sqrt{L/C}/R$$

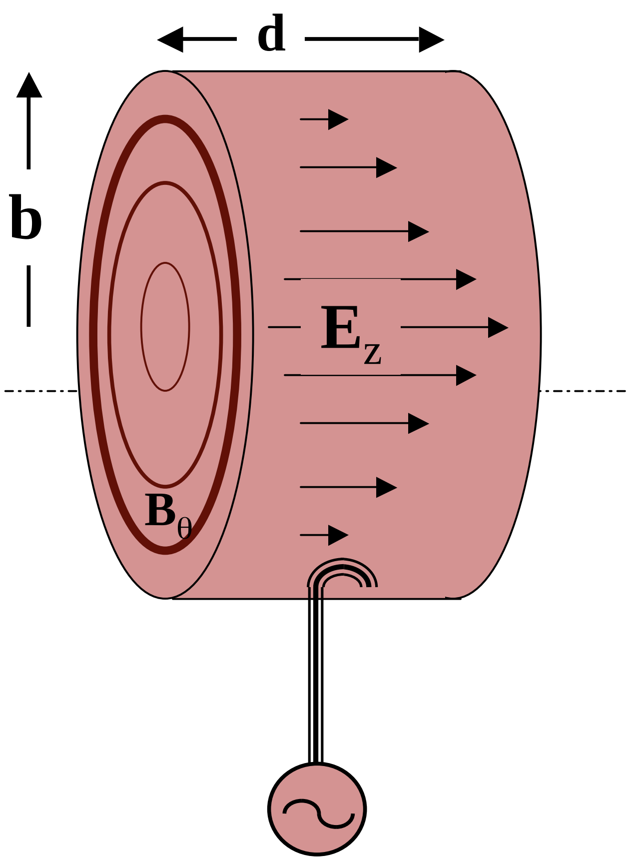


Expanding outer region raises Q

Narrowing gap raises shunt impedance

Source: Humphries, Charged Particle Accelerators

Properties of the RF pillbox cavity



$$\sigma_{walls} = \infty$$

- We want lowest mode: with only $\mathbf{E}_z$ & $\mathbf{B}_\theta$
- Maxwell's equations are:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \frac{1}{c^2}\frac{\partial}{\partial t}E_z \quad \text{and} \quad \frac{\partial}{\partial r}E_z = \frac{\partial}{\partial t}B_\theta$$

- Take derivatives

$$\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\right] = \frac{\partial}{\partial t}\left[\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}\right] = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\frac{\partial}{\partial r}\frac{\partial E_z}{\partial r} = \frac{\partial}{\partial r}\frac{\partial B_\theta}{\partial t}$$

==>

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

For a mode with frequency ω

✺ $$E_z(r,t) = E_z(r)\, e^{i\omega t}$$

✺ Therefore, $$E_z'' + \frac{E_z'}{r} + \left(\frac{\omega}{c}\right)^2 E_z = 0$$

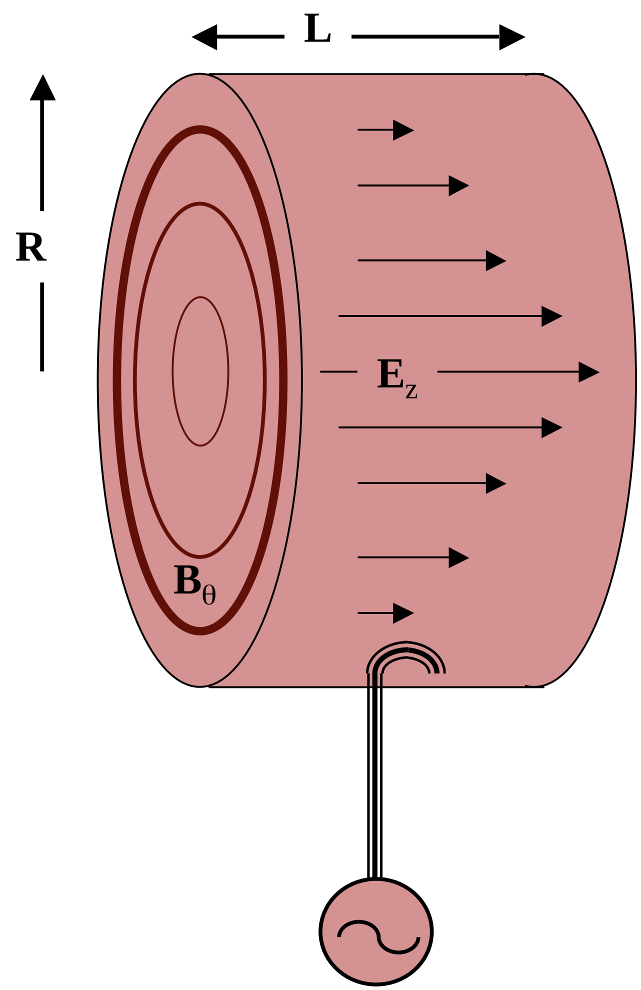
➙ (Bessel's equation, 0 order)

✺ Hence, $$E_z(r) = E_o\, J_o\left(\frac{\omega}{c} r\right)$$

✺ For conducting walls, $E_z(R) = 0$, therefore

$$\frac{2\pi f}{c} b = 2.405$$

Simple consequences of pillbox model

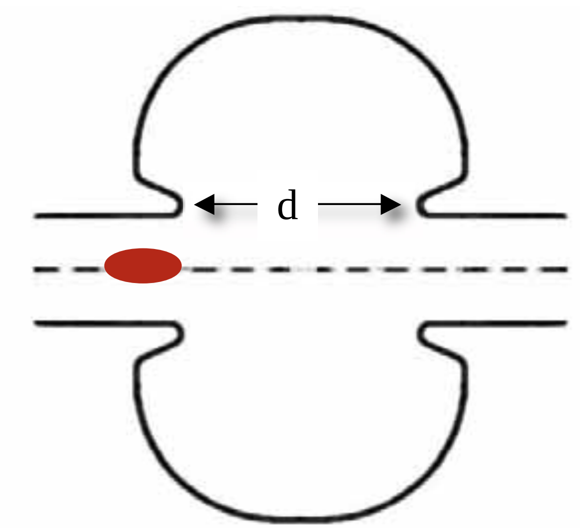


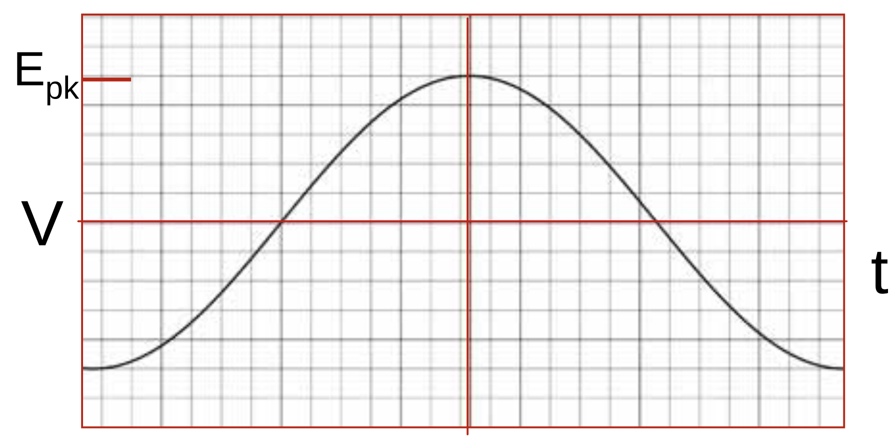
- Increasing R lowers frequency
  ==> Stored Energy, $\mathcal{E} \sim \omega^{-2}$
- $\mathcal{E} \sim E_z^2$
- Beam loading lowers E_z for the next bunch
- Lowering ω lowers the fractional beam loading
- Raising ω lowers $Q \sim \omega^{-1/2}$
- If time between beam pulses,
  $$T_s \sim Q/\omega$$
  almost all $\mathcal{E}$ is lost in the walls

Cavity figures of merit

- The voltage varies during time that bunch takes to cross gap
  - ➞ reduction of the peak voltage by Γ (transt time factor)

$$\Gamma = \frac{\sin\left(\vartheta/2\right)}{\vartheta/2} \quad \text{where} \quad \vartheta = \omega d/\beta c$$





For maximum acceleration $T_{\rm cav} = \frac{d}{c} = \frac{T_{\rm rf}}{2}$ ==> Γ = 2/π

Figure of merit from circuits - Q

$$Q = \frac{\omega_o \; 0 \; Energy\ stored}{Time\ average\ power\ loss} = \frac{2\pi \; 0 \; Energy\ stored}{Energy\ lost\ per\ cycle}$$

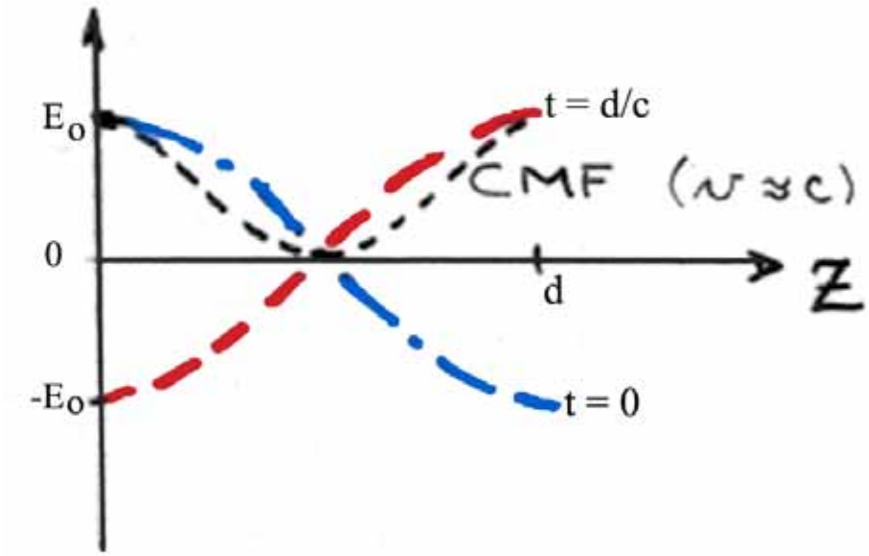
$$\mathrm{E} = \frac{\mu_o}{2} \int_v |H|^2 dv = \frac{1}{2} L\ I_o I_o^*$$

$$\langle \mathrm{P} \rangle = \frac{R_{surf}}{2} \int_s |H|^2 ds = \frac{1}{2} I_o I_o^* R_{surf}$$

$$R_{surf} = \frac{1}{Conductivity \; 0 \; Skin\ depth} \sim \omega^{1/2}$$

$$\therefore \quad Q = \frac{\sqrt{L/C}}{R_{surf}} = \left(\frac{\Delta\omega}{\omega_o} \right)^{-1}$$

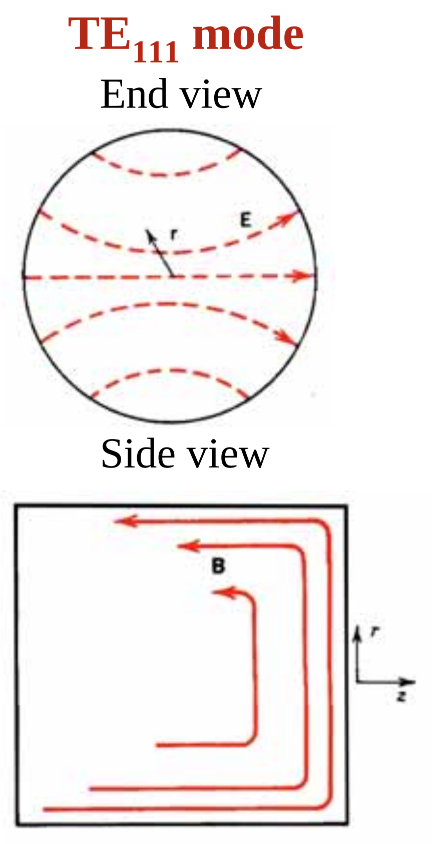
Compute the voltage gain correctly

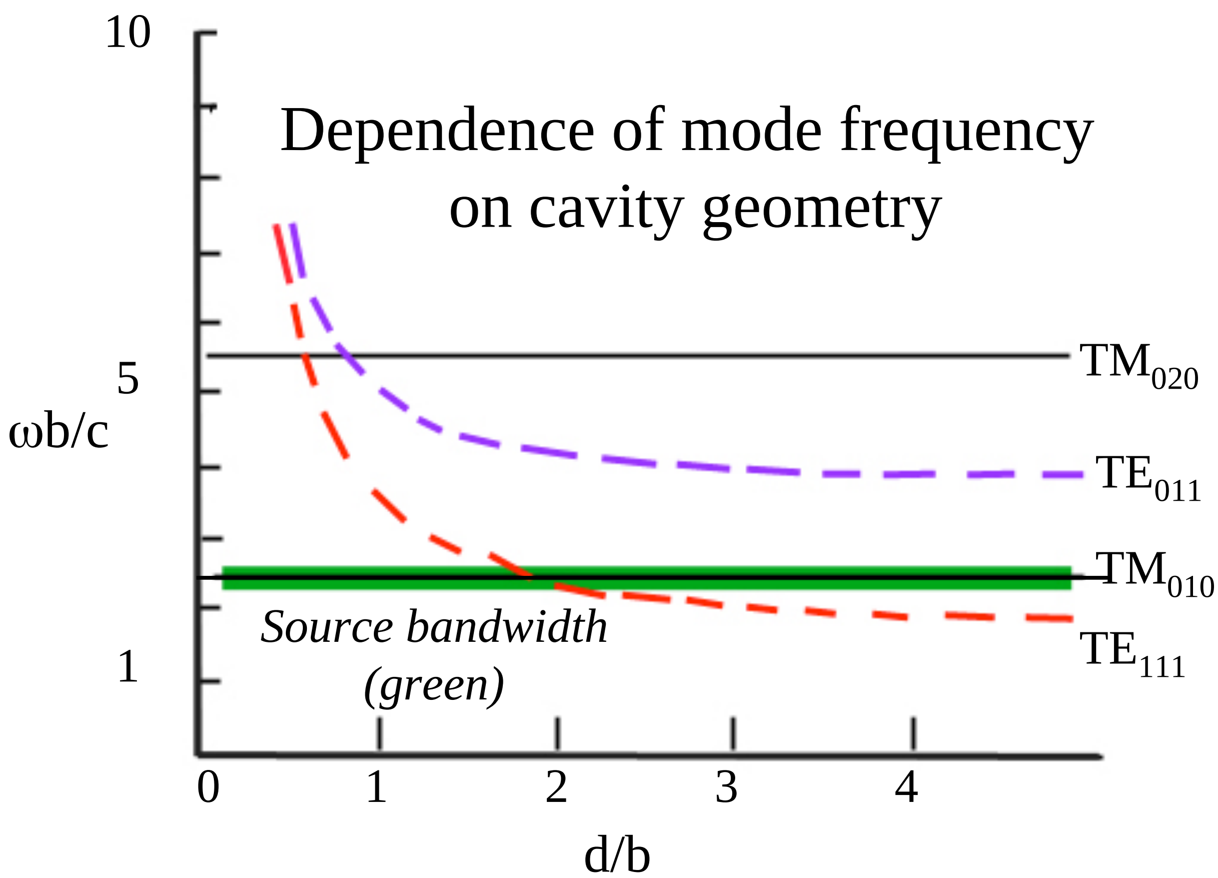


The voltage gain seen by the beam can computed in the co-moving frame, or we can use the transit-time factor, Γ & compute V at fixed time

$$V_o^2 = \Gamma \int_{z_1}^{z_2} E(z)dz$$

Keeping energy out of higher order modes





Choose cavity dimensions to stay far from crossovers

Figure of merit for accelerating cavity: power to produce the accelerating field

Resistive input (shunt) impedance at ω_o relates power dissipated in walls to accelerating voltage

$$R_{in} = \frac{\langle V^2(t) \rangle}{P} = \frac{V_o^2}{2P} = Q\sqrt{L/C}$$

Linac literature commonly defines "shunt impedance" without the "2"

$$R_{in} = \frac{V_o^2}{P} \sim \frac{1}{R_{surf}}$$

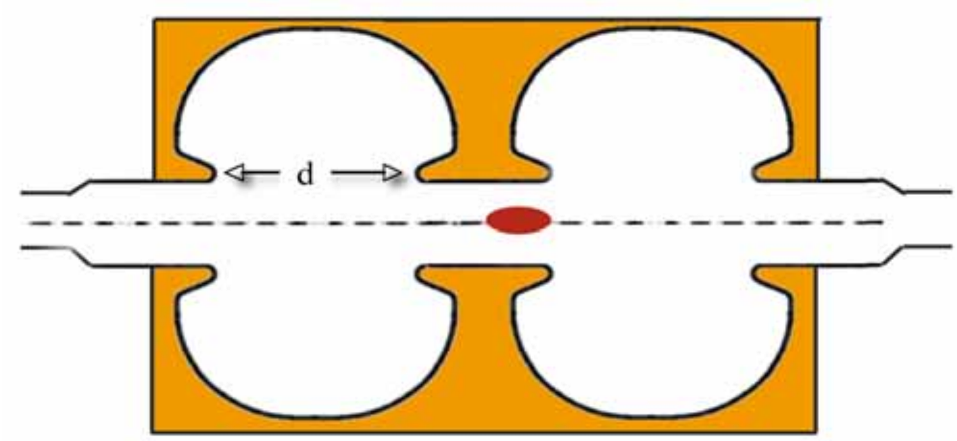
Typical values 25 - 50 MΩ

Unit 4 - Lecture 10

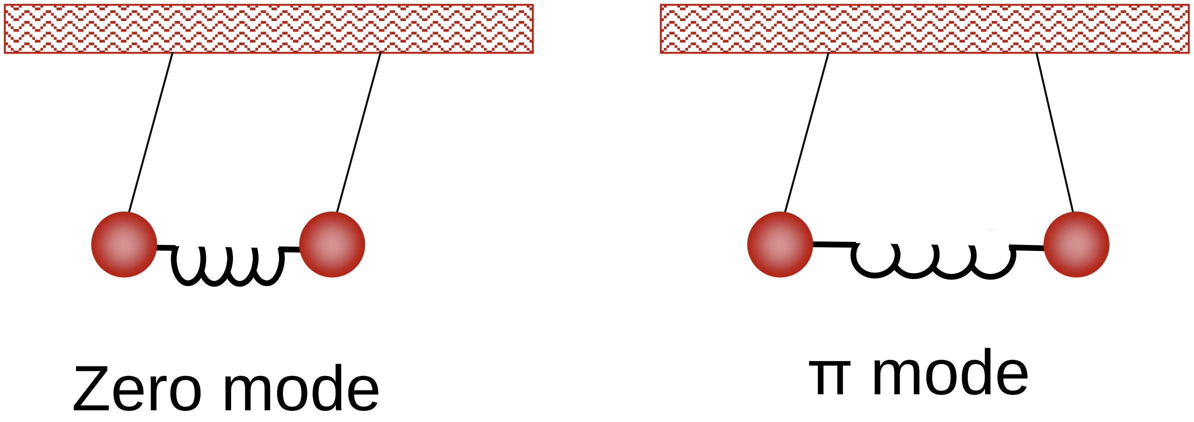
RF-accelerators: Standing wave linacs

William A. Barletta
Director, United States Particle Accelerator School
Dept. of Physics, MIT

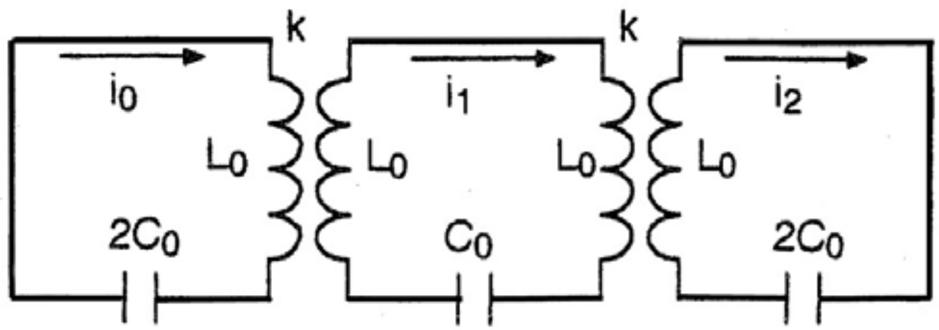
Linacs cells are linked to minimize cost



==> coupled oscillators ==>multiple modes



Example of 3 coupled cavities



$$x_0\left(1-\frac{\omega_0^2}{\Omega^2}\right)+x_1 k=0 \qquad \text{oscillator } n=0$$

$$x_1\left(1-\frac{\omega_0^2}{\Omega^2}\right)+(x_0+x_2)\frac{k}{2}=0 \qquad \text{oscillator } n=1$$

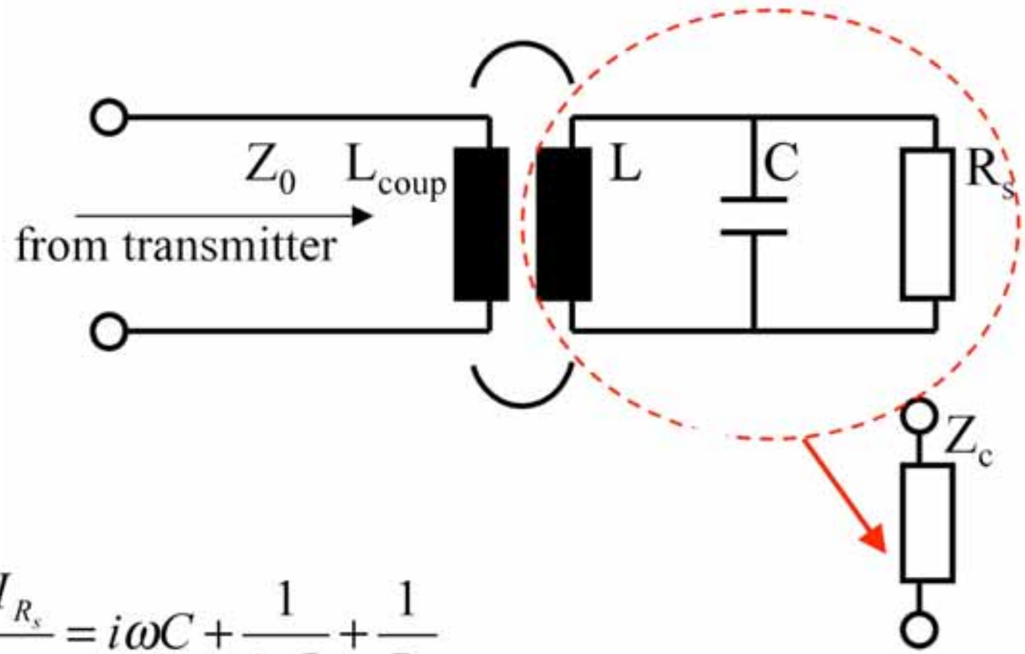
$$x_2\left(1-\frac{\omega_0^2}{\Omega^2}\right)+x_1 k=0 \qquad \text{oscillator } n=2$$

$x_j = i_j\sqrt{2L_o}$ and $\Omega =$ normal mode frequency

Lumped circuit of a transmission line coupled cavity without beam

$$\omega_0^2 = \frac{1}{LC}$$ resonance frequency of RF-structure

$$f_0 \approx 50\,MHz - 3\,GHz$$



$$\frac{1}{Z_c} = Y_c = \frac{I_L + I_c + I_{R_s}}{U_{Z_c}} = i\omega C + \frac{1}{i\omega L} + \frac{1}{R_s}$$

$$Z_c = \frac{R_s}{1 + i\frac{R_s}{\omega L}\left(\frac{\omega^2}{\omega_0^2} - 1\right)} = \frac{R_s}{1 + iQ_0\frac{\omega_0}{\omega}\left(\frac{\omega^2}{\omega_0^2} - 1\right)} = \frac{R_s}{1 + iQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

norm. detuning ξ

$$Q_0 = \frac{R_s}{\omega_0 L} = R_s \omega_0 C$$

unloaded quality factor of cavity

At resonance, the rf source & the beam have the following effects

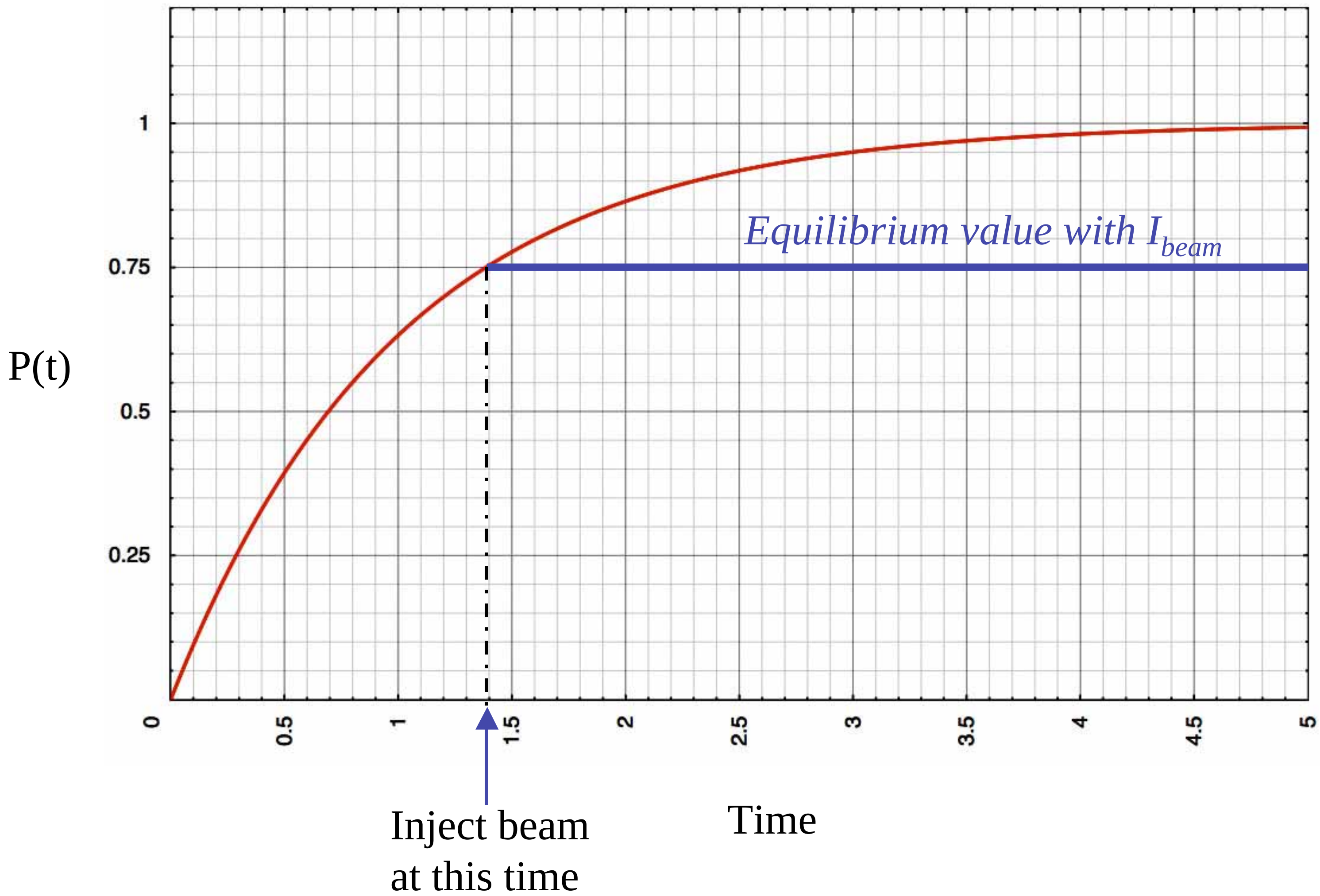
- The accelerating voltage is the sum of these effects

$$V_{accel} = \sqrt{R_{shunt} P_{gen}} \left[\frac{2\sqrt{\beta}}{1+\beta} \left(1 - \frac{K}{\sqrt{\beta}} \right) \right] = \sqrt{R_{shunt} P_{wall}}$$

where $K = \frac{I_{dc}}{2} \sqrt{R_{shunt}/P_{gen}}$ is the "loading factor"

- ==> V_{acc} decreases linearly with increasing beam current

Power flow in standing wave linac



Comparison of SC and NC RF

Superconducting RF

- High gradient
  ==> 1 GHz, meticulous care

- Mid-frequencies
  ==> Large stored energy, $\mathbf{E}_s$

- Large $\mathbf{E}_s$
  ==> very small ΔE/E

- Large Q
  ==> high efficiency

Normal Conductivity RF

- High gradient
  ==> high frequency (5 - 17 GHz)

- High frequency
  ==> low stored energy

- Low $\mathbf{E}_s$
  ==> ~10x larger ΔE/E

- Low Q
  ==> reduced efficiency