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August 2002

Formulae for the waves in a waveguide terminated by a beam-loaded cavity

Waveguide impedance transformed to cavity is

$$Z_{\mathrm{WG}} = R/Q \cdot Q_{\mathrm{ext}} ,$$

where *R/Q* is the cavity specific impedance and Q_{ext} is the external quality factor. The cavity coupling factor is then defined by the ratio of cavity intrinsic quality factor Q_0 and its external *Q*:

$$\beta = \frac{Q_0}{Q_{\mathrm{ext}}} ,\ \beta + 1 = \frac{Q_0}{Q_{\mathrm{L}}} ,$$

here Q_{L} is the cavity loaded quality factor. The waveguide is terminated by the cavity with impedance

$$Z_{\mathrm{c}} = R_{\mathrm{c}} + X_{\mathrm{c}} = \frac{R/Q \cdot Q_0}{1 + i \tan\psi} ,\ \tan\psi = 2Q_0 \frac{\Delta\omega}{\omega} ,$$

where ψ is the cavity tuning angle and $\Delta\omega$ is the cavity resonance detuning from the RF frequency, and in parallel by the beam with admittance

$$Y_{\mathrm{b}} = G_{\mathrm{b}} + B_{\mathrm{b}} = \frac{I_{\mathrm{b}}}{V_{\mathrm{c}}} e^{i\varphi_0} ,$$

ϕ_0 is the beam phase. Then the total load is

$$G = \frac{1}{R/Q \cdot Q_0} + \frac{I_{\mathrm{b}}}{V_{\mathrm{c}}} \cos\varphi_0 ,$$

$$B = \frac{\tan\psi}{R/Q \cdot Q_0} + \frac{I_{\mathrm{b}}}{V_{\mathrm{c}}} \sin\varphi_0 ,$$

$$Y = G + B = \frac{1}{Z} .$$

The reflection coefficient for such load is

$$\Gamma_{\mathrm{V}} = \frac{V_{\mathrm{refl}}}{V_{\mathrm{forw}}} = \frac{Z/Z_{\mathrm{WG}} - 1}{Z/Z_{\mathrm{WG}} + 1} = \frac{1 - Y \cdot Z_{\mathrm{WG}}}{1 + Y \cdot Z_{\mathrm{WG}}} = \frac{1 - \tilde{Y}}{1 + \tilde{Y}} ,$$

The cavity voltage is determined by the forward and reflected waves at the load:

$$V_{\mathrm{c}} = V_{\mathrm{forw}} + V_{\mathrm{refl}} = V_{\mathrm{forw}} (1 + \Gamma_{\mathrm{V}}) = \frac{2}{1 + \tilde{Y}} \cdot V_{\mathrm{forw}} ,$$

One can now get following expressions for the forward and reflected waves:

$$V_{\mathrm{forw}} = \frac{V_{\mathrm{c}}}{2}\cdot\left(1+\tilde{Y}\right) = \frac{V_{\mathrm{c}}}{2}\left[1+\frac{1}{\beta}+\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{V_{\mathrm{c}}}\cos\varphi_0 + i\frac{\tan\psi}{\beta} + i\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{V_{\mathrm{c}}}\sin\varphi_0\right] =$$

$$= \frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{2}\left(\cos\varphi_0 + i\sin\varphi_0\right) + \frac{V_{\mathrm{c}}}{2}\left(\frac{\beta+1}{\beta} + i\frac{\tan\psi}{\beta}\right) =$$

$$= \frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{2}\left(\cos\varphi_0 + i\sin\varphi_0\right) + \frac{V_{\mathrm{c}}}{2}\frac{\beta+1}{\beta}\left(1+i\tan\psi'\right)$$

$$V_{\mathrm{refl}} = \frac{V_{\mathrm{c}}}{2}\cdot\left(1-\tilde{Y}\right) = \frac{V_{\mathrm{c}}}{2}\left[1-\frac{1}{\beta}-\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{V_{\mathrm{c}}}\cos\varphi_0 - i\frac{\tan\psi}{\beta} - i\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{V_{\mathrm{c}}}\sin\varphi_0\right] =$$

$$= -\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{2}\left(\cos\varphi_0 + i\sin\varphi_0\right) + \frac{V_{\mathrm{c}}}{2}\left(\frac{\beta-1}{\beta} - i\frac{\tan\psi}{\beta}\right) =$$

$$= -\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{2}\left(\cos\varphi_0 + i\sin\varphi_0\right) + \frac{V_{\mathrm{c}}}{2}\frac{\beta+1}{\beta}\left(\frac{\beta-1}{\beta+1} - i\tan\psi'\right)$$

$$P_{\mathrm{forw}} = \frac{\left|V_{\mathrm{forw}}\right|^2}{Z_{\mathrm{WG}}} = \frac{V_{\mathrm{c}}^{\,2}}{4R/Q\cdot Q_{\mathrm{ext}}}\cdot\left|1+\frac{1}{\beta}+\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{V_{\mathrm{c}}}\cos\varphi_0 + i\frac{\tan\psi}{\beta} + i\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{V_{\mathrm{c}}}\sin\varphi_0\right|^2 =$$

$$= \frac{V_{\mathrm{c}}^{\,2}}{4R/Q\cdot Q_{\mathrm{ext}}}\cdot\left\{\left[\frac{\beta+1}{\beta}+\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{V_{\mathrm{c}}}\cos\varphi_0\right]^2 + \left[\frac{\tan\psi}{\beta}+\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}}{V_{\mathrm{c}}}\sin\varphi_0\right]^2\right\} =$$

$$= \frac{V_{\mathrm{c}}^{\,2}}{4R/Q\cdot Q_{\mathrm{ext}}}\cdot\frac{(\beta+1)^2}{\beta^2}\cdot\left\{\left[1+\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{L}}}{V_{\mathrm{c}}}\cos\varphi_0\right]^2 + \left[\tan\psi'+\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{L}}}{V_{\mathrm{c}}}\sin\varphi_0\right]^2\right\}$$

$$\tan\psi' = 2Q_{\mathrm{L}}\frac{\Delta\omega}{\omega}\ .$$

To compensate the reactive part of the beam impedance, the cavity has to be detuned so that

$$I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{ext}}\sin\varphi_0 + V_{\mathrm{c}}\frac{\beta+1}{\beta}\tan\psi' = 0$$

$$\tan\psi' = -\frac{I_{\mathrm{b}}\,R/Q\cdot Q_{\mathrm{L}}\sin\varphi_0}{V_{\mathrm{c}}}$$

or

$$\Delta\omega = -\frac{I_{\mathrm{b}}\,R/Q\cdot\omega\cdot\sin\varphi_0}{2V_{\mathrm{c}}}\ .$$

Then matched or reflection-free condition will be reached at the beam current

$$I_{\mathrm{b}}\, R/Q \cdot Q_{\mathrm{ext}} \cos\varphi_0 = V_{\mathrm{c}} \cdot \frac{\beta - 1}{\beta}$$
$$I_{\mathrm{b}} = \frac{V_{\mathrm{c}}}{R/Q \cdot Q_{\mathrm{ext}} \cos\varphi_0} \frac{\beta - 1}{\beta} \quad .$$

This corresponds to forward power

$$P_{\mathrm{forw}} = \frac{V_{\mathrm{c}}^{\;2}}{R/Q \cdot Q_{\mathrm{ext}}} \quad .$$