Unit 4 - Lectures 11 & 12
Acceleration by RF waves

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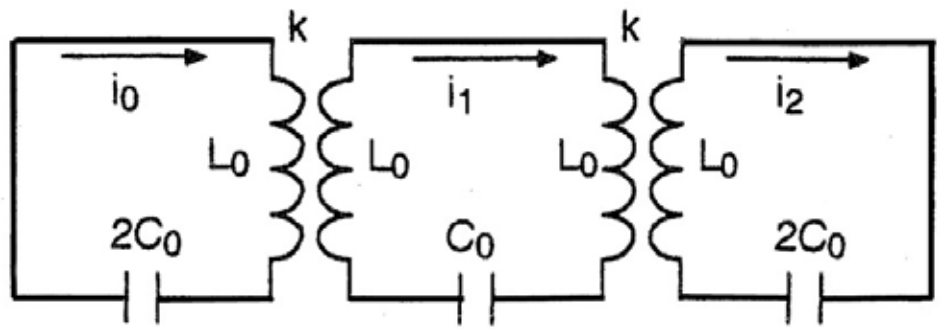
Sources: USPAS Course notes by F. Sannibale
High Energy Electron Linacs by P. Wilson

Example of 3 coupled cavities



$$x_0\left(1-\frac{\omega_0^2}{\Omega^2}\right)+x_1 k=0 \qquad \text{oscillator } n=0$$

$$x_1\left(1-\frac{\omega_0^2}{\Omega^2}\right)+(x_0+x_2)\frac{k}{2}=0 \qquad \text{oscillator } n=1$$

$$x_2\left(1-\frac{\omega_0^2}{\Omega^2}\right)+x_1 k=0 \qquad \text{oscillator } n=2$$

$x_j = i_j\sqrt{2L_o}$ and $\Omega =$ normal mode frequency

Write the coupled circuit equations in matrix form

$$\mathbf{L}\mathbf{x}_q = \frac{1}{\Omega_q^2}\mathbf{x}_q \quad \text{where} \quad \mathbf{L} = \begin{pmatrix} 1/\omega_o^2 & k/\omega_o^2 & 0 \\ k/2\omega_o^2 & 1/\omega_o^2 & k/2\omega_o^2 \\ 0 & k/\omega_o^2 & 1/\omega_o^2 \end{pmatrix} \quad \text{and} \quad \mathbf{x_q} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- Compute eigenvalues & eigenvectors to find the three normal modes

$$\text{Mode q} = 0: \text{ zero mode} \qquad \Omega_0 = \frac{\omega_o}{\sqrt{1+k}} \qquad \mathbf{x_0} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\text{Mode q} = 1: \ \pi/2 \text{ mode} \qquad \Omega_1 = \omega_o \qquad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\text{Mode q} = 2: \ \pi \text{ mode} \qquad \Omega_2 = \frac{\omega_o}{\sqrt{1-k}} \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Exercise:

B fields can change the trajectory of a particle

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$$

Show that B fields cannot change its energy

We will cast our discussion in terms of the E field

Electromagnetic waves

From Maxwell equations, we can derive

$$\nabla^2 E_i = \frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} \qquad i = x, y, z$$

for electromagnetic waves in free space (no charge or current distributions present).

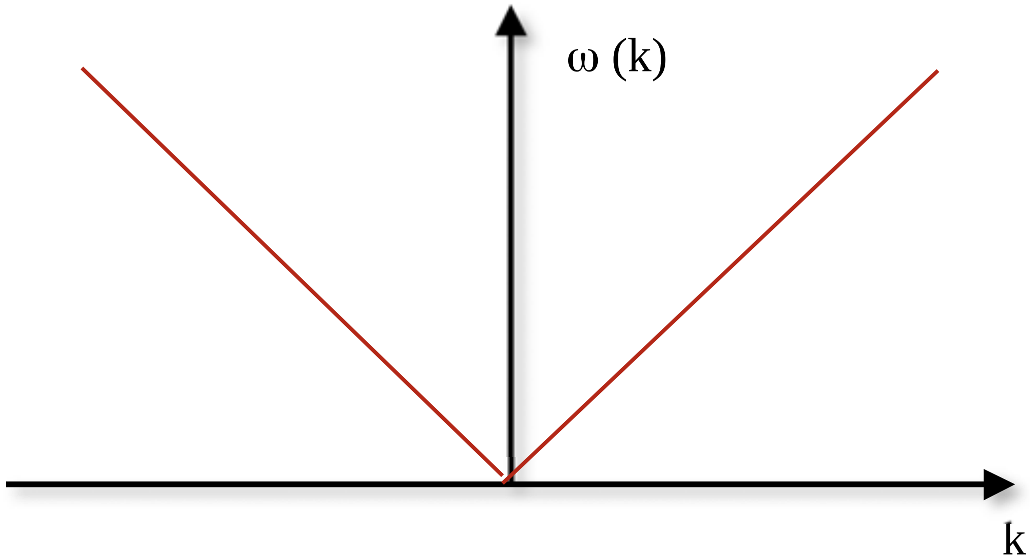
The plane wave is a particular solution of the EM wave equation

$$\overline{E} = \overline{E}_o e^{i(\omega t - ks)} = \overline{E}_o \left[\cos\underbrace{(\omega t - ks)}_{\text{Phase of the wave } = \phi} + i \sin(\omega t - ks)\right]$$

when

$$\omega = c\, k$$

Dispersion (Brillouin) diagram for a monochromatic plane wave



The phase of this plane wave is constant for

$$\frac{d\phi}{dt} = \omega - k\frac{ds}{dt} \equiv \omega - kv_{ph} = 0$$

or

$$v_{ph} = \frac{\omega}{k} = c$$

Plane wave representation of EM waves

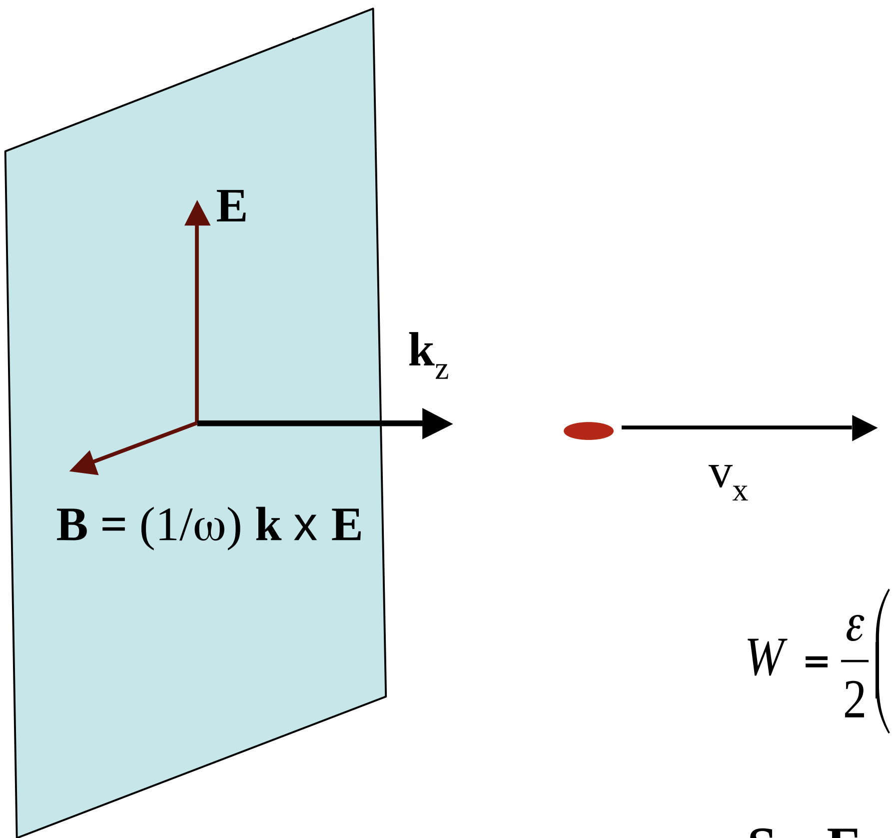
- In more generality, we can represent an arbitrary wave as a sum of plane waves:

$$\overline{E} = \sum_{n=-\infty}^{\infty} \overline{E}_{no} e^{i(n\omega_0 t - ks)} \qquad \overline{E} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, f(\omega) e^{i(\omega t - ks)}$$

Periodic Case

Non-periodic Case

Exercise: Can the plane wave accelerate the particle in the x-direction?



$$W = \frac{\varepsilon}{2}\left(E^2 + \frac{1}{\varepsilon\mu}B^2\right) = \frac{\varepsilon}{2}\left(E^2 + c^2B^2\right) = \varepsilon E^2$$

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{1}{\mu}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu c}E^2 = \sqrt{\frac{\varepsilon}{\mu}}E^2$$

Fields in waveguides

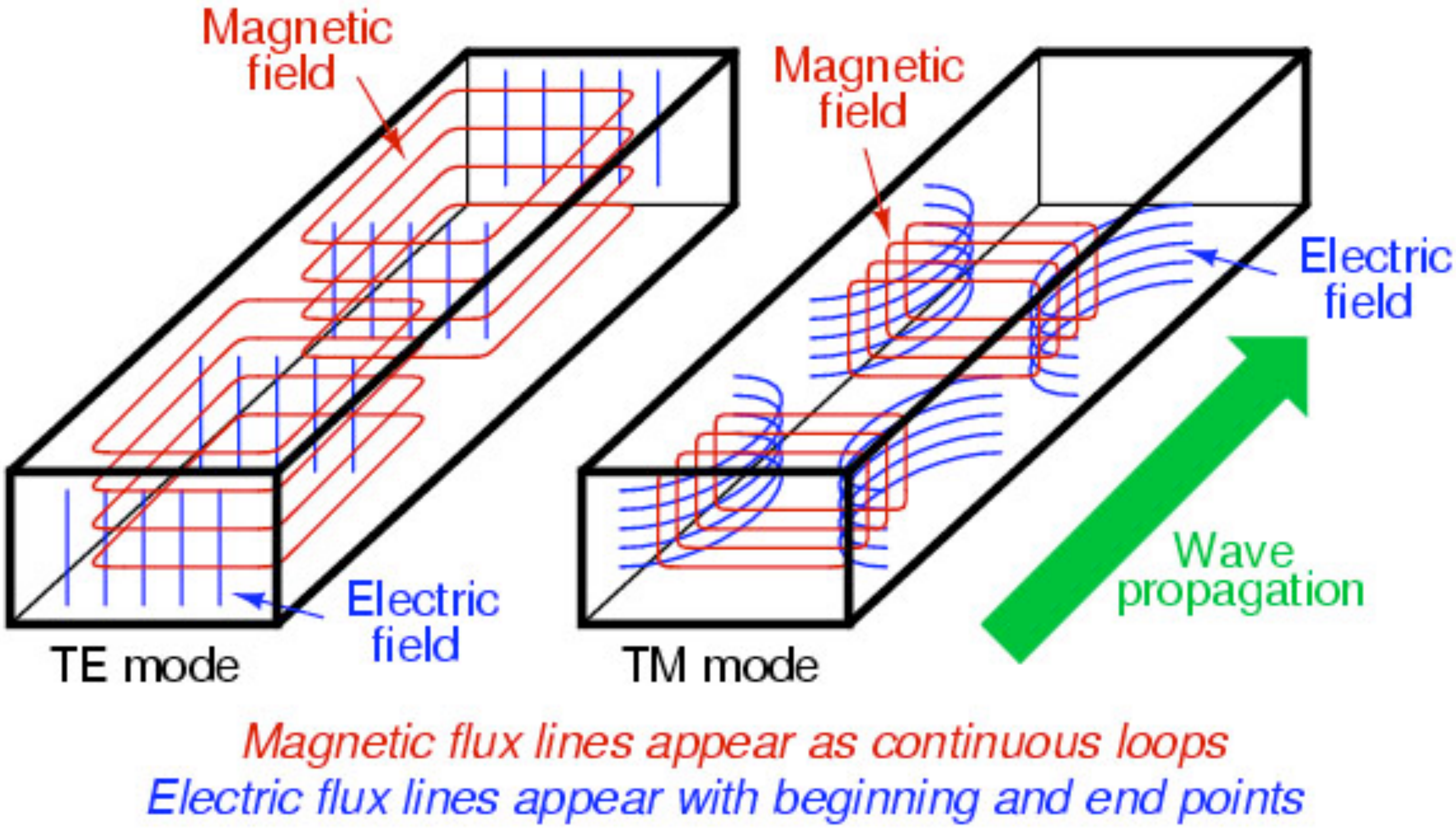


Figure source: www.opamp-electronics.com/tutorials/waveguide
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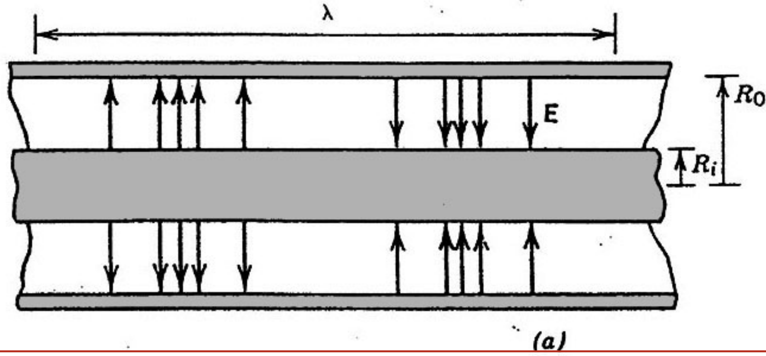
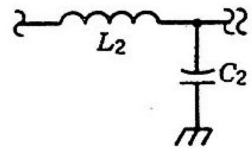
Can the accelerating structure be a simple (smooth) waveguide?

- Assume the answer is "yes"
- Then $\mathbf{E} = \mathbf{E}(r,\theta)\, e^{i(\omega t - kz)}$ with $\omega/k = v_{ph} < c$
- Transform to the frame co-moving at $v_{ph} < c$
- Then,
  - ➙ The structure is unchanged (by hypothesis)
  - ➙ E is static (v_{ph} is zero in this frame)
  - ==> By Maxwell's equations, H =0
  - ==> $\nabla \circ \mathbf{E} = 0$ and $\mathbf{E} = -\nabla\phi$
  - ➙ But ϕ is constant at the walls (metallic boundary conditions)
  - ==> $\mathbf{E} = 0$

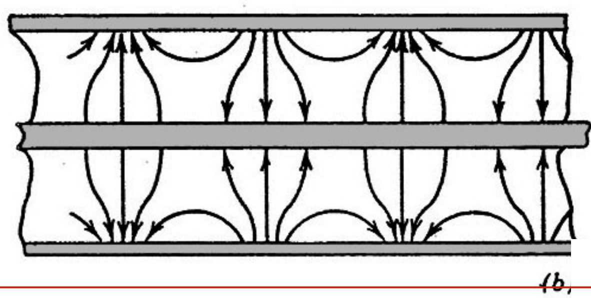
The assumption is false, smooth structures have $v_{ph} > c$

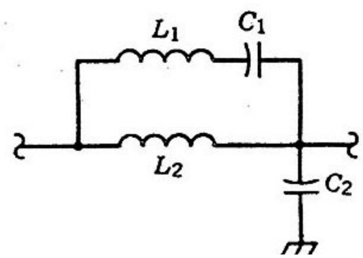
Propagating modes & equivalent circuits

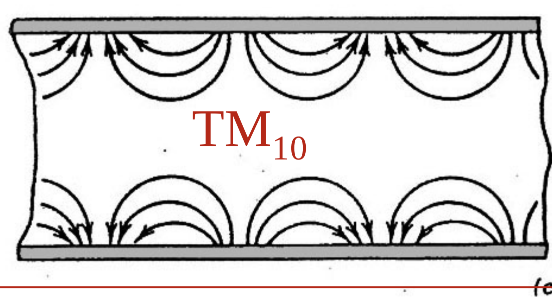


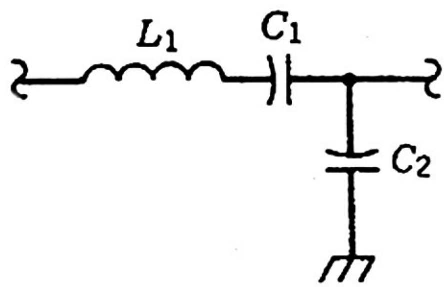


All frequencies can propagate



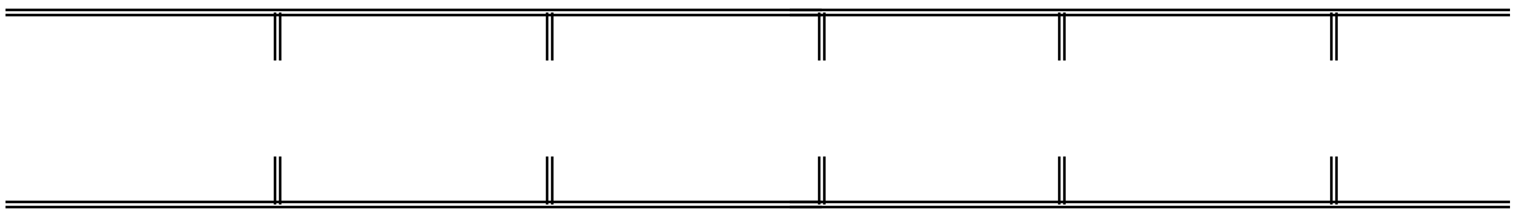






Propagation is cut-off at low frequencies

To slow the wave, add irises



In a transmission line the irises

a) Increase capacitance, C

b) Leave inductance ~ constant

c) ==> lower impedance, Z

d) ==> lower v_{ph}

Similar for TM01 mode in the waveguide

$$\frac{\omega}{k} = \frac{1}{\sqrt{LC}}$$

$$Z = \frac{L}{C}$$

Traveling wave structures

Consider a periodic structure of period p along the z-axis. By Floquet's theorem, at a given ω the fields at z & z+p differ only by a complex constant

$$\mathbf{E}(r,\phi,z,t) = \mathbf{E}_p(r,\phi,z)\, e^{-\gamma z} e^{j\omega t}$$

where

$$\gamma = jk + \alpha \quad \text{and} \quad \mathbf{E}_p \text{ is periodic}$$

Then

$$\mathbf{E}(r,\phi,z,t) = \sum_{n=-\infty}^{\infty} \mathbf{E}_p(r,\phi)\, e^{-\alpha z} e^{j(\omega t - k_n z)}$$

with

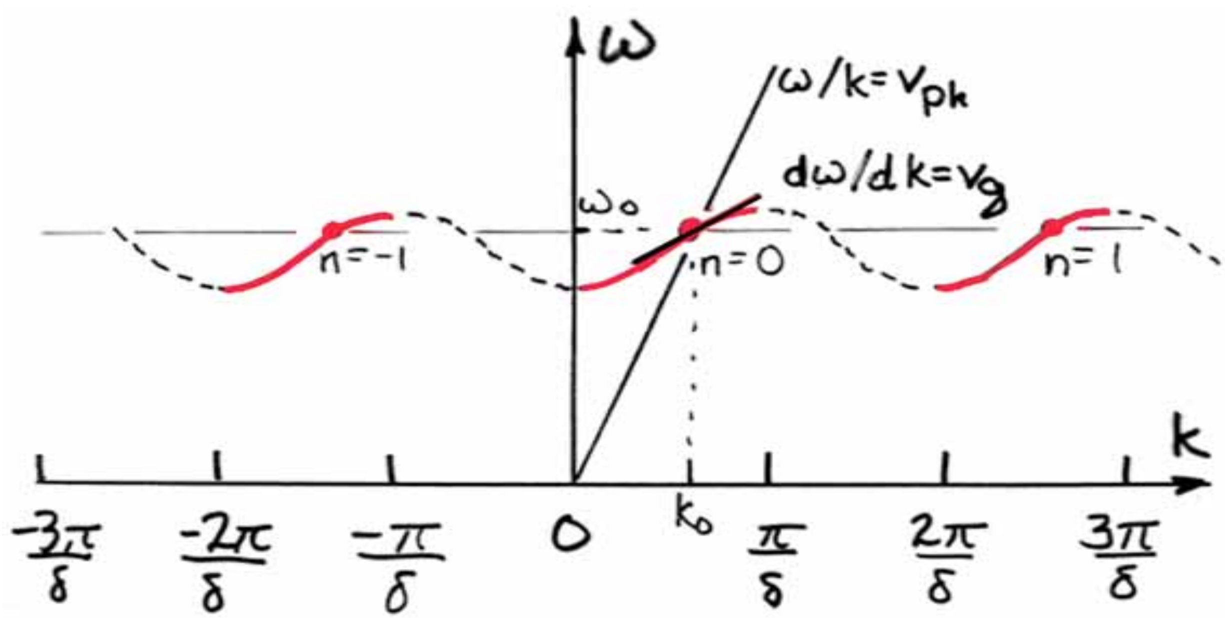
$$k_n = k_o + \frac{2\pi n}{p}$$

and

$$\mathbf{E}_n(r,\phi) = \frac{1}{p}\int_z^{z+p} \mathbf{E}_p(r,\phi,z)\, e^{j\left(\frac{2\pi n}{p}\right)z}\, dz$$

Traveling waves in periodic structures

- The traveling wave is a sum of spatial harmonics
- Each harmonic has
  - ➙ a propagation constant k_n
  - ➙ a phase velocity $v_{ph,n} = \omega/k_n$
  - ➙ a group velocity $v_g = d\omega/dk$



Typical RF accelerating structures have axial symmetry

- Natural coordinates are cylindrical coordinates

- Write the wave equation for E_z

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

+

$$E_z = E_{0z}(r,\theta) e^{\pm i\omega t} e^{\pm ikt}$$

+

$$\nabla^2 E_z = -\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 E_{0z}}{\partial r^2} + \frac{1}{r}\frac{\partial E_{0z}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_{0z}}{\partial \theta^2} + \left(\frac{\omega^2}{c^2} - k^2\right) E_{0z} = 0$$

Assume that the azimuthal component of the field has periodicity *n*

$$E_{0z}(r,\theta) = \tilde{E}_{0z}(r)e^{\pm in\theta}$$

$$\Rightarrow \quad \frac{\partial^2 \tilde{E}_{0z}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{E}_{0z}}{\partial r} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{n^2}{r^2}\right)\tilde{E}_{0z} = 0$$

This equation has a general solution in Bessel functions

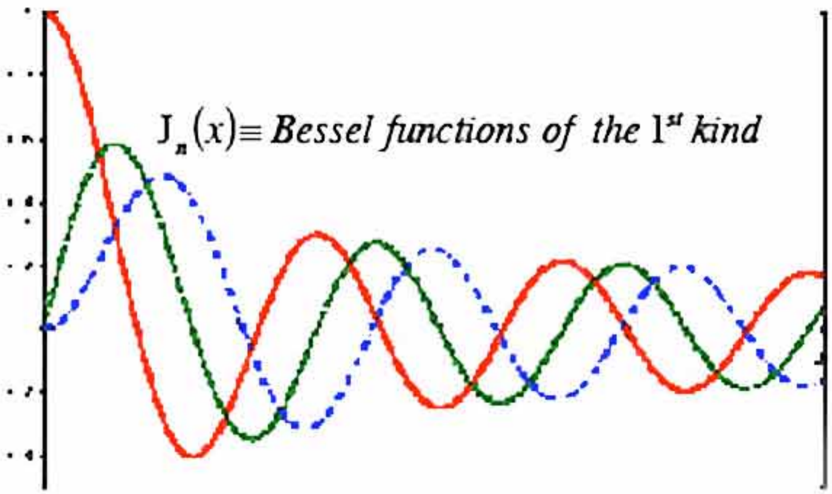
$$\boxed{\tilde{E}_{0z} = AJ_n(k_c r) + BY_n(k_c r)}$$

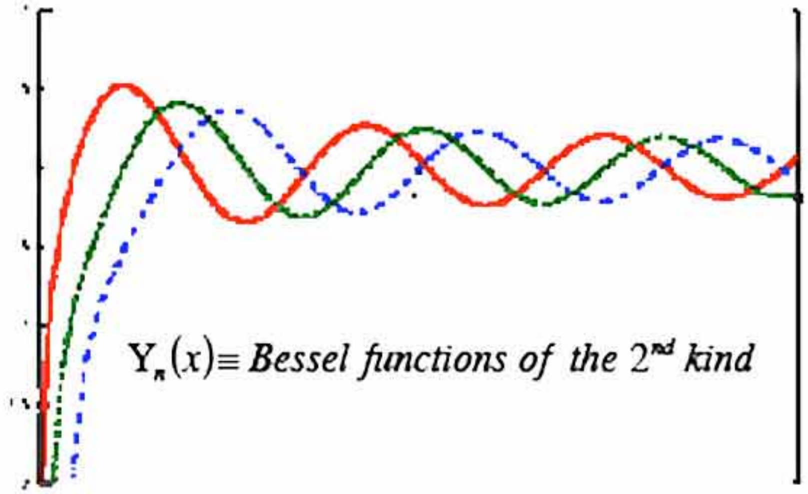
where

$$k_c^2 \equiv \frac{\omega^2}{c^2} - k^2$$

is the cutoff wave number

General solution of the wave equation





Remembering that the field must be finite at r = 0, we eliminate the terms in Y_n

$$E_z(r,\theta,z,t) = \cos m\theta \sum_{n=-\infty}^{\infty} A_n J_m(X_n r) e^{i(\omega t - k_n z)}$$

with

$$X_n = \left(\omega/c\right)^2 - k_n^2$$

Near the axis of the wave guide, the field has the form

$$E_z(r,\theta,z,t) = \cos m\theta \sum_{n=-\infty}^{\infty} A_n r^m e^{i(\omega t - k_n z)}$$

For the lowest synchronous mode, m = 0,

E_z is independent of r (particle trajectory)

For ion accelerators, design is complicated by the fact that $v_{ion} < c$ and changing. Therefore the structure must change to assure phase stability.

We will restrict attention to high energy injection, $v_{ion} \sim c$

Modes of propagation

- **Transverse Electric (TE)**
  - Longitudinal E-field component, $E_z = 0$
- **Transverse Magnetic (TM)**
  - $B_z = 0$
- **Transverse Electro-Magnetic (TEM)**
  - E_z ; H_z = 0 everywhere
  - Note: Hollow wave guide, whose walls are perfect conductors, cannot support propagation of TEM waves.
- The accelerating modes are the TM modes

Notation:

T_{nm} *where n = periodicity in θ, m = periodicity in r*

The cutoff frequency in the waveguide

From the definition of cutoff wavenumber:

$$k^2 = \frac{\omega^2}{c^2} - k_C^2$$

By defining: $\omega_C = ck_C$ **Cutoff (angular) frequency**

$$E_z = J_0(k_C r) A_F \, e^{i(\omega t - kz)}$$

$$\omega > \omega_C \Rightarrow k^2 > 0 \Rightarrow k \text{ is real} \Rightarrow \text{the wave propagates}$$

$$\omega < \omega_C \Rightarrow k^2 < 0 \Rightarrow k \text{ is immaginary} \Rightarrow \begin{matrix}\text{the wave does not propagate} \\ \text{and decreases exponentially}\end{matrix}$$

$$\omega = \omega_C \Rightarrow k = 0 \Rightarrow \text{the wave does not propagates and does not depend on } z$$

Definitions: Phase & group velocities

$$E_z = E_0 \cos(\omega t - kz) \qquad \varphi = \omega t - kz$$

$$\frac{d\varphi}{dt} = \omega - k\frac{dz}{dt} = 0$$

$$v_P = \frac{\omega}{k}$$ **Phase Velocity**

$$v_P = \frac{\omega}{k} = \frac{\omega}{\sqrt{(\omega/c)^2 - k_C^2}} = \frac{\omega}{\sqrt{(\omega/c)^2 - (\omega_C/c)^2}} = \frac{c}{\sqrt{1-(\omega_C/\omega)^2}} > c$$

For propagating waves $v_P > c$ → **No acceleration is possible!**

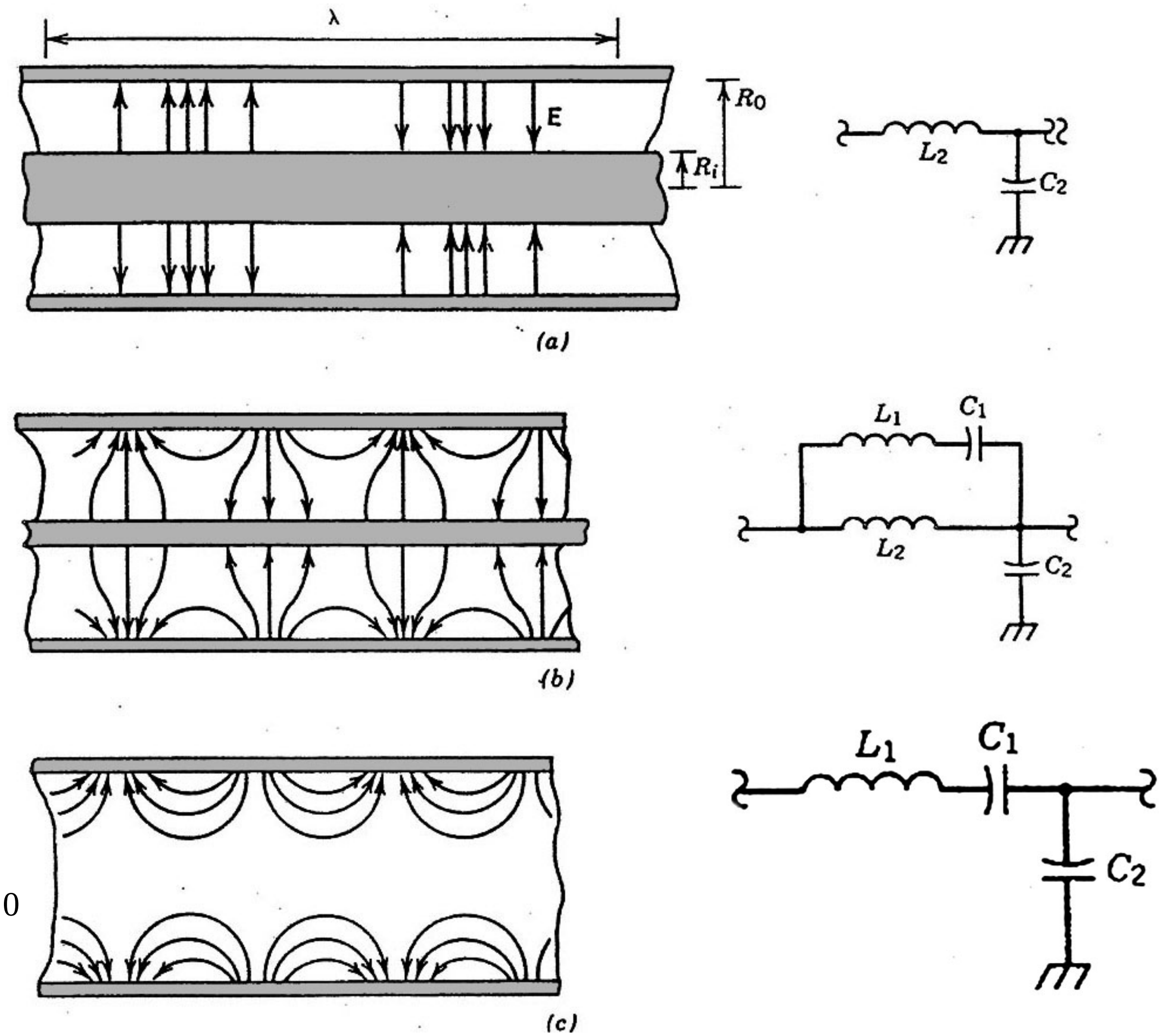
Group Velocity

$$v_G = \frac{d\omega}{dk}$$

$$\omega = c\sqrt{k_C^2 + k^2}$$

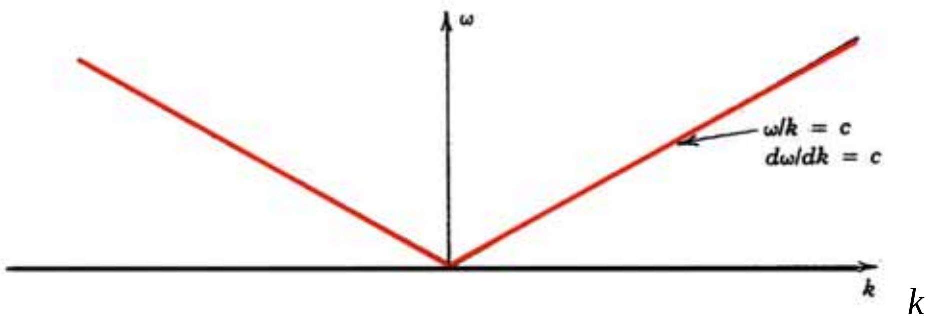
$$v_G = \frac{d\omega}{dk} = \frac{ck}{\sqrt{k_C^2 + k^2}} = \frac{c}{\sqrt{1+k_C^2/k^2}} < c$$

Propagating modes & equivalent circuits

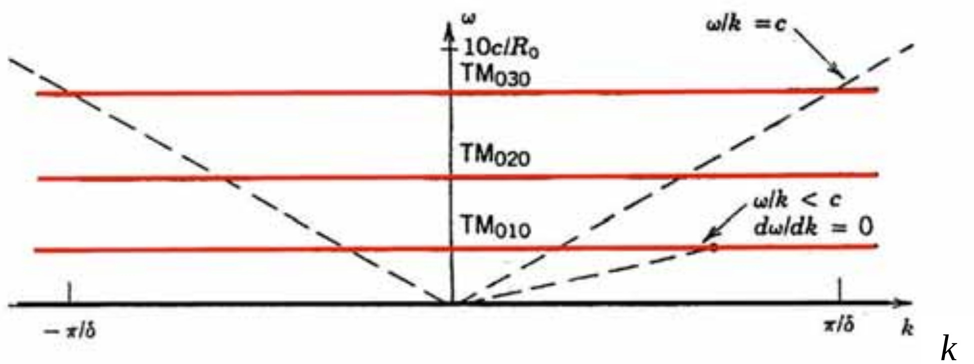


TM_{10}

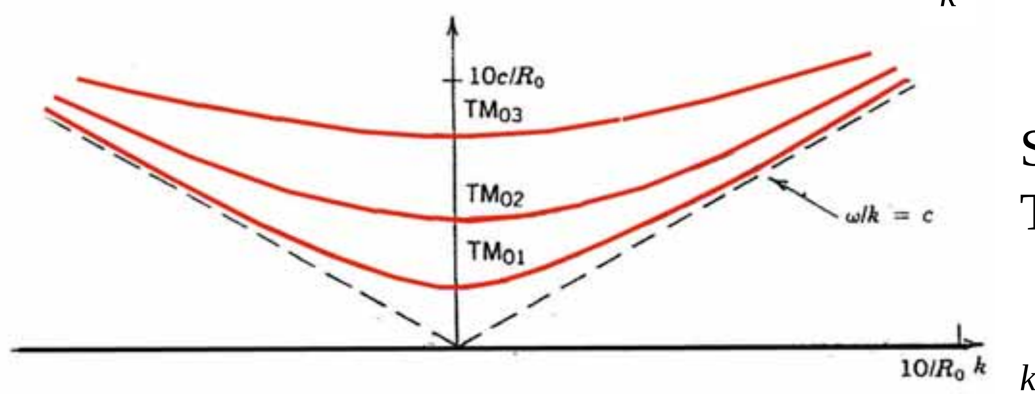
Dispersion diagrams



Transmission line
TEM mode

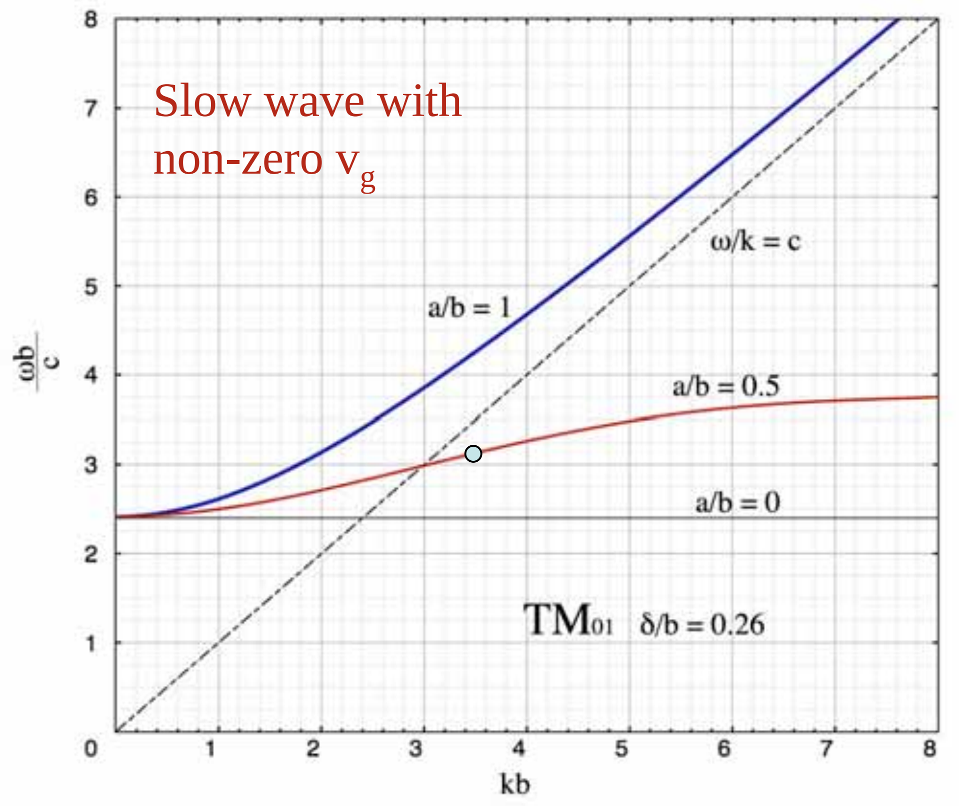


Weakly coupled pillboxes
TM_{0n0} modes

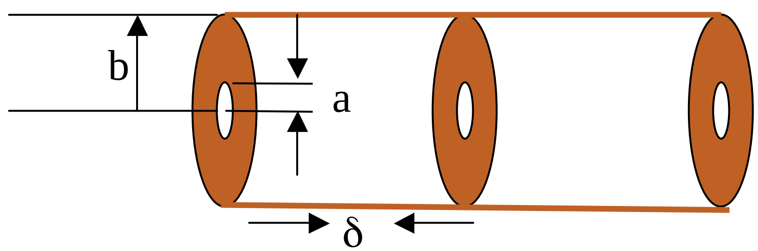


Smooth waveguide,
TM_{0n} modes

Dispersion relation for SLAC structure



Small changes in *a* lead to large reduction in v_g



Notation

$\beta_g = v_g/c$ = Relative group velocity

E_a = Accelerating field (MV/m)

E_s = Peak surface field (MV/m)

P_d = Power dissipated per length (MW/m)

P_t = Power transmitted (MW/m)

w = Stored energy per length (J/m)

Structure parameters for TW linacs

$$r_{shunt} = \frac{E_a^2}{\left| \frac{dP_t}{dz} \right|} \quad (\text{M}\Omega/\text{m})$$

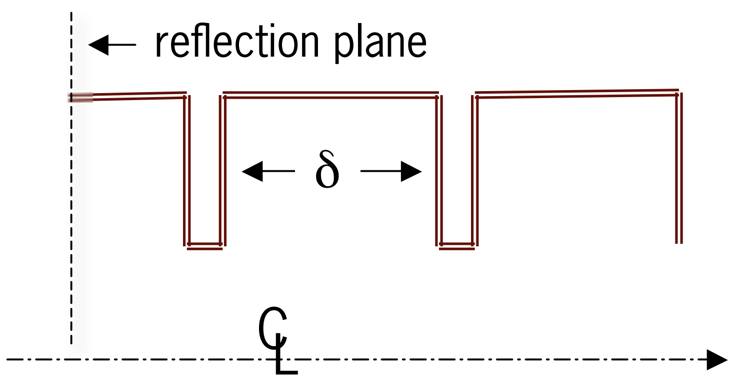
$$Q = \frac{w\omega}{\left| \frac{dP_t}{dz} \right|}$$

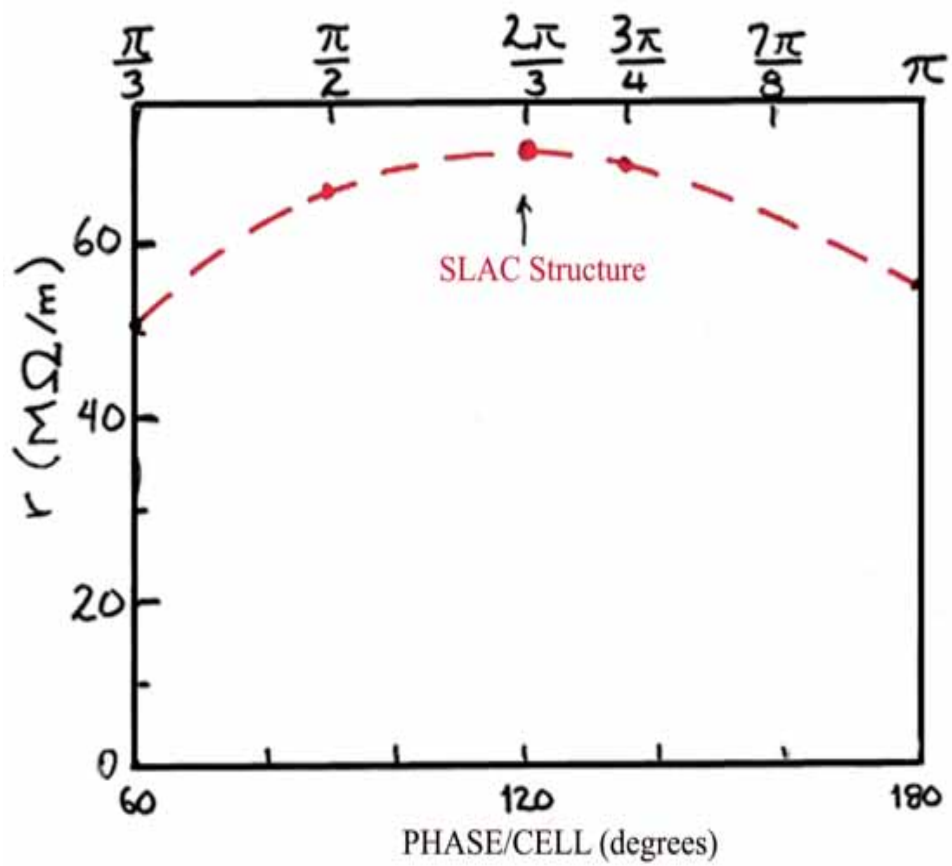
$$\frac{r_{shunt}}{Q} = \frac{E_a^2}{w\omega}$$

$$s = \frac{E_a^w}{w} = \text{ Elastance } (\text{M}\Omega/\text{m}/\mu\text{s})$$

$$W_{acc} = \text{ emergy/length for acceleration}$$

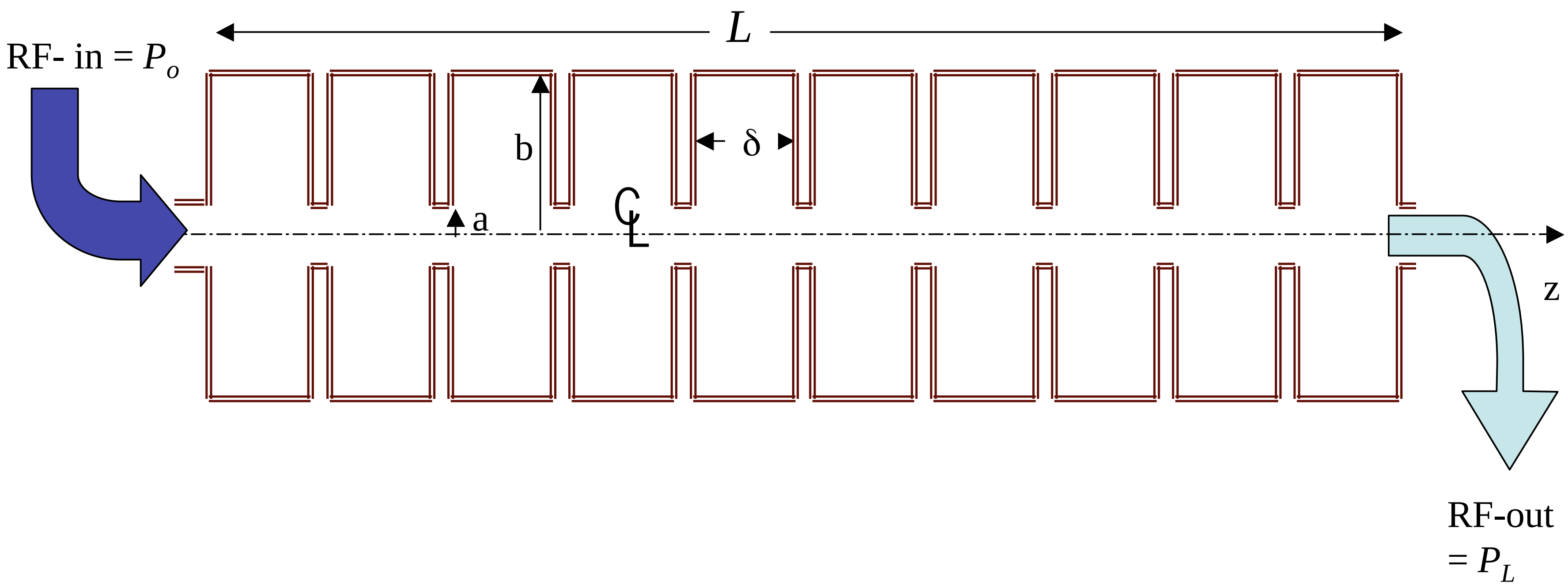
Variation of shunt impedance with cell length





Calculations by D. Farkas (SLAC)

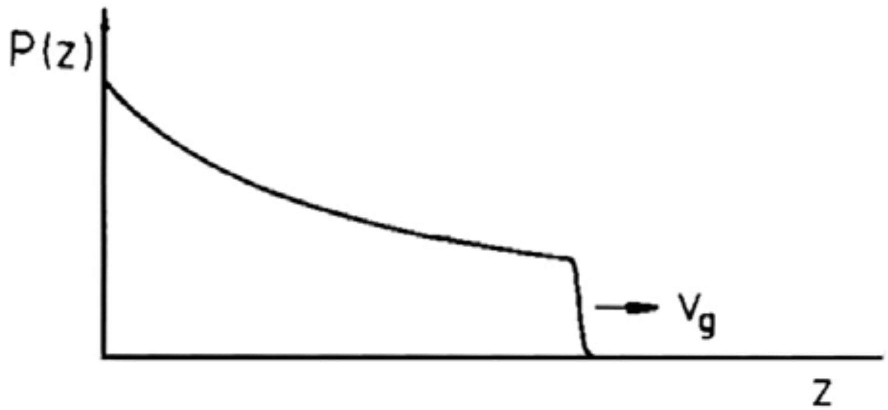
The constant geometry structure

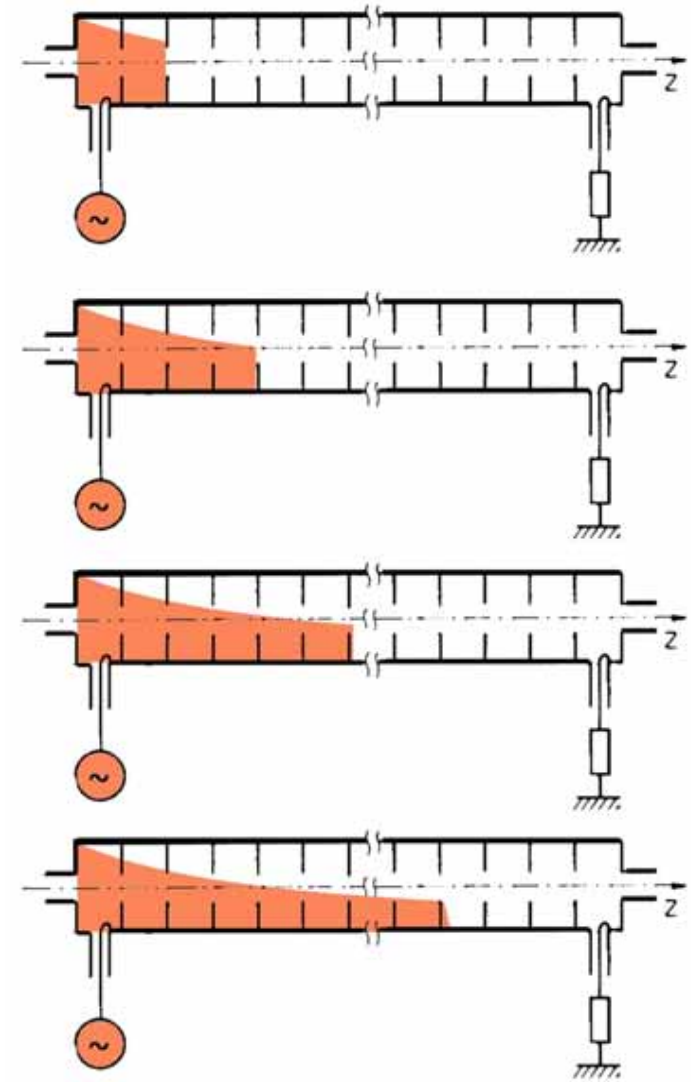


In a structure with a constant geometry, the inductance & capacitance per unit length are constant

==> **constant impedance structure**

Filling the traveling wave linac





Energy flow in the structure

$$v_{energy} = v_{group} \qquad \left(\beta_g = {v_g}/{c} \right)$$

From the definition of Q

$$\frac{dP_t}{dz} = -\frac{\omega P_t}{v_g Q} \equiv -2\alpha P_t$$

and

$$\frac{dE_a}{dz} = -\alpha E_a$$

where the antenuation length is defined as

$$\alpha \equiv \frac{\omega}{2 v_g Q}$$

Then

$$E_a^2 = r_{shunt}\left|\frac{dP_t}{dz}\right| = 2\alpha\, r_{shunt} P_t$$

Constant impedance structure

- A structure with constant structure parameters along its length is called a

Constant Impedance Structure

$$E_a(z) = E_o e^{-\alpha z} \quad \& \quad P_a(z) = P_o e^{-2\alpha z}$$

- For a structure of length L the attenuation parameter is

$$\tau = \alpha L = \frac{\omega L}{2 v_g Q}$$

Acceleration in a constant impedance structure

$$\frac{dP_t}{dz} = -\frac{\omega P_t}{v_g Q} \equiv -2\alpha P_t \quad \text{and} \quad \frac{dE_a}{dz} = -\alpha E_a$$

$$E_a(z) = E_o e^{-\alpha z}$$

$$P_t(z) = P_o e^{-2\alpha z}$$

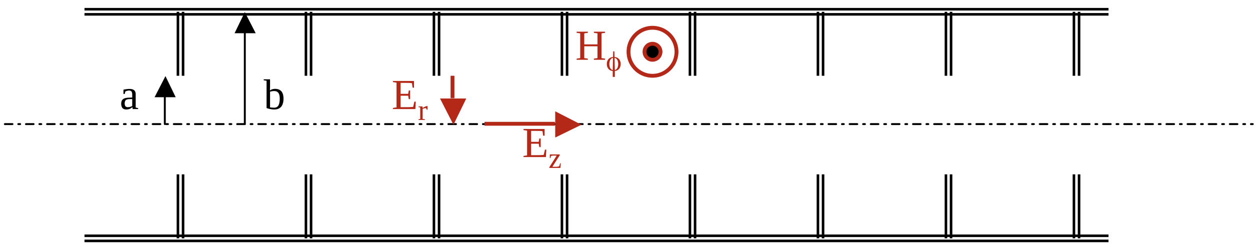
Transmitted power & accelerating gradient *decrease exponentially* along the structure.

$$E_a(L) = E_o e^{-\alpha L} \equiv E_o e^{-\tau}$$
$$P_t(L) = P_o e^{-2\alpha L} \equiv P_o e^{-2\tau}$$

where $\tau \equiv \alpha L = \frac{\omega L}{2 v_g Q}$

Can we do better by varying the structure geometry?

Power flow through the structure



In the region of the aperture,

$$E_r \propto r \quad \text{and} \quad H_\phi \propto r$$

==> momentum flux is $\Pi \propto E \times H \propto r^2$

The power flowing through the structure is

$$\therefore \quad P_t = \int_0^a \Pi \, r dr \propto a^4 \quad \Rightarrow \quad v_g \propto a^4$$

==> Small variations in *a* lead to large variations in v_g and P_t

The constant gradient structure

- Rapid variation of P_t ==> we can make E_a constant by varying P_t as α^{-1}

$$E_a^2 = r_{shunt}\left|\frac{dP_t}{dz}\right| = 2\alpha\, r_{shunt} P_t$$

- As r_{sh} varies very weakly with the iris size. Then,

$$\left|\frac{dP_t}{dz}\right| = const \qquad \Longrightarrow$$

$$P_t(z) = P_o - (P_o - P_L)(z/L)$$

==> $$\frac{P(z)}{P_o} = 1 - \left(z/L\right)\left(1 - e^{-2\tau}\right)$$

How to vary v_g in the CG structure

- Compute $$\left|\frac{dP_t}{dz}\right| = -\frac{P_o}{L}\left(1-e^{-2\tau}\right)$$
- Recall that $$\frac{dP_t}{dz} = -\frac{\omega P_t}{v_g Q} \equiv -2\alpha P_t$$
- So, $$v_g = -\frac{\omega P_t}{Q\frac{dP_t}{dz}} = \frac{\omega L P_t}{QP_o\left(1-e^{-2\tau}\right)}$$

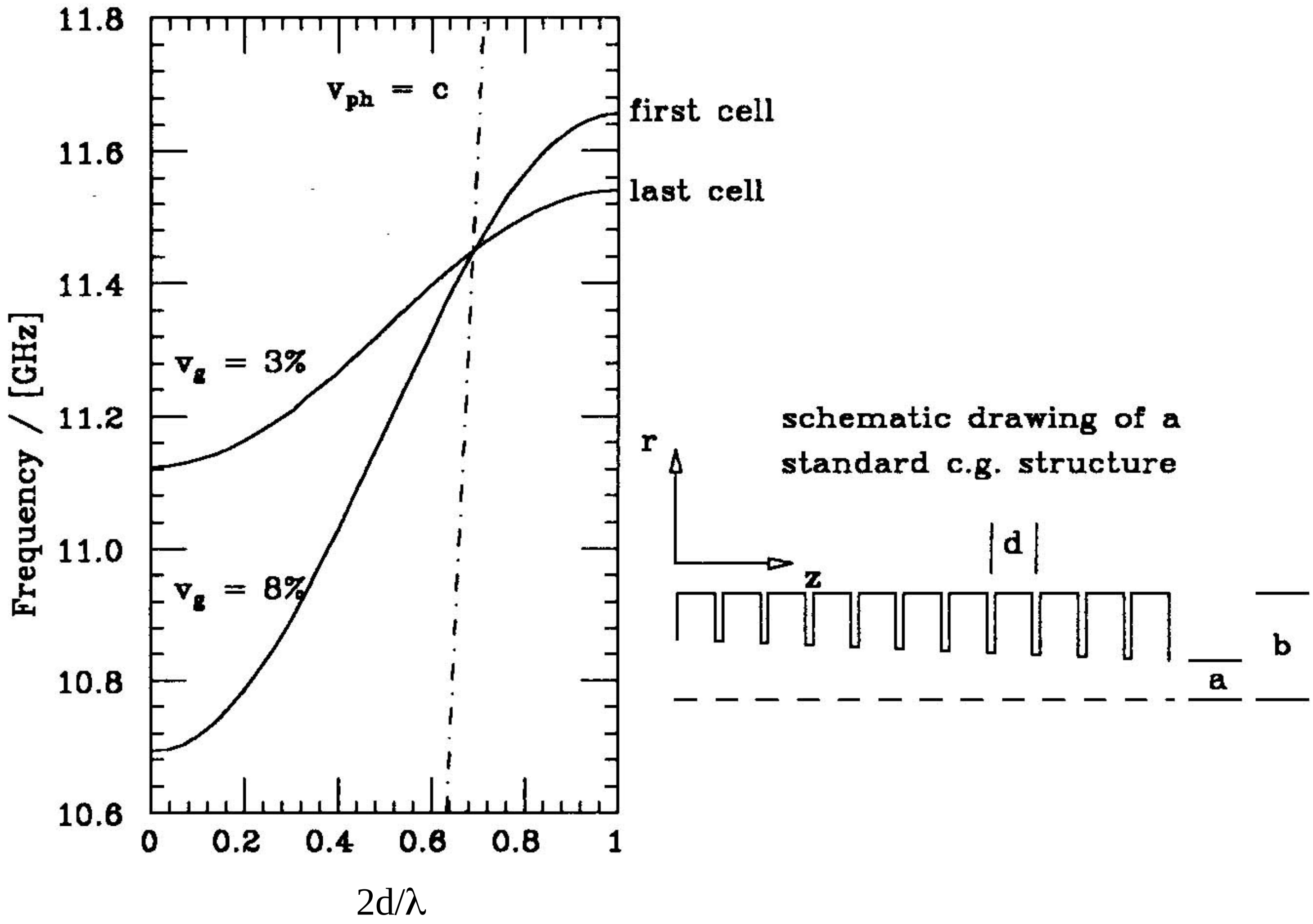
$$v_g = \frac{\omega L P_t}{QP_o\left(1-e^{-2\tau}\right)} = \frac{\omega L}{Q}\frac{\left(1-\frac{z}{L}\left(1-e^{-2\tau}\right)\right)}{\left(1-e^{-2\tau}\right)}$$

==> make the irises smaller

Advantages of the CG structure

- Uniform thermal load along structure
  - ➞ In CZ structure load can vary by 10:1
- Higher average (breakdown limited) accelerating gradient
  - ➞ Higher no-load voltage gain
  - ➞ Higher efficiency under beam loading
- For equal attenuation parameter, equal fill time & equal stored energy
- Disadvantage: mechanically more complex ==> more expensive

Example of CG-structure at 11.4 GHz



Accelerator technology: scaling disk-loaded waveguide structures

- ✺ Based on calculations by D. Farkas (SLAC)
- ✺ Does not include the effect of the thickness of the disk
- ✺ Scaling relations ~10% optimistic compared with measurements

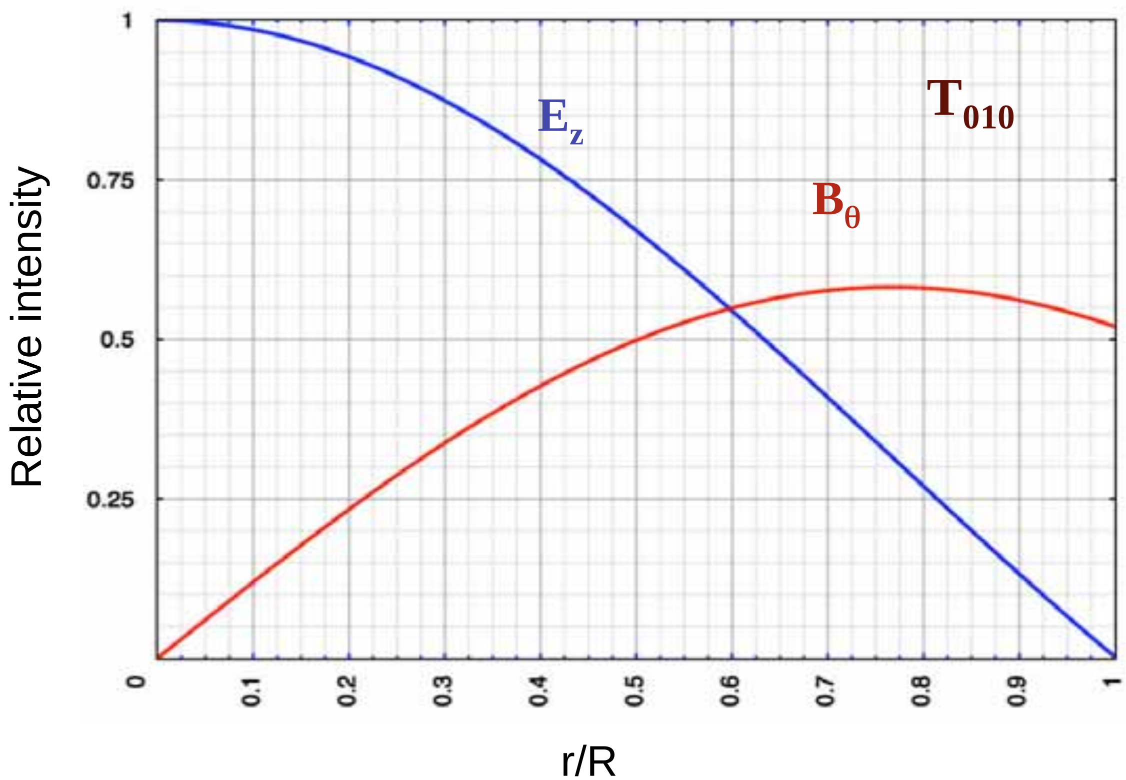
A few more definitions

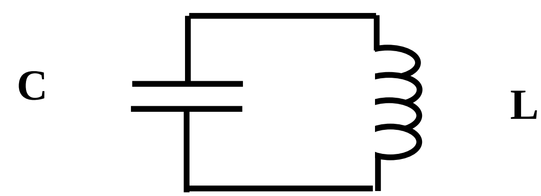
$$\text{Structure efficiency (h)} \equiv \frac{\text{energy available for acceleration}}{\text{energy input}}$$

$$\text{Attenuation time } (T_o) \equiv \frac{Q}{\pi f_{rf}}$$

$$\text{Attenuation parameter } (\tau) \equiv \alpha L = \frac{\omega L}{2 v_g Q}$$

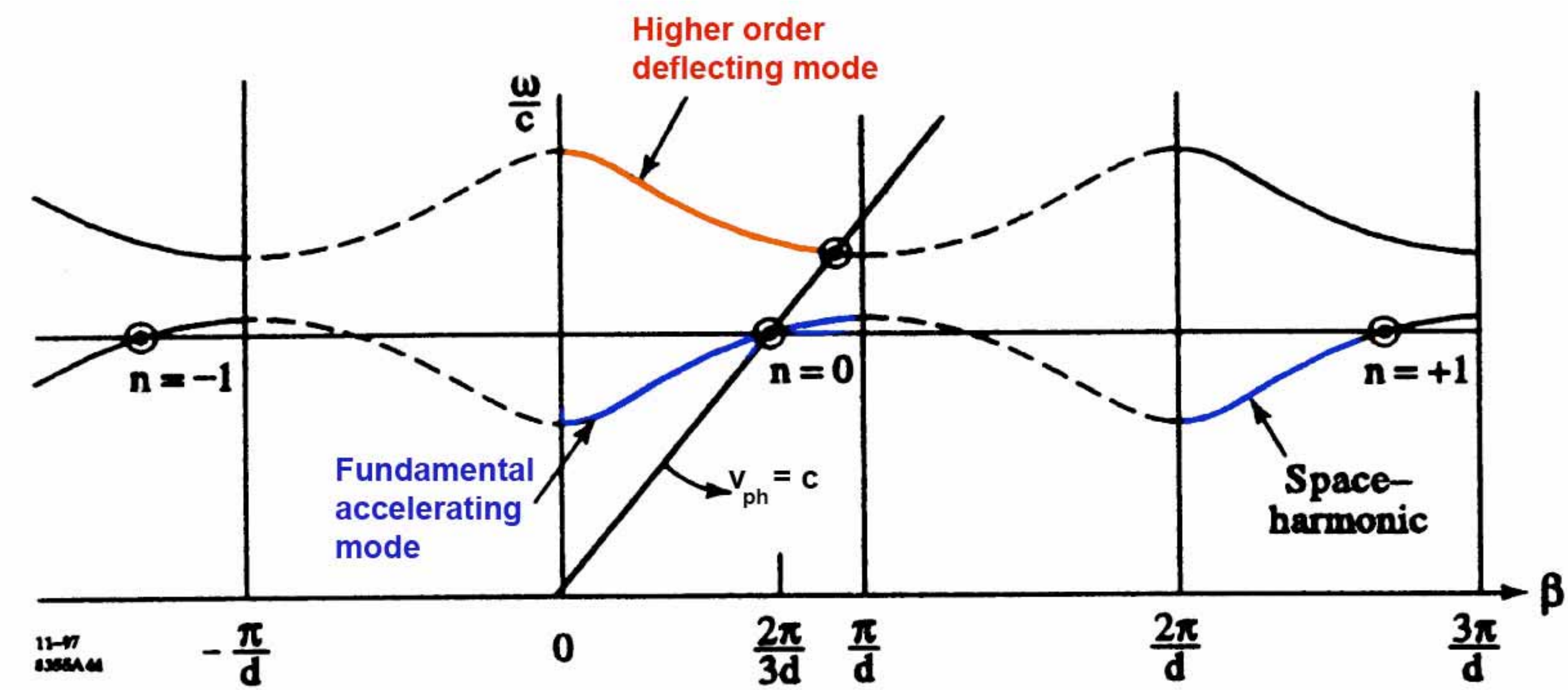
Pillbox E-fields : T_{on1o} mode



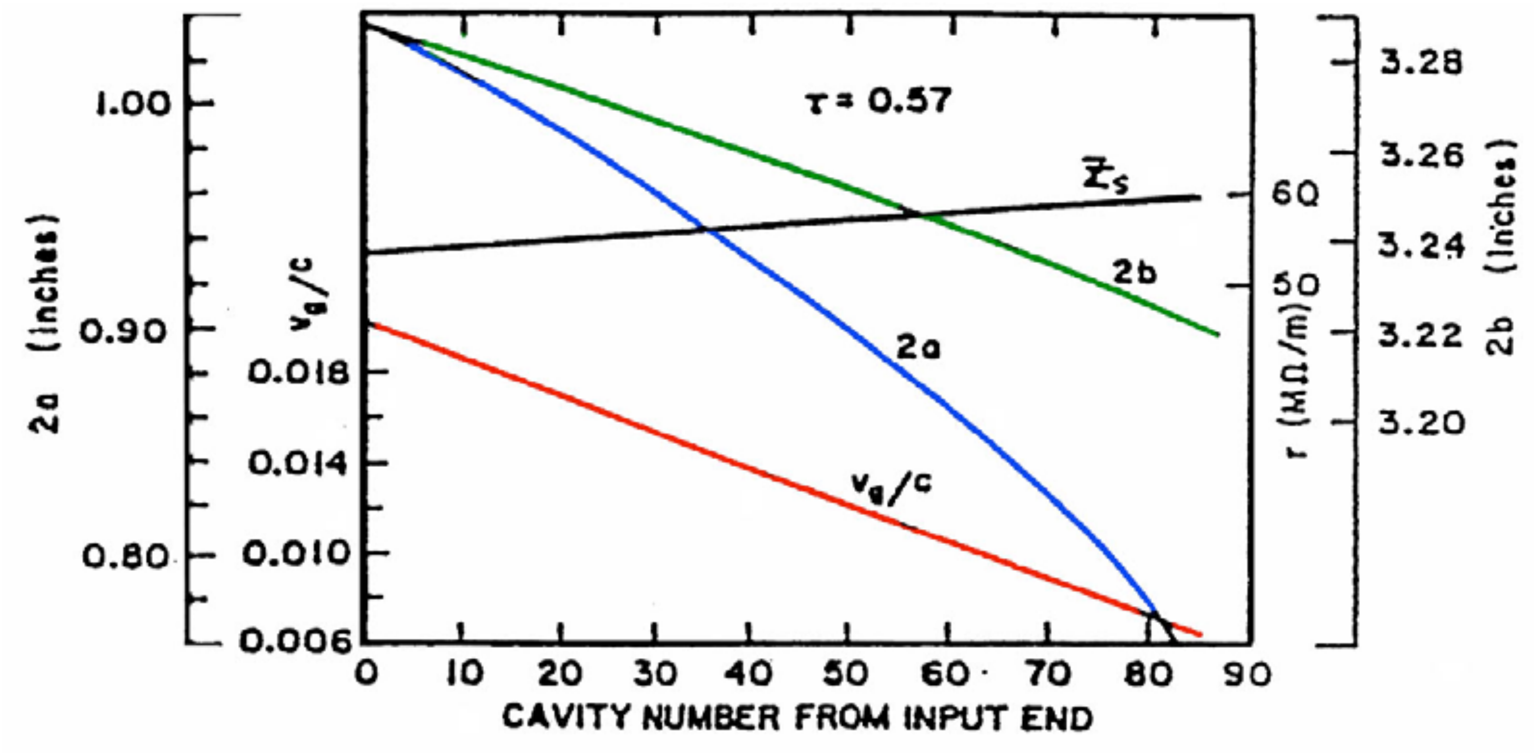


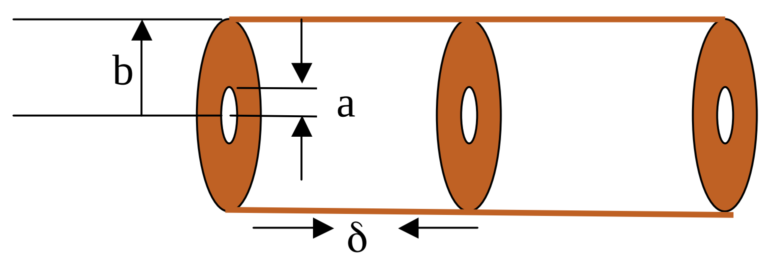
$$\frac{2\pi f_{rf}}{c} b = 2.405 \quad \Rightarrow \quad b(cm) = \frac{11.48}{f_{rf}(Ghz)}$$

Dispersion diagram for the SLAC structure

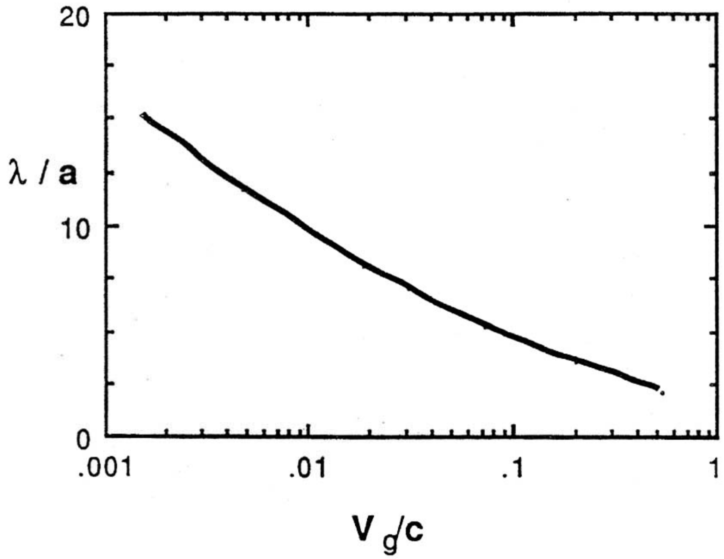


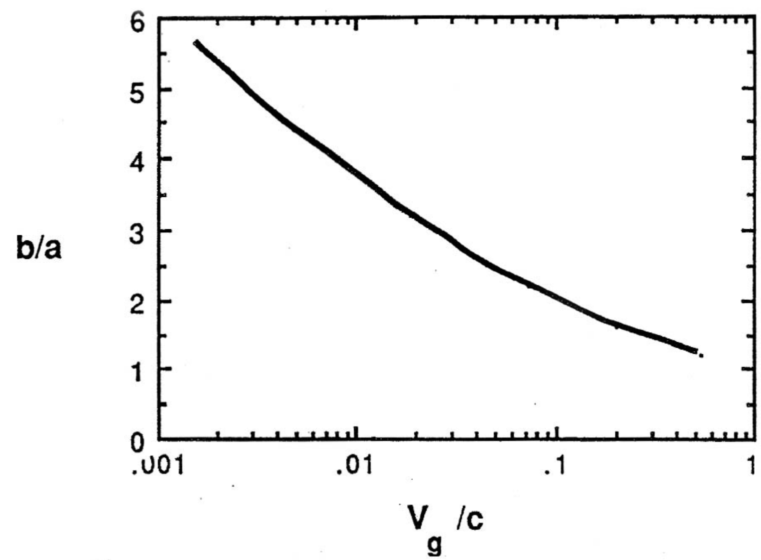
SLAC CG structure characteristics





Variation of v_g with aperture



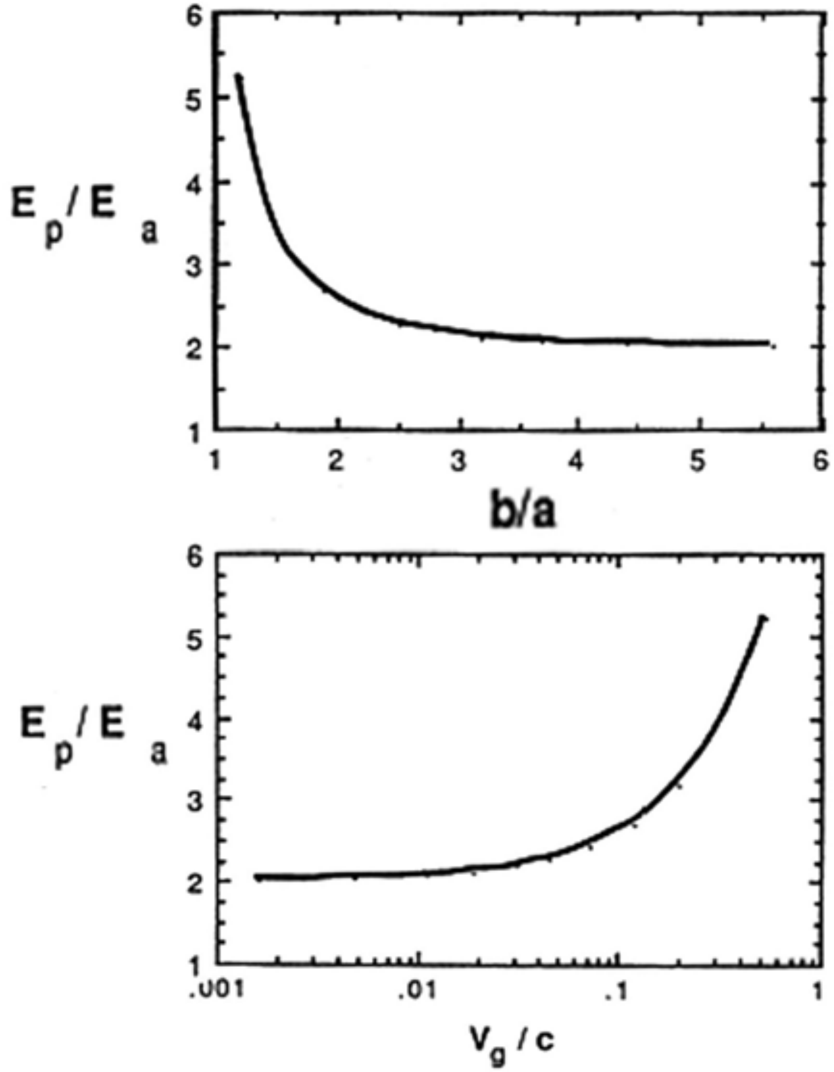


$$\beta_g = \mathrm{Exp}\left[3.1 - 2.4\left(\frac{\lambda}{a}\right)^{1/2} - 0.9\left(\frac{a}{\lambda}\right)\right]$$

$$b \approx a\left[1.04 - 0.29 \ln \beta_g + 0.68 \ln^2 \beta_g\right]$$

Fits to TWAP code calculations by D. Farkas (SLAC)

Variation of peak field with iris aperture



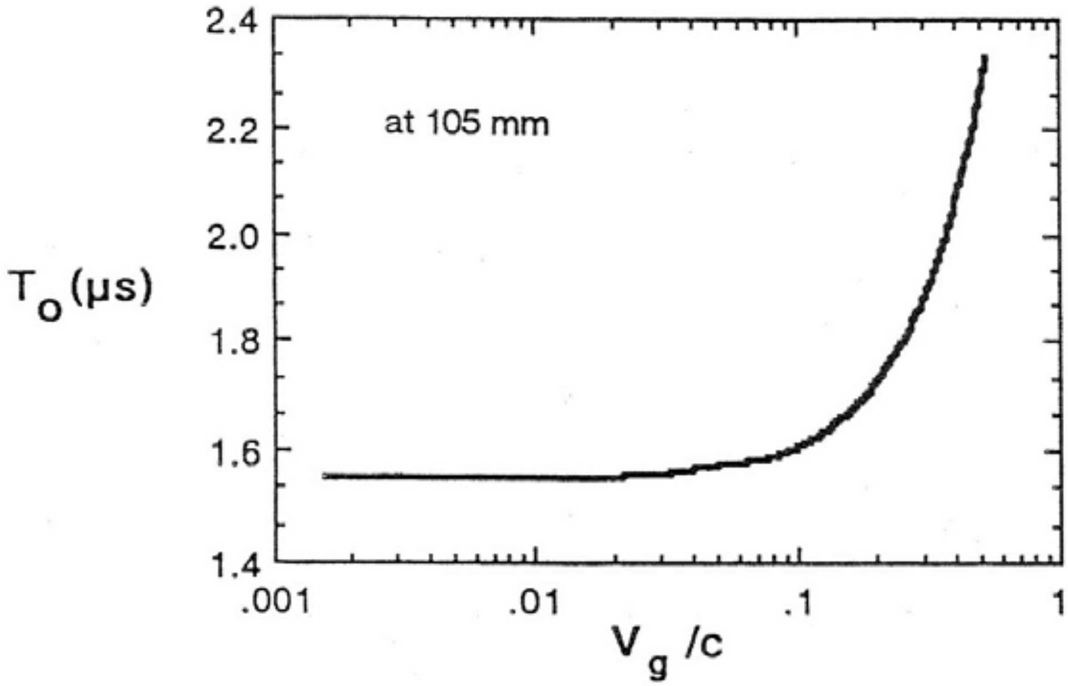
In the region of the aperture

$$E_r \sim r$$

$$E_z \approx \text{constant}$$

$$E_a = \frac{E_{peak}}{2 + 6\beta_g}$$

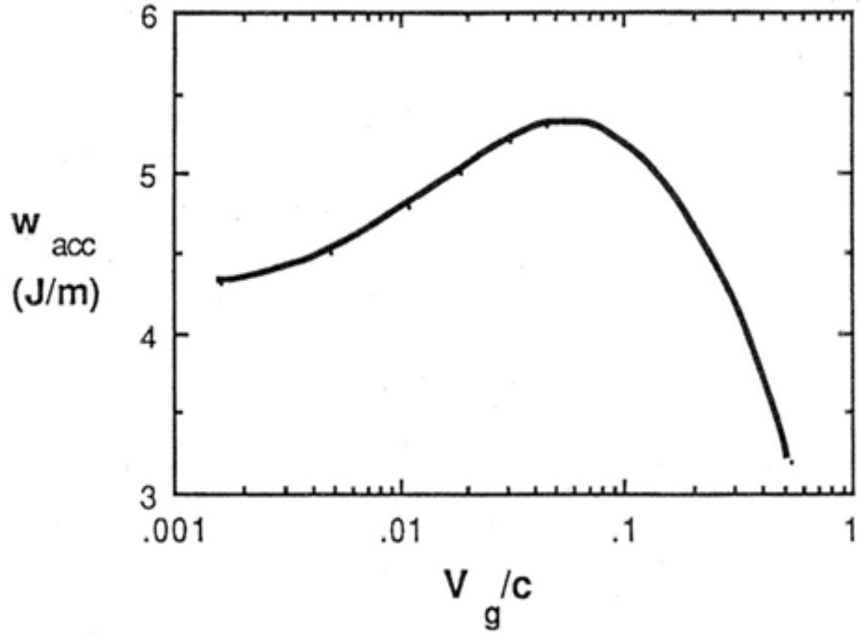
Variation of Q with group velocity



$$T_o = 1.45\ \mu s \left(1 + 1.29\ \beta_g^{3/2}\right)\left(\frac{\lambda}{105\ \text{mm}}\right)^{3/2}$$

$$Q = \pi f_{rf} T_o$$

Variation of elastance with group velocity



$$s_t = \frac{E_{peak}^2}{w_{acc}} = 5.7 \times 10^{14} \text{ V/mC} \left(\frac{10\text{mm}}{a}\right)^2 \beta_g^{0.4}$$

$$E_a = 1.8 \times 10^8 \text{ V/m} \left(2 + 6\,\beta_g\right)^{-1} \left(\frac{F_{safety}}{0.66}\right)\left(\frac{f_{rf}}{2.87 \text{ GHz}}\right)^{1/2} T_{fill}^{-1/4}$$

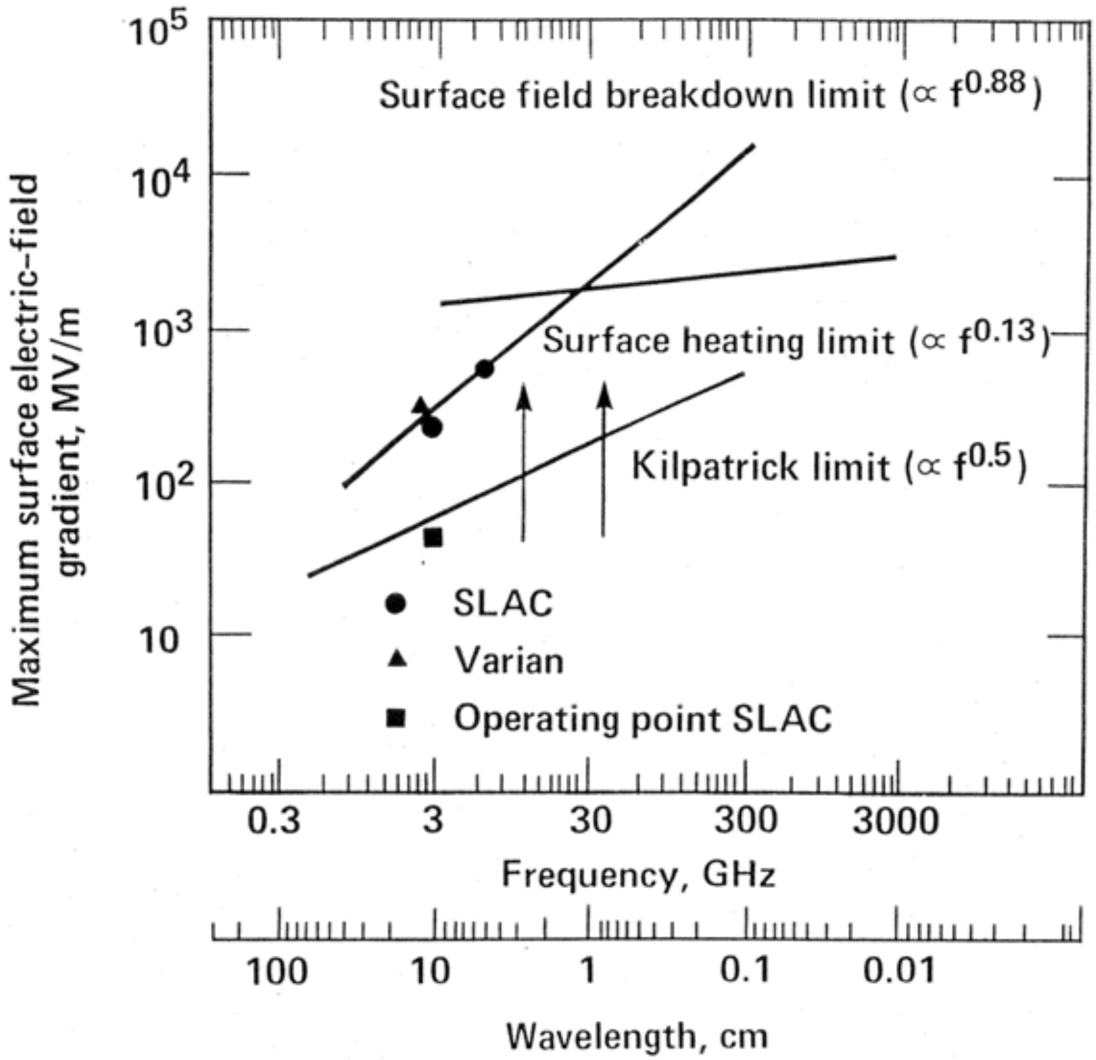
Choice of rf determines linac size

- Higher frequency allows for higher breakdown fields
- Operate linac at 66% of breakdown limit do avoid dark currents stimulating beam instabilities
- Accelerating field is reduced from the peak field by structure geometry
  - → Opening iris reduces fill time, gradient, and wake fields
- Smaller structure size reduces RF energy needed to fill structure for high field strength

Peak power scaling:

$$P \approx 74\ \mathrm{MW/m}\ (E_a\ /\ 100\ \mathrm{MeVm})\ (h\tau\ )^{-1}\ (\lambda\ /\ 105\ \mathrm{mm})^{1/2}$$

Surface field breakdown behavior



Power scaling in TW linacs using the Farkas relations

- For a given E_a and β_g

$$w_{acc} \sim f_{rf}^{-2}$$

and

$$T_o \sim f_{rf}^{-3/2}$$

Therefore

$$\mathrm{P}_{\mathrm{rf}} \sim w_{acc}/T_o \sim f_{rf}^{-1/2}$$

- But higher frequency permits higher E_a

$$E_a \sim f_{rf}^{-1/2}\, T_o^{-1/4} \sim f_{rf}^{1/2}\, f_{rf}^{3/8}$$

$$\boxed{E_a \sim f_{rf}^{7/8}}$$

For a fixed final energy

- The shortest accelerator has

$$P_{rf} \sim E_a^{\,2} f_{rf}^{-1/2} \sim f_{rf}^{7/4} f_{rf}^{-1/2}$$

$$\boxed{P_{rf} \sim f_{rf}^{5/4}}$$

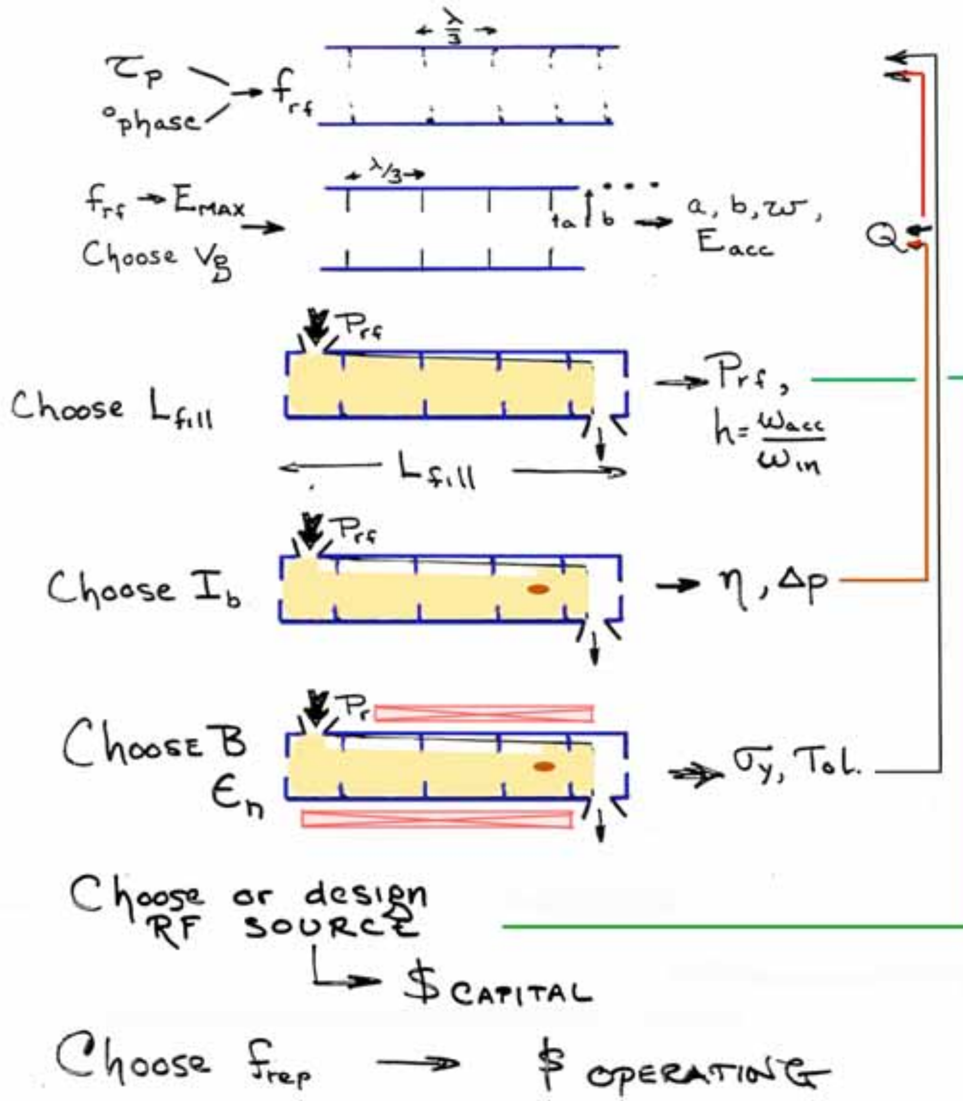
- But

$$W_{rf} \sim E_a^{\,2} w_{acc,o} \sim f_{rf}^{7/4} f_{rf}^{-2}$$

$$\boxed{W_{rf} \sim f_{rf}^{-1/8}}$$

- Do we pay for joules or watts of rf-power?

Steps in designing a TW linac



Why not go to extremely high frequency?

- Cost of accelerating structures
- Power source availability
- Beam loading
  - ➙ Process of transferring energy from the cavity to the beam
- Wakefields

If you can kick the beam, the beam can kick you

End of unit 5

Brief discussion about costs

Exercise: B fields can change the trajectory of a particle but not its energy

$$\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)$$

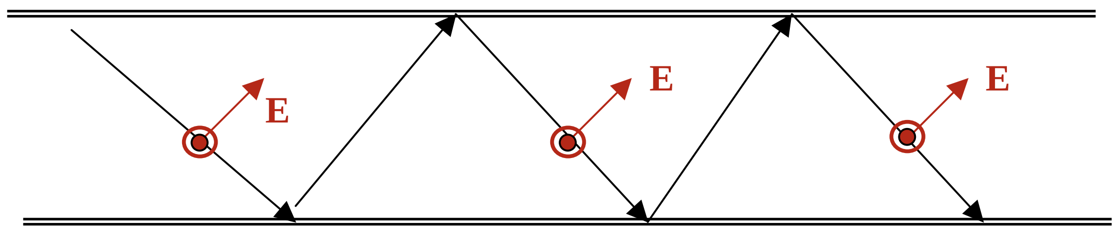
$$W = \int \mathbf{F} \cdot d\mathbf{l} = q\left(\int \mathbf{E}\cdot d\mathbf{l} + \underbrace{\int \frac{1}{c}\mathbf{v}\times\mathbf{B}\cdot d\mathbf{l}}_{=\,0}\right)$$

$\therefore$

$$\Delta E = W = q\int \mathbf{E}\cdot d\mathbf{l}$$

Typically we need a longitudinal E-field to accelerate particles in vacuum

- Example: the standing wave structure in a pillbox cavity
- What about traveling waves?
  - ➙ Waves guided by perfectly conducting walls can have E_{long}



- But first, think back to phase stability
  - ➙ To get continual acceleration the wave & the particle must stay in phase
  - ➙ Therefore, we can accelerate a charge with a wave with a synchronous phase velocity, $v_{ph} \approx v_{particle} < c$