Unit 4 - Lecture 9
RF-accelerators:
RF-cavities

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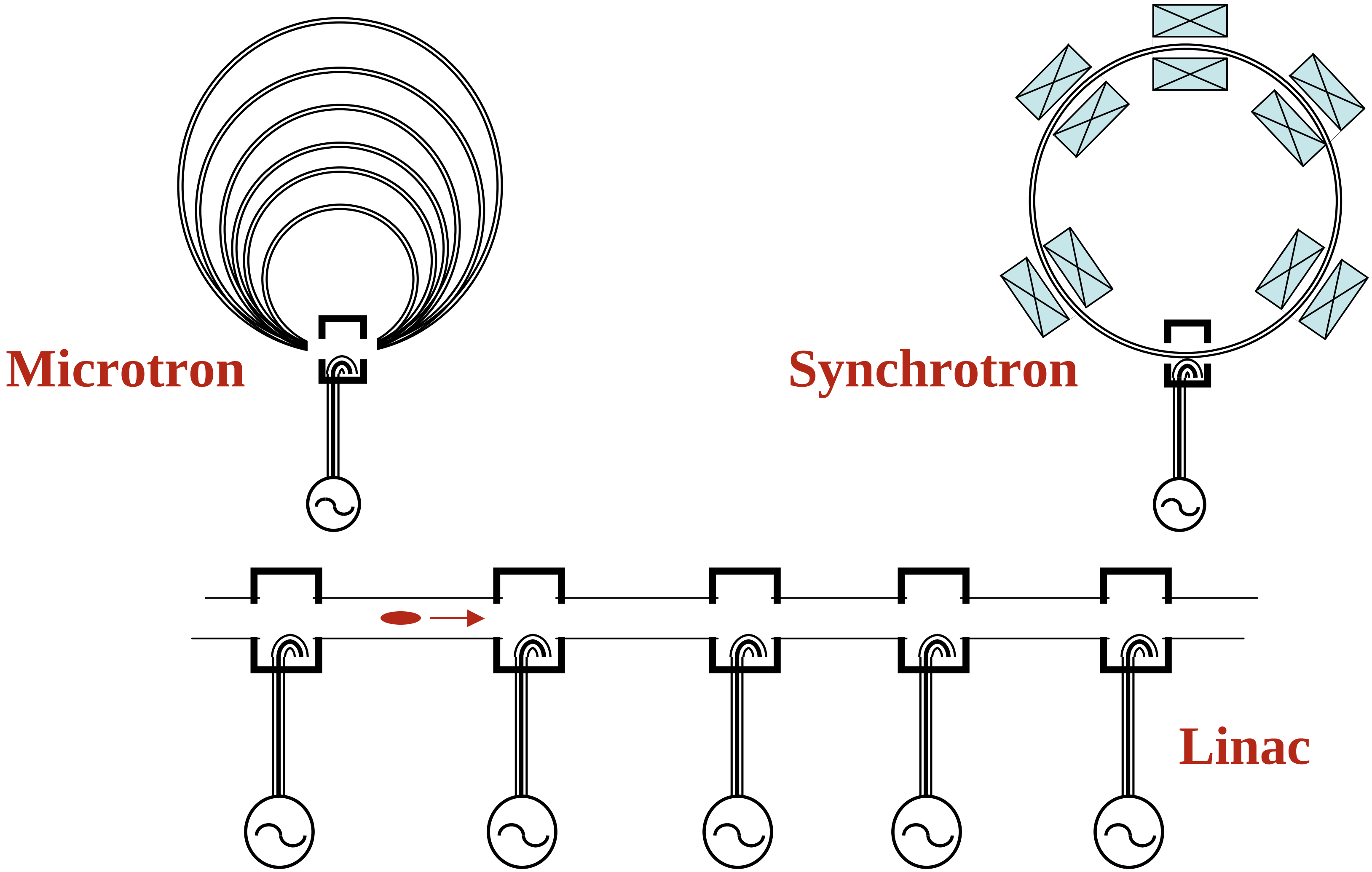
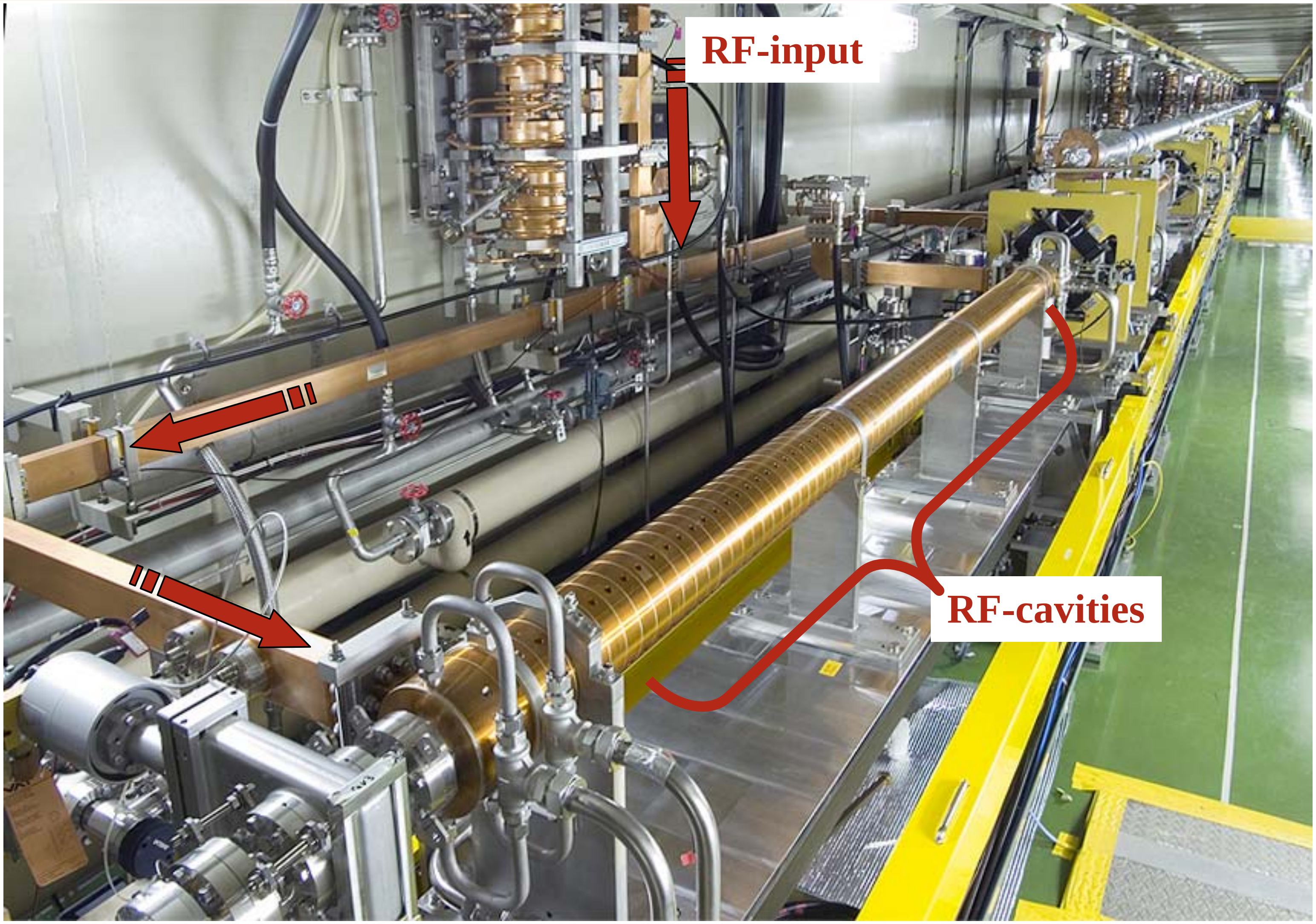
RF-cativties for acceleration

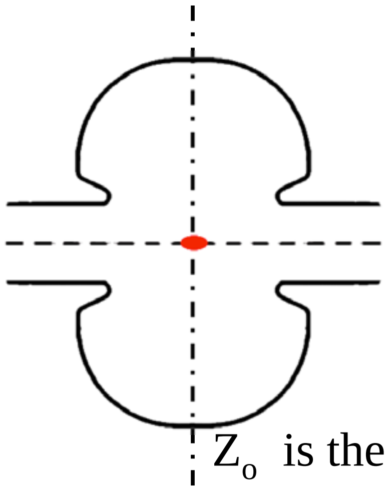


S-band (~3 GHz) RF linac



RF cativties: Basic concepts

- Fields and voltages are complex quantities.
  - For standing wave structures use phasor representation



$$\tilde{V} = Ve^{i\omega t} \quad \text{where} \quad V = \left|\tilde{V}\right|$$

At t = 0 particle receives maximum voltage gain

Z_o is the reference plane

- For cavity driven externally, phase of the voltage is

$$\theta = \omega t + \theta_o$$

- For electrons $v \approx c$; therefore $z = z_o + ct$

Basic principles and concepts

- Superposition
- Energy conservation
- Orthogonality (of cavity modes)
- Causality

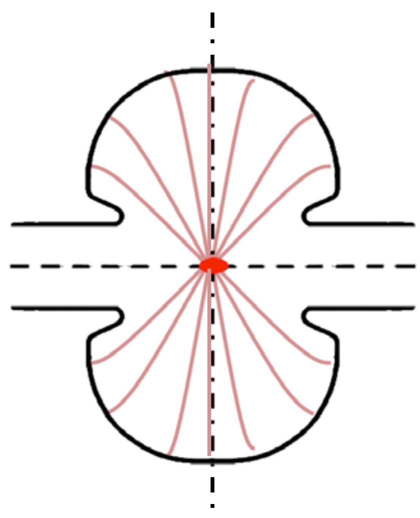
Basic principles: Reciprocity & superposition

- If you can kick the beam, the beam can kick you

==>

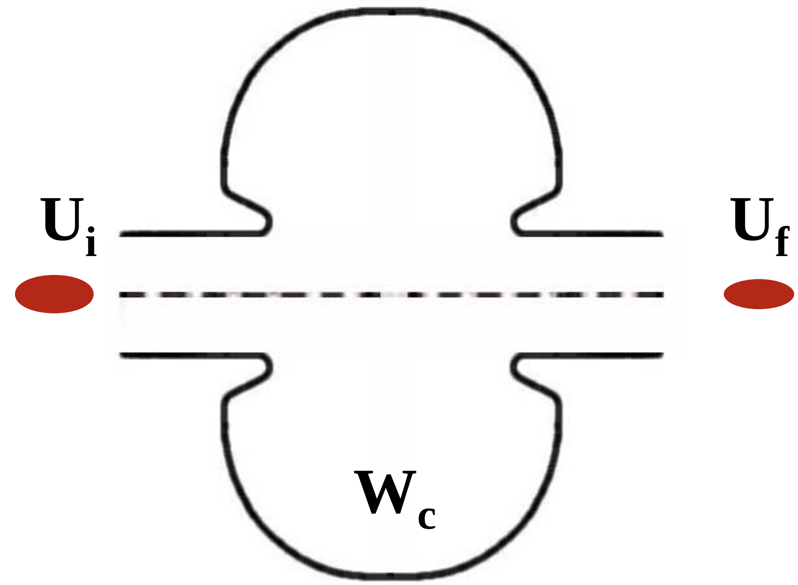
$$\text{Total cavity voltage} = V_{generator} + V_{beam\text{-}induced}$$

$$\text{Fields in cavity} = \mathbf{E}_{generator} + \mathbf{E}_{beam\text{-}induced}$$



Basic principles: Energy conservation

- Total energy in the particles and the cavity is conserved
  - ➞ Beam loading



$$\Delta W_c = U_i - U_f$$

Basics: Orthogonality of normal modes

- Each mode in the cavity can be treated independently in computing fields induced by a charge crossing the cavity.
- The total stored energy is equals the sum of the energies in the separate modes.
- The total field is the phasor sum of all the individual mode fields at any instant.

Basic principles: Causality

- There can be no disturbance ahead of a charge moving at the velocity of light.
- In a mode analysis of the growth of the beam-induced field, the field must vanish ahead of the moving charge for each mode.

Example: Differential superposition

- A point charge q induces a voltage Vo passing through a cavity, what voltage is induced by a Gaussian bunch of charge q?
- A differential charge induces the differential voltage

$$d\tilde{V} = \tilde{V}_o \frac{dq}{q} = V_o e^{j\omega_o t} \frac{dq}{q}$$

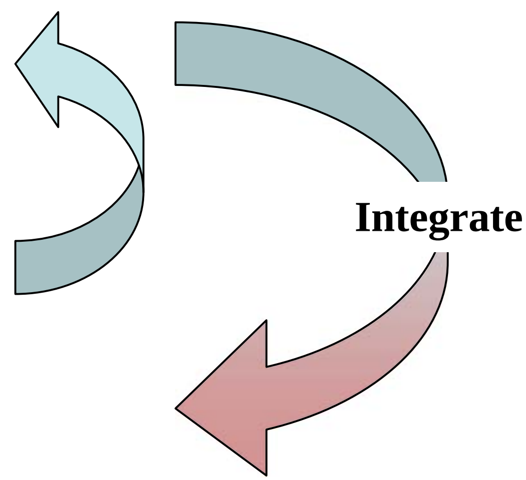
- Say dq passes z = 0 at t_o; at time t the induced voltage will be

$$d\tilde{V} = \frac{V_o}{q} e^{j\omega_o (t-t_o)} dq(t_o)$$

- The bunch has a Gaussian distribution in time

$$dq(t_o) = \frac{q}{\sqrt{2\pi\sigma}} e^{-t_o^2/2\sigma^2} dt_o$$

$$V = V_o e^{j\omega_o t} e^{-\omega_o^2 \sigma^2/2} dt_o$$

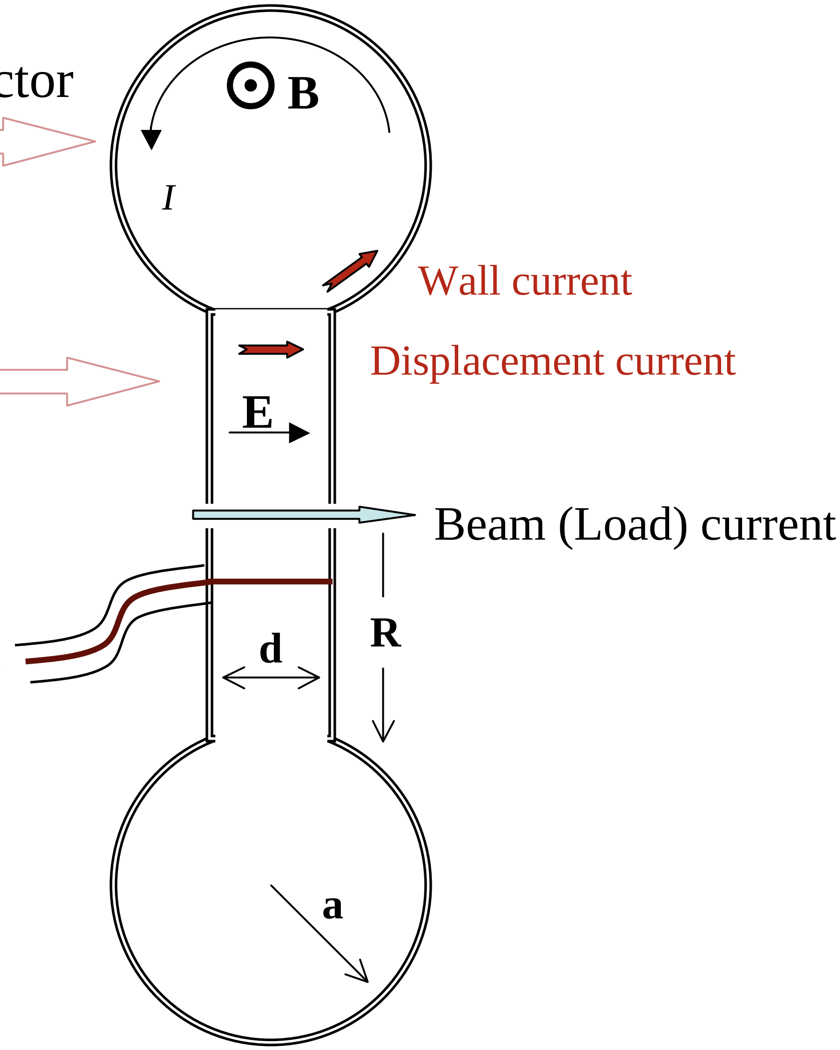


Basic components of an RF cavity

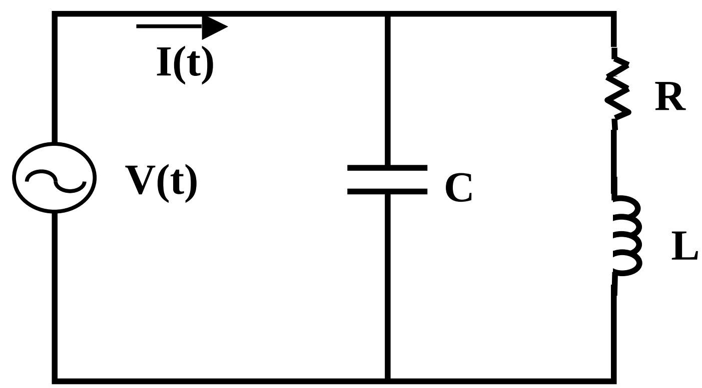
Outer region: Large, single turn Inductor

Central region: Large plate Capacitor

Power feed from rf - generator



Lumped circuit analogy of resonant cavity



$$Z(\omega) = \left[j\omega C + (j\omega L + R)^{-1} \right]^{-1}$$

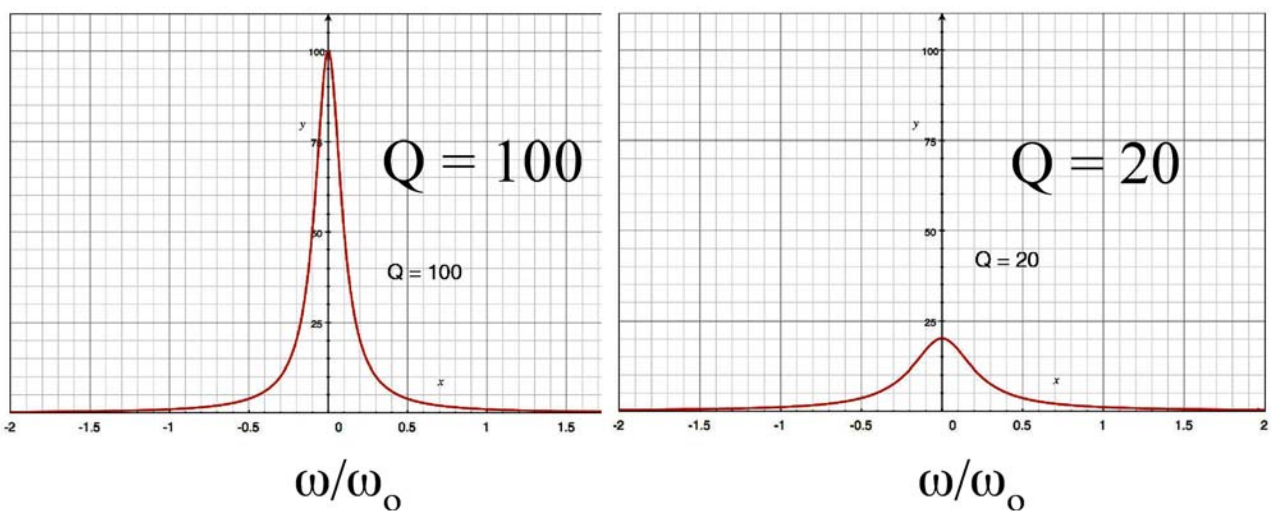
$$Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R)\, j\omega C + 1} = \frac{(j\omega L + R)}{(1 - \omega^2 LC) + j\omega RC}$$

The resonant frequency is $\omega_o = 1/\sqrt{LC}$

Q of the lumped circuit analogy

Converting the denominator of Z to a real number we see that

$$|Z(\omega)| \sim \left[\left(1 - \frac{\omega^2}{\omega_o^2}\right)^2 + (\omega RC)^2 \right]^{-1}$$



The width is $\frac{\Delta\omega}{\omega_0} = \frac{R}{\sqrt{L/.C}}$

More basics from circuits - Q

$$Q = \frac{\omega_o \circ \textit{ Energy stored}}{\textit{Time average power loss}} = \frac{2\pi \circ \textit{Energy stored}}{\textit{Energy per cycle}}$$

$$\mathscr{E} = \frac{1}{2} L\, I_o I_o^* \quad \text{and} \quad \langle \mathscr{P} \rangle = \langle \mathrm{i}^2(t) \rangle R = \frac{1}{2} I_o I_o^* R_{surface}$$

$$\therefore \; Q = \frac{\sqrt{L/C}}{R} = \left(\frac{\Delta\omega}{\omega_o} \right)^{-1}$$

Translate circuit model to a cavity model: Directly driven, re-entrant RF cavity

Outer region: Large, single turn Inductor

$$L = \frac{\mu_o \pi a^2}{2\pi(R+a)}$$

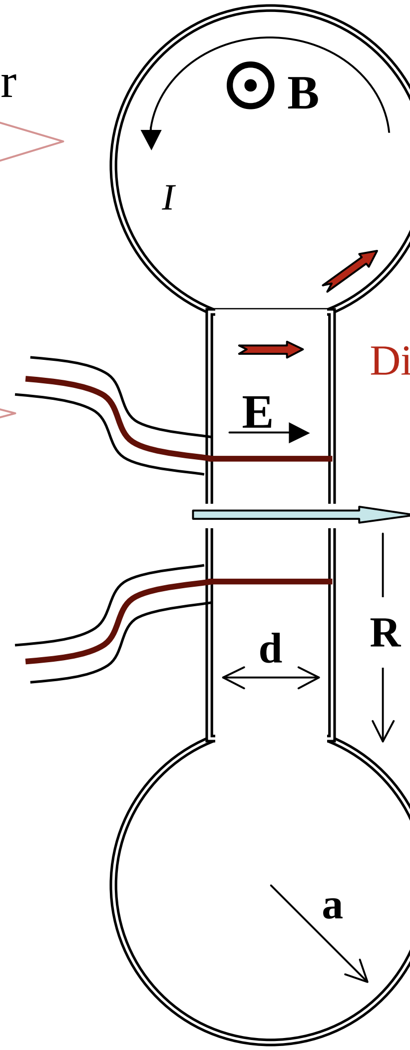
Central region: Large plate Capacitor

$$C = \varepsilon_o \frac{\pi R^2}{d}$$

$$\omega_o = 1\Big/\sqrt{LC} = c\left[\frac{2((R+a)d}{\pi R^2 a^2}\right]^{1/2}$$

Q – set by resistance in outer region

$$Q = \sqrt{L/C}\Big/R$$

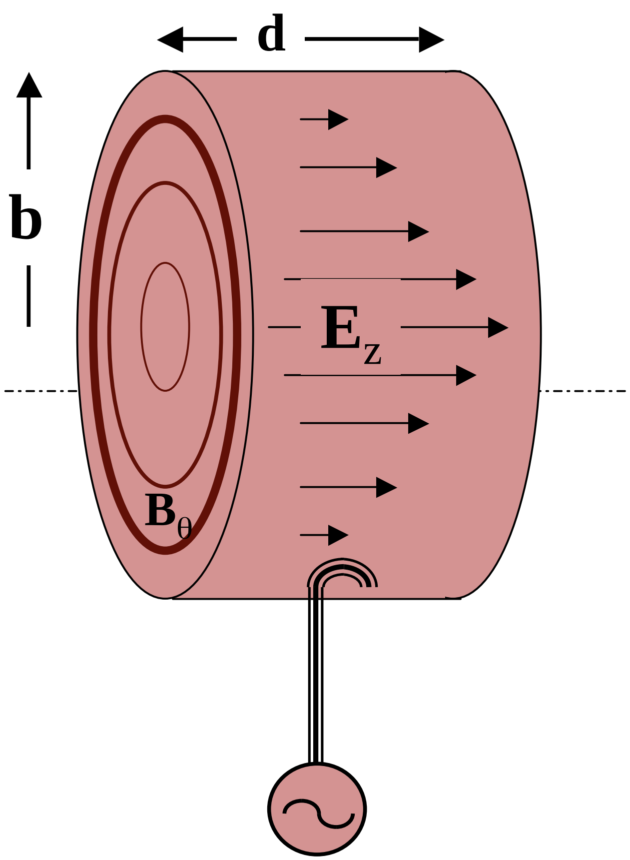


Expanding outer region raises Q

Narrowing gap raises shunt impedance

Source: Humphries, Charged Particle Accelerators

Properties of the RF pillbox cavity



$$\sigma_{walls} = \infty$$

- We want lowest mode: with only $\mathbf{E}_z$ & $\mathbf{B}_\theta$
- Maxwell's equations are:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right) = \frac{1}{c^2}\frac{\partial}{\partial t}E_z \quad \text{and} \quad \frac{\partial}{\partial r}E_z = \frac{\partial}{\partial t}B_\theta$$

- Take derivatives

$$\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(rB_\theta\right)\right] = \frac{\partial}{\partial t}\left[\frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r}\right] = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

$$\frac{\partial}{\partial r}\frac{\partial E_z}{\partial r} = \frac{\partial}{\partial r}\frac{\partial B_\theta}{\partial t}$$

==> $$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}$$

For a mode with frequency ω

- $$E_z(r,t) = E_z(r)\; e^{i\omega t}$$

- Therefore, $$E_z'' + \frac{E_z'}{r} + \left(\frac{\omega}{c}\right)^2 E_z = 0$$

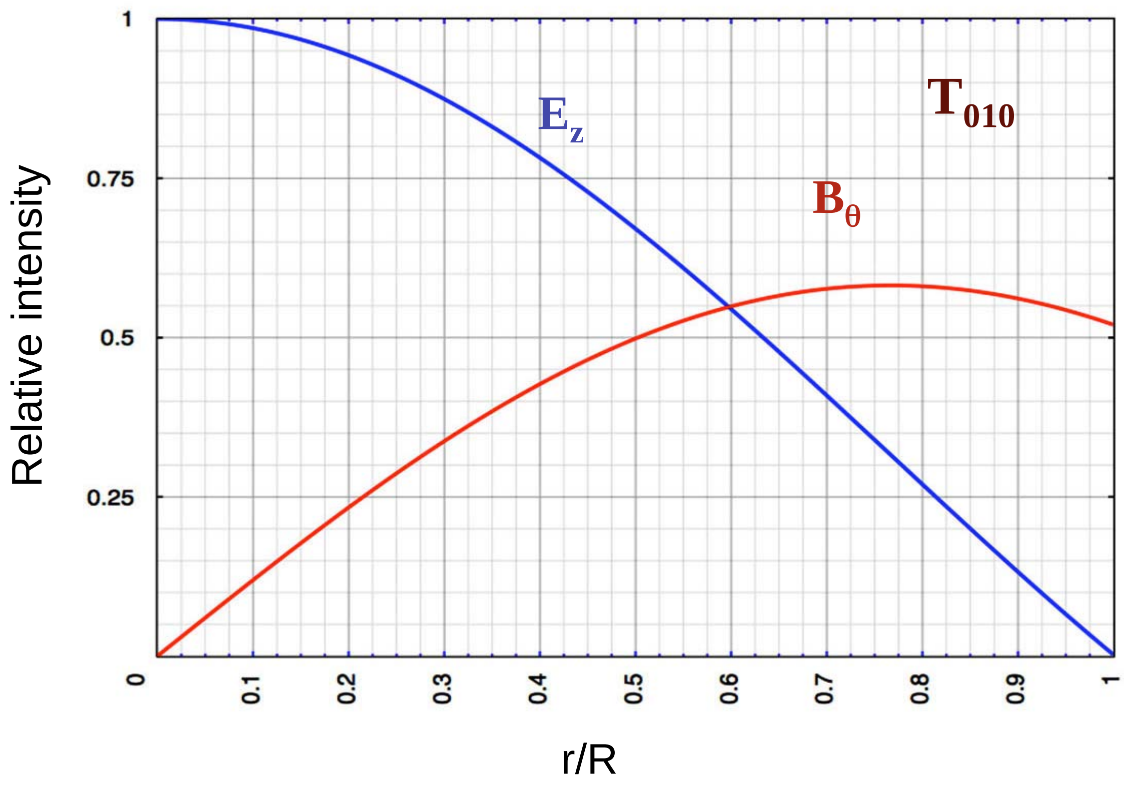
  ➞ (Bessel's equation, 0 order)

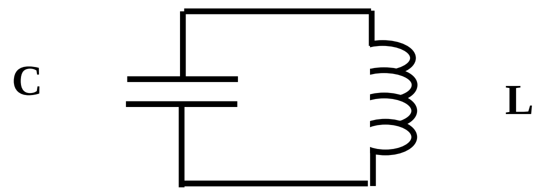
- Hence, $$E_z(r) = E_o\, J_o\left(\frac{\omega}{c} r\right)$$

- For conducting walls, $E_z(R) = 0$, therefore

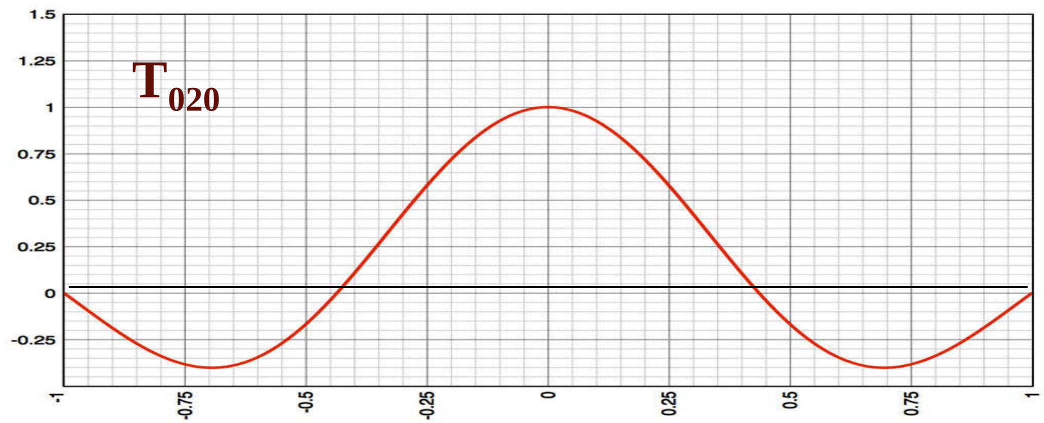
$$\frac{2\pi f}{c} b = 2.405$$

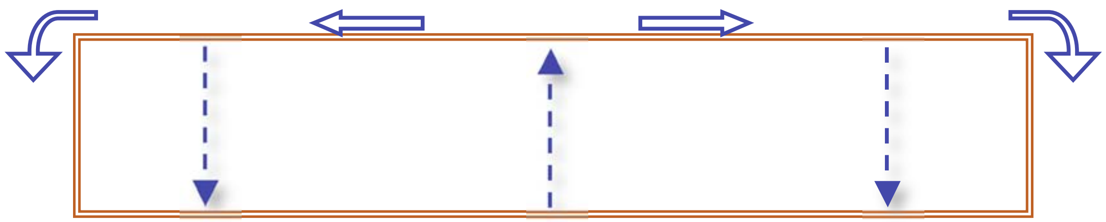
E-fields & equivalent circuit: T_{on1o} mode

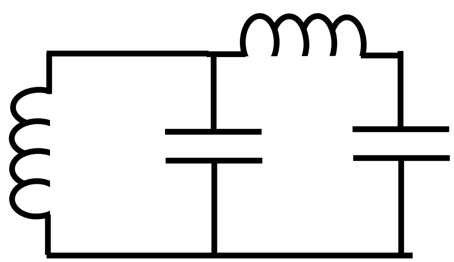




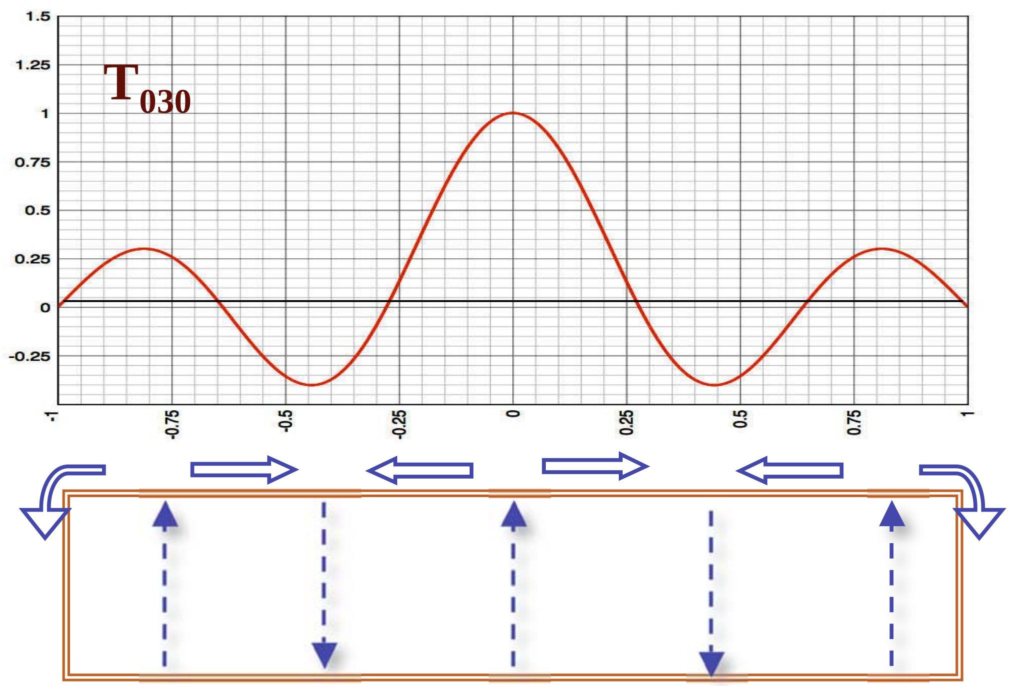
E-fields & equivalent circuits for T_{020} modes

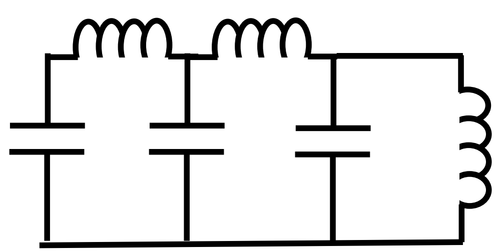






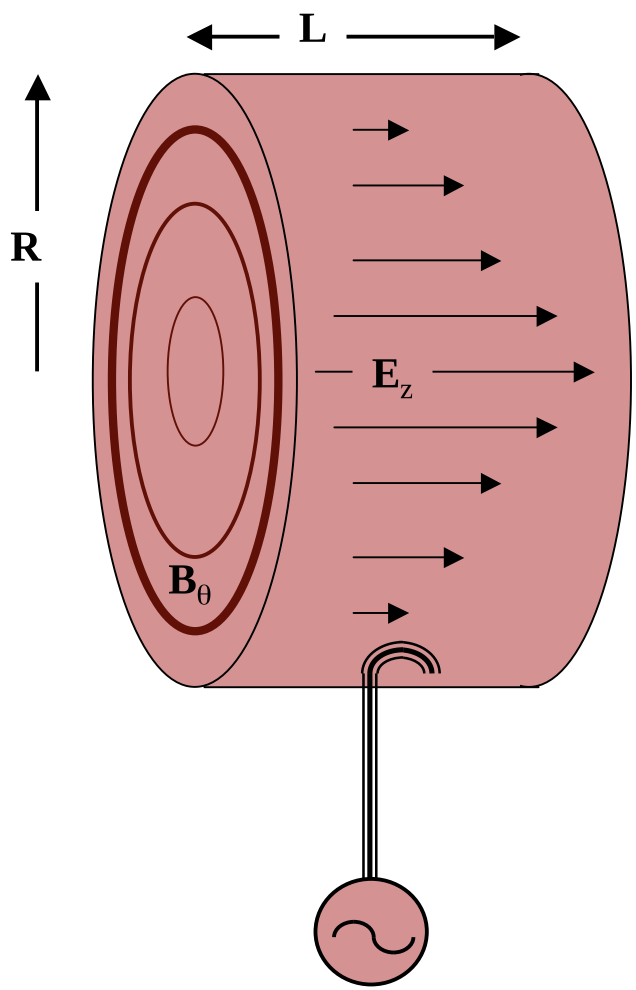
E-fields & equivalent circuits for T_{ono} modes





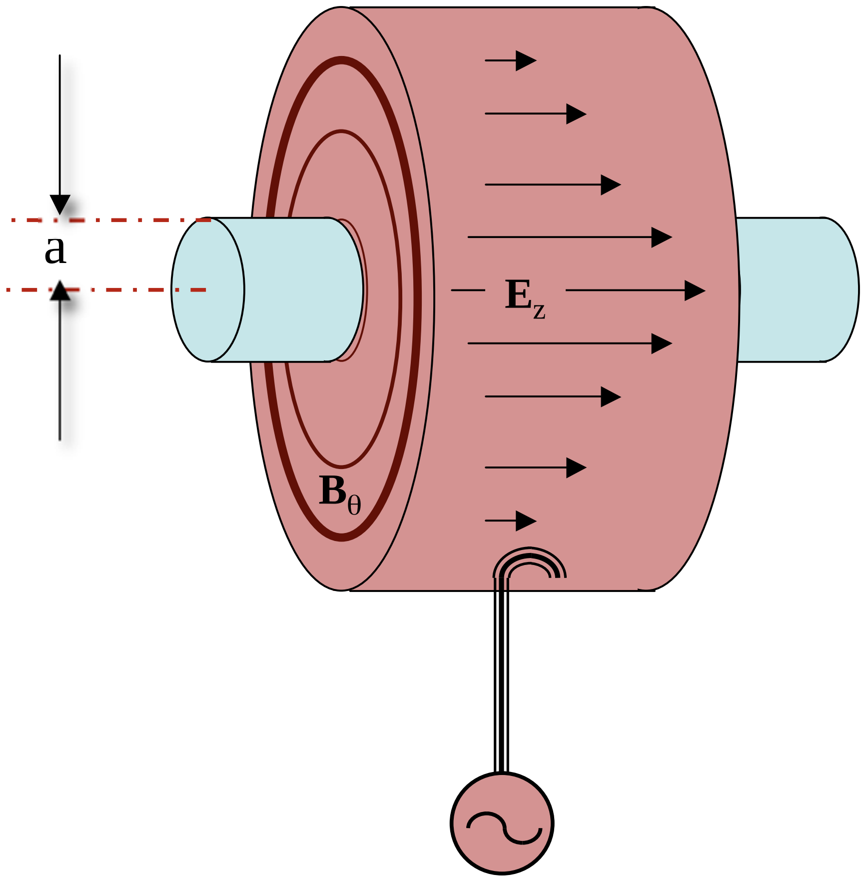
T_{0n0} has n coupled, resonant circuits; each L & C reduced by 1/n

Simple consequences of pillbox model



- Increasing R lowers frequency
  ==> Stored Energy, $\mathcal{E} \sim \omega^{-2}$
- $\mathcal{E} \sim E_z^{\;2}$
- Beam loading lowers E_z for the next bunch
- Lowering ω lowers the fractional beam loading
- Raising ω lowers $Q \sim \omega^{-1/2}$
- If time between beam pulses,
  $T_s \sim Q/\omega$
  almost all $\mathcal{E}$ is lost in the walls

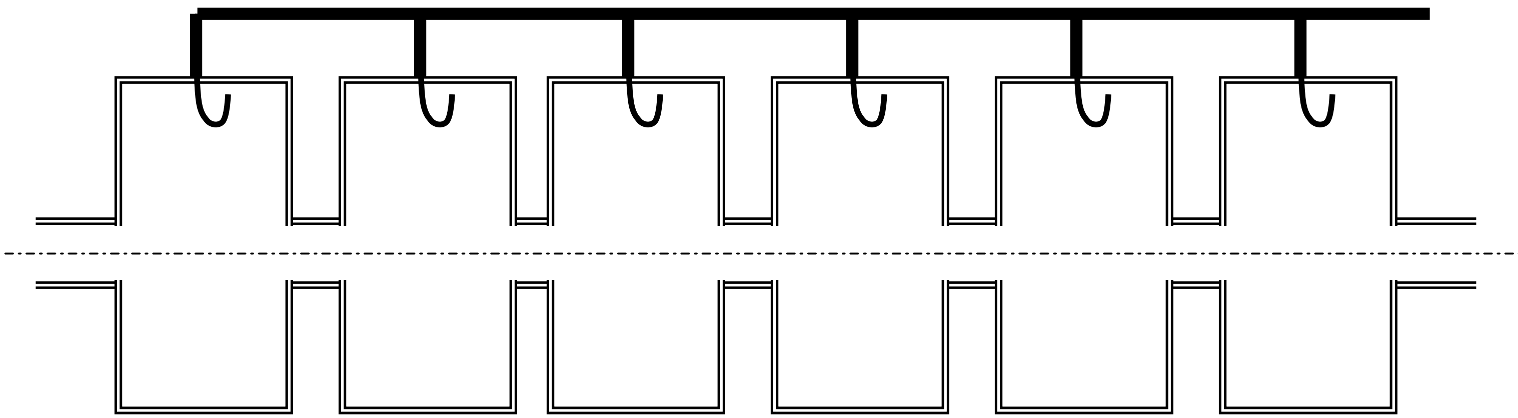
The beam tube makes the field modes (& cell design) more complicated



- Peak E no longer on axis
  - ➞ $E_{pk} \sim 2 - 3 \times E_{acc}$
  - ➞ $FOM = E_{pk}/E_{acc}$
- ω_o more sensitive to cavity dimensions
  - ➞ Mechanical tuning & detuning
- Beam tubes add length & €'s w/o acceleration
- Beam induced voltages ~ a^{-3}
  - ➞ Instabilities

Cavity figures of merit

Make the linac with a series of pillbox cavities

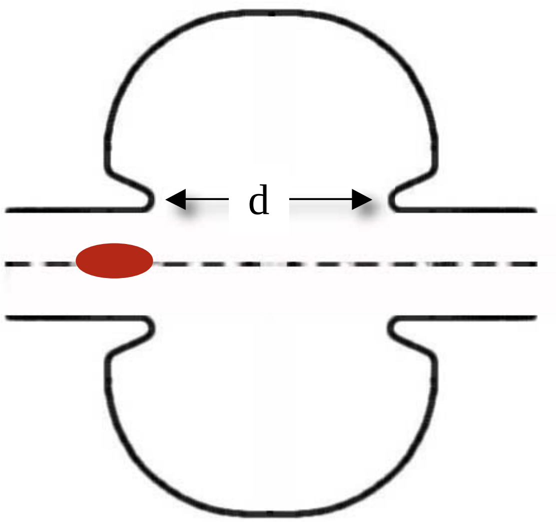


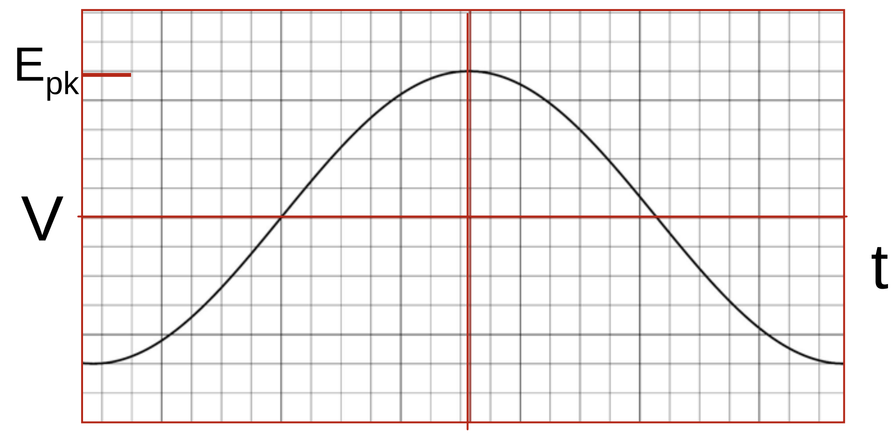
Power the cavities so that $E_z(z,t) = E_z(z)e^{i\omega t}$

Figure of Merit: Accelerating voltage

- The voltage varies during time that bunch takes to cross gap
  - ➞ reduction of the peak voltage by Γ (transt time factor)

$$\Gamma = \frac{\sin\left(\vartheta/2\right)}{\vartheta/2} \quad \text{where} \quad \vartheta = \omega d/\beta c$$





For maximum acceleration $T_{cav} = \frac{d}{c} = \frac{T_{rf}}{2}$ ==> Γ = 2/π

Figure of merit from circuits - Q

$$Q = \frac{\omega_o \circ \; Energy\ stored}{Time\ average\ power\ loss} = \frac{2\pi \circ Energy\ stored}{Energy\ lost\ per\ cycle}$$

$$\mathscr{E} = \frac{\mu_o}{2}\int_v |H|^2 dv = \frac{1}{2} L\, I_o I_o^*$$

$$\langle \mathscr{P} \rangle = \frac{R_{surf}}{2}\int_s |H|^2 ds = \frac{1}{2} I_o I_o^* R_{surf}$$

$$R_{surf} = \frac{1}{Conductivity \circ Skin\ depth} \sim \omega^{1/2}$$

$$\therefore \; Q = \frac{\sqrt{L/C}}{R_{surf}} = \left(\frac{\Delta\omega}{\omega_o}\right)^{-1}$$

Measuring the energy stored in the cavity allows us to measure

- We have computed the field in the fundamental mode

$$U = \int_0^d dz \int_0^b dr 2\pi r \left(\varepsilon E_o^2 / 2 \right) J_1^2(2.405 r/b)$$

$$= b^2 d \left(\varepsilon E_o^2 /2 \right) J_1^2(2.405)$$

- To measure Q we excite the cavity and measure the E field as a function of time

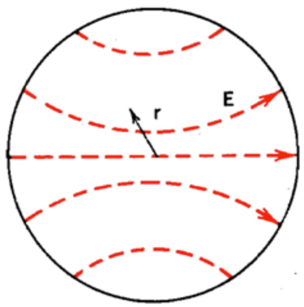
- Energy lost per half cycle = UπQ

- Note: energy can be stored in the higher order modes that deflect the beam

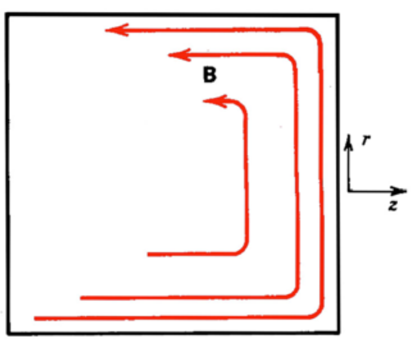
Keeping energy out of higher order modes

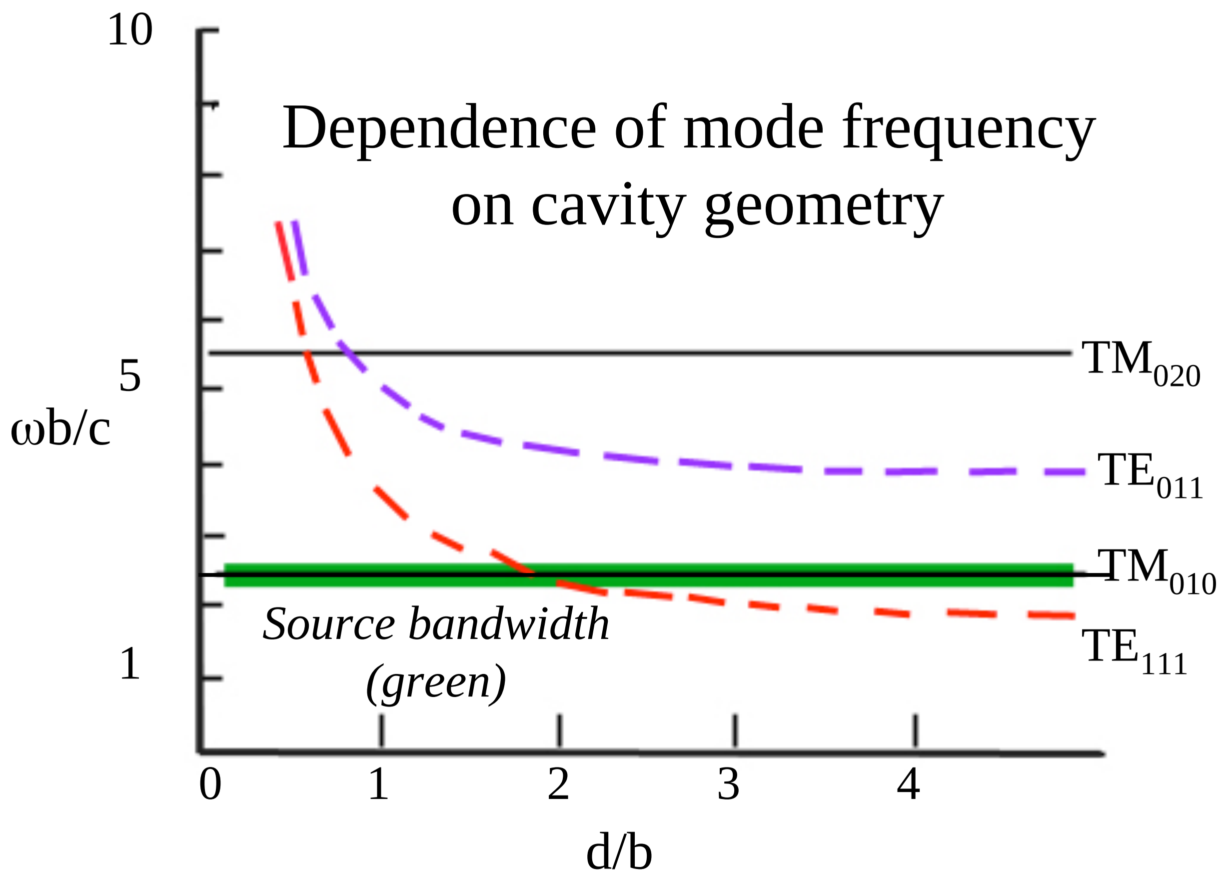
TE_{111} mode

End view



Side view





Choose cavity dimensions to stay far from crossovers

Figure of merit for accelerating cavity: power to produce the accelerating field

Resistive input (shunt) impedance at ω_o relates power dissipated in walls to accelerating voltage

$$R_{in} = \frac{\langle V^2(t) \rangle}{\mathscr{P}} = \frac{V_o^2}{2\mathscr{P}} = Q\sqrt{L/C}$$

Linac literature commonly defines "shunt impedance" without the "2"

$$\mathscr{R}_{in} = \frac{V_o^2}{\mathscr{P}} \sim \frac{1}{R_{surf}}$$

Typical values 25 - 50 MΩ

Computing shunt impedance

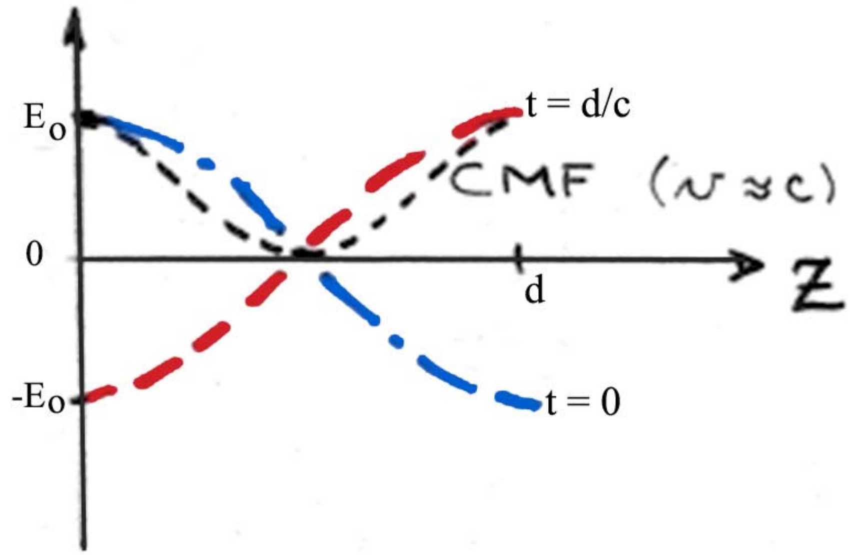
$$\mathscr{R}_{in} = \frac{V_o^2}{\mathscr{P}}$$

$$\langle \mathscr{P} \rangle = \frac{R_{surf}}{2} \int_s |H|^2 ds$$

$$R_{surf} = \frac{\mu\omega}{2\sigma_{dc}} = \pi Z_o \frac{\delta_{skin}}{\lambda_{rf}} \quad \text{where } Z_o = \sqrt{\mu_o / \varepsilon_o} = 377\Omega$$

The on-axis field E and surface H are generally computed with a computer code such as SUPERFISH for a complicated cavity shape

Compute the voltage gain correctly



The voltage gain seen by the beam can computed in the co-moving frame, or we can use the transit-time factor, Γ & compute V at fixed time

$$V_o^2 = \Gamma \int_{z_1}^{z_2} E(z)dz$$

Exercise: Pillbox array

- Derive the Q and R_{sh} for the pillbox cavity as a function of the dimensions of the cavity and the frequency of the fundamental mode

[redacted]

Note on previous slide

$$Z(\omega) = \frac{1}{j\omega C + (j\omega L + R)^{-1}} = \frac{(j\omega L + R)}{(j\omega L + R)\, j\omega C + 1} = \frac{(j\omega L + R)}{(1 - \omega^2 LC) + j\omega RC}$$

$$= \frac{(j\omega L + R)}{\left(1 - \dfrac{\omega^2}{\omega_o^2}\right)^2 + j\omega RC} = \frac{(j\omega L + R)\left[\left(1 - \dfrac{\omega^2}{\omega_o^2}\right) - j\omega RC\right]}{\left(1 - \dfrac{\omega^2}{\omega_o^2}\right)^2 + \left(\omega RC\right)^2} = \frac{j\omega\left[L\left(1 - \dfrac{\omega^2}{\omega_o^2}\right) - R^2 C\right] + R\left(1 - \dfrac{\omega^2}{\omega_o^2}\right) + \dfrac{\omega^2}{\omega_o^2} R}{\left(1 - \dfrac{\omega^2}{\omega_o^2}\right)^2 + \left(\omega RC\right)^2}$$

$$\frac{j\omega\left[L\left(1 - \dfrac{\omega^2}{\omega_o^2}\right) - R^2 C\right] + R}{\left(1 - \dfrac{\omega^2}{\omega_o^2}\right)^2 + \left(\omega RC\right)^2} = \frac{1}{\left(1 - \dfrac{\omega^2}{\omega_o^2}\right)^2 + \left(\omega RC\right)^2}\left[R + j\omega\left(L\left(1 - \dfrac{\omega^2}{\omega_o^2}\right) - R^2 C\right)\right]$$

$$\Rightarrow \left|Z\right| = \frac{1}{\left(1 - \dfrac{\omega^2}{\omega_o^2}\right)^2 + \left(\omega RC\right)^2}\left[R^2 + \omega^2\left(L\left(1 - \dfrac{\omega^2}{\omega_o^2}\right) - R^2 C\right)^2\right]^{1/2}$$