Radiation by Charged Particles: a Review

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(based on a lecture by Fernando Sannibale)

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Introduction

The scope of this lecture is to give a quick review of the physics of radiation from charged particles.

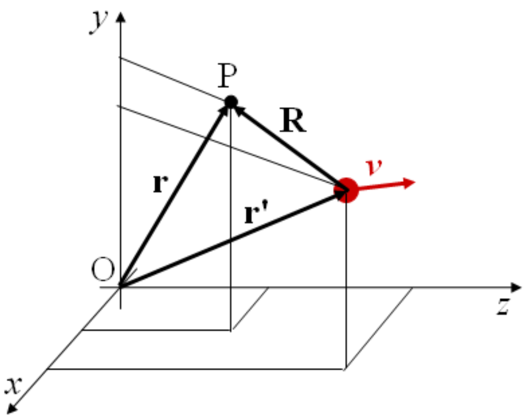
A basic knowledge of electromagnetism laws is assumed.

The classical approach is briefly described, main formulas are given but generally not derived.
The detailed derivation can be found in any classical electrodynamics book and it is beyond the scope of this course.

A semi-classical approach by Max Zolotorev is also presented that gives an "intuitive" view of the radiation process.

The Field of a Moving Charged Particle

A particle with charge q is moving along the trajectory $\mathbf{r}'(t)$, the vector $\mathbf{r}$ defines the observation point P. $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ is the vector with magnitude equal to the distance between the particle and the observation point.



The particle at the time t generate a Coulomb potential that will contribute to the potential at the point P at a later time t given by (cgs units):

$$t = \tau + \frac{R(\tau)}{c}$$

$$d\varphi(\mathbf{r},t) = \frac{q}{R(\tau)}\delta[\tau - t + R(\tau)/c]dt \qquad R(\tau) = |\mathbf{R}(\tau)| = |\mathbf{r} - \mathbf{r}'(\tau)|$$

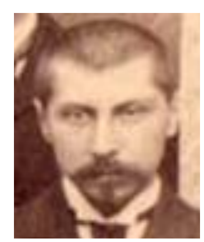
So the total potential at the point P at the time t is given by:

$$\varphi(\mathbf{r},t) = q\int_{-\infty}^{\infty}\frac{\delta[\tau - t + R(\tau)/c]}{R(\tau)}d\tau = q\int_{-\infty}^{\infty}\frac{\delta[\tau - t + |\mathbf{r} - \mathbf{r}'(\tau)|/c]}{|\mathbf{r} - \mathbf{r}'(\tau)|}d\tau$$

Lienard-Wiechert Potentials

And analogously for the vector potential:

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{c}\int_{-\infty}^{\infty}\mathbf{v}\frac{\delta[\tau - t + R(\tau)/c]}{R(\tau)}d\tau = \frac{q}{c}\int_{-\infty}^{\infty}\mathbf{v}\frac{\delta[\tau - t + |\mathbf{r} - \mathbf{r}'(\tau)|/c]}{|\mathbf{r} - \mathbf{r}'(\tau)|}d\tau$$



Accelerated Particles Radiate

The field components can be calculated from the Lienard-Wiechert potentials and the relations:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi \qquad \mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{R} = R\,\mathbf{n} \;\; with \;\; |\mathbf{R}| = R$$

$$\mathbf{E} = \frac{q}{\gamma^2 R^2 (1-\mathbf{n}\cdot\boldsymbol{\beta})^3}(\mathbf{n}-\boldsymbol{\beta}) + \frac{q}{cR(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\frac{d\boldsymbol{\beta}}{dt}\right] \quad with \;\; \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \;\; \gamma = \left(1-\beta^2\right)^{-1/2}$$

$$\mathbf{B} = \mathbf{n}\times\mathbf{E} \;\;\Rightarrow\;\; \mathbf{B} \; is \;\; perpendicular \;\; to \;\; \mathbf{E}$$

where the quantities on the RHS of the expressions are calculated at $t = t - R(t)/c$.

The first term of the electric field depends on the particle speed and converges to the Coulomb field when v goes to zero.

The second term is non zero only if the particle is accelerated.
Charged particles when accelerated radiate electromagnetic waves.

When the observation direction $\mathbf{n}$ is parallel to the particle trajectory β and the acceleration $d\beta/dt$ is perpendicular to β, the resulting electric field is parallel to the acceleration. If $d\beta/dt$ is parallel to $\mathbf{R}$ there is no radiation.

Emission by Relativistic Electron in Free Space

The radiated electric field can be expressed in frequency domain:

$$\mathbf{E}_\omega = \frac{q}{c}\int_{-\infty}^{+\infty} \frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times d\boldsymbol{\beta}/dt\right] + cR^{-1}\gamma^{-2}(\mathbf{n}-\boldsymbol{\beta})}{R\cdot(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\exp\left[i\omega(\tau + R/c)\right]d\tau$$ **L. D. Landau**

$$\mathbf{E}_\omega = \frac{iq\omega}{c}\int_{-\infty}^{+\infty} R^{-1}[\boldsymbol{\beta} - [1 + ic/(\omega R)]\,\mathbf{n}]\exp[i\omega(\tau + R/c)]d\tau$$ **I.M.Ternov**

The equivalence of the two expressions can be shown by integration by parts and the quantities on the RHS of the expressions are again calculated at $t = t - R(t)/c$.

Landau also showed that when $r \gg r'$ and $R \sim R_0 = r$ then the vector potential in frequency domain can be written as:

$$\tilde{\mathbf{A}}(\omega) = q\frac{i\omega\exp(ikR)}{cR_0}\oint\exp[i(\omega t - \mathbf{k}r')]d\mathbf{r}' \quad where\ k = \frac{2\pi}{\lambda}$$

$$\tilde{\mathbf{E}}(\omega) = \frac{ic}{\omega}\mathbf{k}\times\left[\tilde{\mathbf{A}}(\omega)\times\mathbf{k}\right]$$

$$\tilde{\mathbf{B}}(\omega) = i\,\mathbf{k}\times\tilde{\mathbf{A}}(\omega)$$

The last integral is calculated on the particle trajectory and shows that for $r \gg r'$, the net radiation is the result of the interference between plane waves emitted by the particle during its motion.

For a relativistic particle in rectilinear motion in a uniform media the interference is fully destructive and no radiation is emitted.

Coherence Lengths and Coherence Volume

By applying the Heisenberg uncertainty principle for the photon case we obtain:

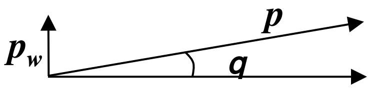
$$\sigma_{pz}\sigma_z \geq \frac{\hbar}{2} \qquad p = \frac{E}{c} = \frac{h\nu}{c} = \frac{\hbar\omega}{c} = \hbar\frac{2\pi}{\lambda} = \hbar k \qquad \sigma_z = c\sigma_\tau$$

$$\sigma_{pz}\sigma_z = \frac{\hbar}{c}\sigma_\omega c\sigma_\tau \quad \Rightarrow \quad \sigma_\omega\sigma_\tau \geq \frac{1}{2} \quad or \quad \boxed{\frac{\sigma_\lambda}{\lambda}\sigma_z \geq \frac{\lambda}{4\pi}}$$

we can define the longitudinal coherence length as

$$\boxed{\sigma_{zc} = \frac{c}{2\sigma_\omega}}$$

$$\sigma_{pw}\sigma_w \geq \frac{\hbar}{2} \qquad p_w = p\sin\theta_w = \frac{\hbar\omega}{c}\sin(\theta_w) \cong \frac{\hbar\omega}{c}\theta_w = \hbar\frac{2\pi}{\lambda}\theta_w \qquad w = x, y$$



$$\sigma_{pw}\sigma_w = \hbar\frac{2\pi}{\lambda}\sigma_\theta\sigma_w \quad \Rightarrow \quad \boxed{\sigma_\theta\sigma_w \geq \frac{\lambda}{4\pi}}$$

and the transverse coherence length as

$$\boxed{\sigma_{wc} = \frac{\lambda}{4\pi\sigma_{\theta w}}}$$

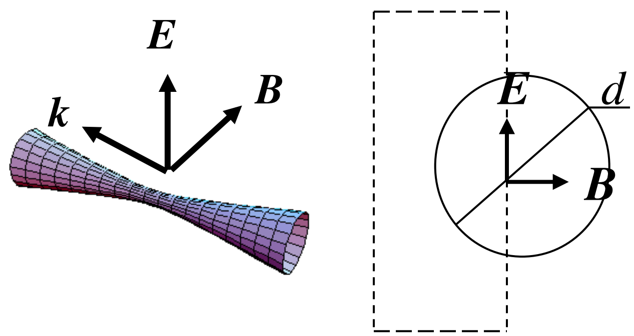
By using the previous results, we can define the volume of coherence V_C in the 6-D phase space

$$\boxed{V_C = (\lambda/4\pi)^3}$$

Two photons inside V_C are indistinguishable, or in other words are in the same coherent state or mode.

Alternative Derivation of the Coherence Lengths

Let us consider a wave focused into a waist of diameter d. Field components and wave vector as in the figure. From Stokes theorem and Faraday law (SI units):



$$\oint \overline{E}\cdot d\overline{l} = \int_S \left(\nabla\times\overline{E}\right)\cdot\overline{n}\; dS = \frac{\partial}{\partial t}\int_S \overline{B}\cdot\overline{n}\; dS$$

If we Integrate over the dotted path, we notice that the integral on the left is not vanishing. This implies that the magnetic field must have a component parallel to k due to diffraction.

$$E\; d \approx d^2\theta_{dif}\frac{\partial B}{\partial t} = B\omega d^2\theta_{dif} = Bckd^2\theta_{dif} \qquad But\;\; E = Bc \Rightarrow \boxed{\theta_{dif}\approx\frac{1}{kd}}$$

One can say that the waist diameter is diffraction limited and d represents the transverse coherence length when q is the radiation angular aperture

$$d_\perp \approx \frac{1}{k\theta}$$

The transform limited length of a pulse with bandwidth $\Delta\omega$ is $t_C = 1/\Delta\omega$, so the longitudinal coherence length is defined as

$$l_\parallel \approx \frac{c}{\Delta\omega}$$

The Coherence Volume for Particles

By applying the Heisenberg uncertainty principle to emittance:

$$\sigma_w \sigma_{pw} \geq \hbar/2 \;\; and \;\; \varepsilon_{nw} = \sigma_w \sigma_{pw} / m_0 c = \beta\gamma \sigma_w \sigma'_w \Rightarrow \boxed{\varepsilon_{nw} \geq \lambda_{Compton}/4\pi} \quad w = x,y,z$$

$$\lambda_{Compton} \equiv Compton \;\; wavelength = h/m_0 c = 2.426 \;\; pm \;\; for \;\; electrons,$$

$$\varepsilon_{nw} \equiv normalized \;\; \text{emittance}, \quad w' = dw/ds$$

This allows to define a 6-D phase space volume V_C ⇒ $$\boxed{V_C = \left(\frac{\lambda_{Compton}}{4\pi}\right)^3}$$

Two particles inside V_C are indistinguishable, or in other words are in the same coherent state.

By analogy with the photon case we can say that V_C is the coherence volume for the particle.

The Degeneracy Parameter

The **degeneracy parameter** δ is defined as the number of particles (photons, electrons, ...) in the volume of coherence V_C

The **limit value of** δ **is infinity for bosons**, and **2 for non polarized-fermions** because of the Pauli exclusion principle.

The relation between **brightness** B and δ is:

$$B = \frac{N}{\varepsilon_{nx}\varepsilon_{ny}\varepsilon_{nz}}$$

$$N \equiv number\ of\ particles$$

$$\delta = B\left(\frac{\lambda_C}{4\pi}\right)^3$$

Typical Degeneracy Parameter Values

Photons (spin 1)

$$\delta = \frac{1}{e^{\frac{h\nu}{kT}} - 1} << 1$$ for thermal sources of radiation in the visible range

$$\delta \approx \alpha N_e \omega \tau_b \approx 10^3$$ for synchrotron sources of radiation in the visible range

$$\delta \approx N_{ph} \approx 3\times 10^{18}$$ for a 1 J laser in the visible range

Electrons (spin 1/2)

$$\delta = 2$$ for electrons in a metal at $T = 0$ °K (maximum allowed for unpolarized electrons)

$$\delta \approx N_e \frac{\lambdabar^3}{\varepsilon_x \varepsilon_y \varepsilon_z} \approx 2\times 10^{-12}$$ for electrons from RF photo guns

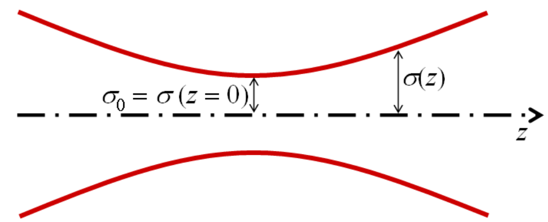
$$\delta \approx 10^{-6}$$ for electrons from needle (field emission) cathodes

Rayleigh Range and Beta Function

For a beam (particles or photons in paraxial approximation) drifting in a free space of length z:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} \Rightarrow \langle x^2 \rangle = \langle (x_0 + z x'_0)^2 \rangle = \langle x_0^2 \rangle + z^2 \langle x_0'^2 \rangle + 2z \langle x_0 x'_0 \rangle$$

Let's assume that the beam for $z = 0$ is in a *waist*



$$\Rightarrow \langle x_0 x'_0 \rangle = 0 \qquad \Rightarrow \langle x^2 \rangle = \langle x_w^2 \rangle + z^2 \langle x_w'^2 \rangle = \langle x_w^2 \rangle \left(1 + z^2 \langle x_w'^2 \rangle / \langle x_w^2 \rangle \right)$$

For particles $\langle x^2 \rangle = \sigma_x^2 = \varepsilon_x \beta_x$ *and* $\langle x'^2 \rangle = \sigma_x'^2 = \varepsilon_x / \beta_x$ $\Rightarrow$ $\sigma_x = \sigma_w \left(1 + z^2/\beta_x^2\right)^{1/2}$

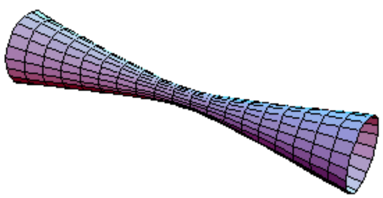
For photons $\langle x^2 \rangle \langle x'^2 \rangle = \sigma_x^2 \sigma_x'^2 = (\lambda / 4\pi)^2$ $\Rightarrow$ $\sigma_x = \sigma_w \left(1 + z^2/z_0^2\right)^{1/2}$

Where we have defined the Rayleigh range as $z_0 = \dfrac{4\pi\sigma_w^2}{\lambda} = \dfrac{\pi w_0^2}{\lambda}$ **and the photon beam size as** $w_0 = 2\sigma_w$

Note that the z_0 in optics plays the same role of β in particle physics

Electron and photon optics: a complete analogy

Light optics
(paraxial approximation)



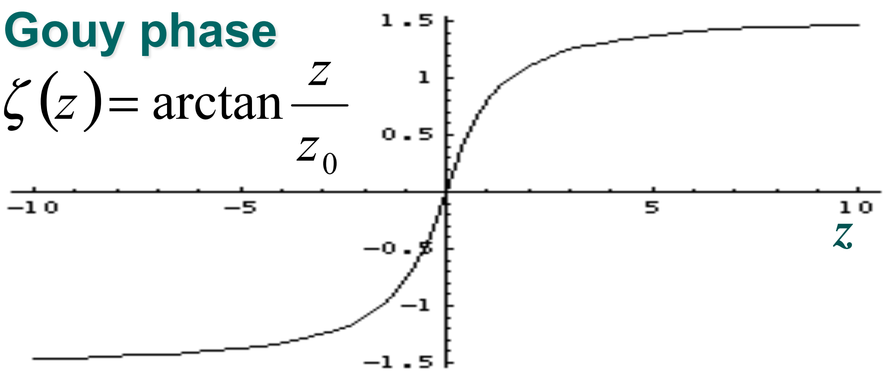
$$\varepsilon_{Ph} = \frac{\lambda}{4\pi}$$

$$\sigma_{wPh}^2 = \varepsilon_{Ph} z_0 \qquad \sigma_{wR}'^2 = \frac{\varepsilon_{Ph}}{z_0}$$

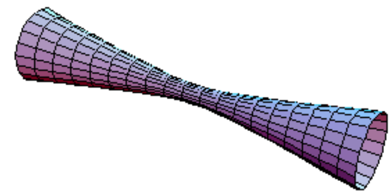
$$\sigma_{Ph}^2(z) = \sigma_{wPh}^2\left(1 + \frac{z^2}{z_0^2}\right)$$

Gouy phase

$$\zeta(z) = \arctan\frac{z}{z_0}$$



Accelerator optics



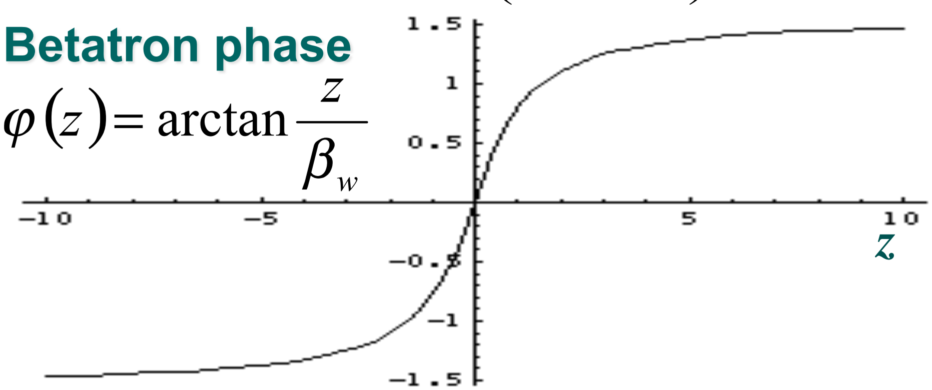
$$\varepsilon_B >> \frac{\lambda_C}{4\pi}$$

$$\sigma_{wB}^2 = \varepsilon_B \beta_w \qquad \sigma_{wB}'^2 = \frac{\varepsilon_B}{\beta_w}$$

$$\sigma_B^2(z) = \sigma_{wB}^2\left(1 + \frac{z^2}{\beta_w{}^2}\right)$$

Betatron phase

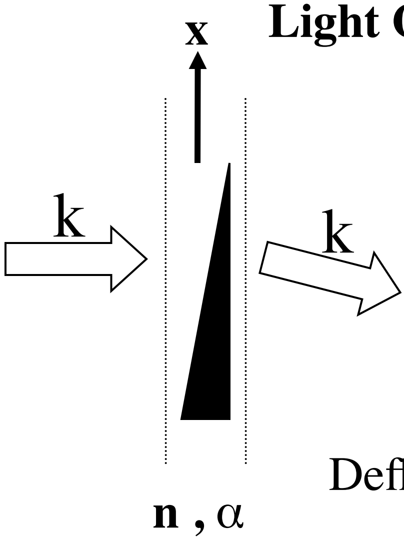
$$\varphi(z) = \arctan\frac{z}{\beta_w}$$



A Complete Symmetry: Particles and Plane Waves

Light Optics

Prism

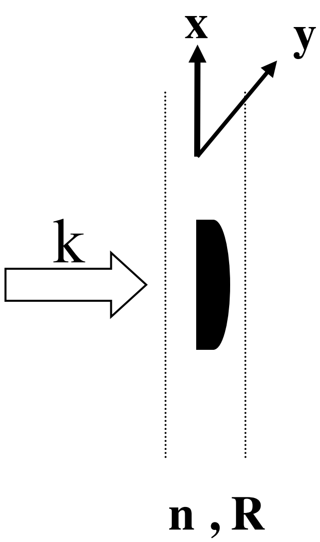


$$e^{i\int n(r)\vec{k}\,d\vec{r} - i\int \omega\, dt}$$

$$e^{i(n-1)\alpha k x - i\omega t}$$

Deflection angle = *(n -1) α*

Lens



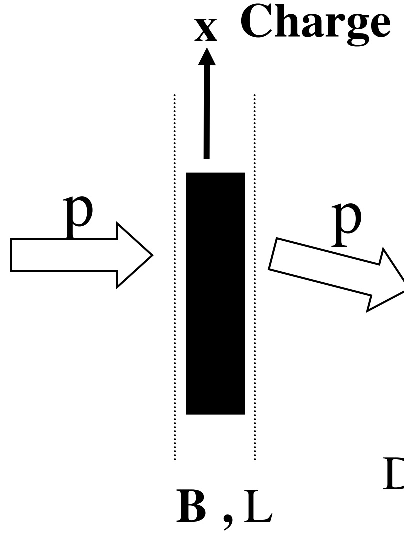
$$e^{i\int n(r)\vec{k}\,d\vec{r} - i\int \omega\, dt}$$

$$e^{i(n-1)k\frac{x^2+y^2}{2R} - i\omega t}$$

$$\frac{1}{F_x} = \frac{1}{F_y} = (n-1)\frac{1}{R}$$

Charge Particle Optics

Bend magnet

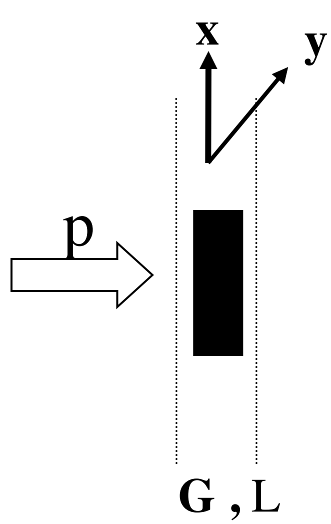


$$e^{\frac{i}{\hbar}\int\left(\vec{p}-\frac{e}{c}\vec{A}\right)d\vec{r} - \frac{i}{\hbar}\int(\mathcal{E}-e\varphi)\,dt}$$

$$e^{\frac{i}{\hbar}\frac{eBL}{pc}px - \frac{i}{\hbar}\mathcal{E}t}$$

Deflection angle = ***eBL/cp***

Quadrupole lens



$$e^{\frac{i}{\hbar}\int\left(\vec{p}-\frac{e}{c}\vec{A}\right)d\vec{r} - \frac{i}{\hbar}\int(\mathcal{E}-e\varphi)\,dt}$$

$$e^{\frac{i}{\hbar}\frac{eGL}{2pc}p(x^2-y^2) - \frac{i}{\hbar}\mathcal{E}t}$$

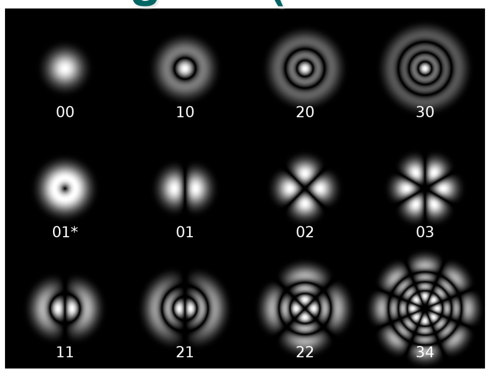
$$\frac{1}{F_x} = -\frac{1}{F_y} = \frac{eGL}{cP}$$

Transverse Modes

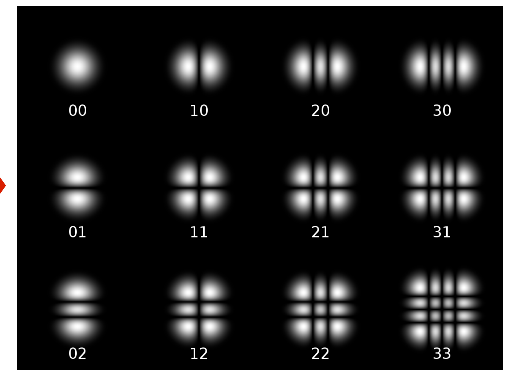
Transverse modes define the intensity profile of photon beams.
Transverse Electro-Magnetic or TEM modes are of particular interest.

These can present cylindrical simmetry (Laguerre-Gaussian modes radially polarized) or rectangular (Hermite-Gaussian modes linearly polarized):

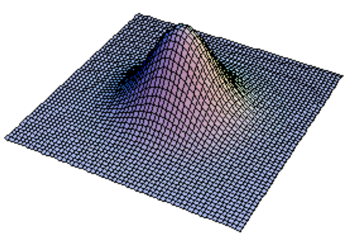
LG_{pq} modes



HG_{pq} modes



Gaussian mode: the fundamental mode for both LG and HG modes



$$U(\mathbf{r})=A_0\frac{W_0}{W(z)}\exp\left[\frac{-\rho^2}{W(z)^2}\right]\exp\left[-ikz-ik\frac{\rho^2}{2R(z)}+i\zeta(z)\right]$$

$$W(z)=W_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} \qquad \zeta(z)=\tan^{-1}\frac{z}{z_0}$$

$$I(\rho,z)=I_0\left[\frac{W_0}{W(z)}\right]^2\exp\left[-\frac{2\rho^2}{W(z)^2}\right]$$

$$R(z)=z\left[1+\left(\frac{z_0}{z}\right)^2\right] \qquad W_0=\left(\frac{\lambda z_0}{\pi}\right)^{1/2}$$

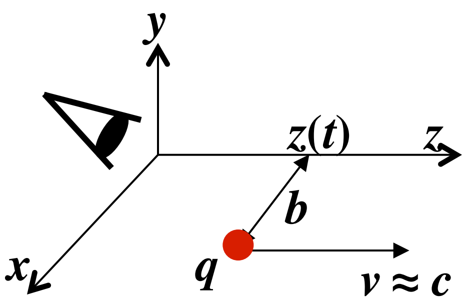
The emittance of the higher order modes is proportional to the number m of transverse spots

$$\varepsilon\approx m\frac{\lambda}{4\pi}=\frac{m}{2k}$$

Weizsäcker-Williams Method of Virtual Photons

The method exploits the fact that the field of a relativistic particle is very similar to the one of a plane wave.

Because of this, the particle can be replaced by virtual photons (plane wave) that with their field represent the field of the particle.



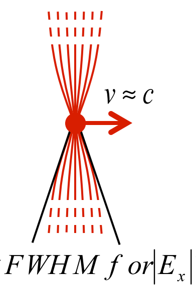
In the particle rest frame (cgs units):

$$\bar{E}' = \frac{q}{r'^3}\bar{r}'$$

$$E'_x = \frac{qb}{\left[b^2 + z'^2(t')\right]^{3/2}} = \frac{qb}{\left[b^2 + (ct')^2\right]^{3/2}} \qquad E'_z = \frac{qct'}{\left[b^2 + (ct')^2\right]^{3/2}} \qquad E'_y = 0$$

and in the laboratory frame:

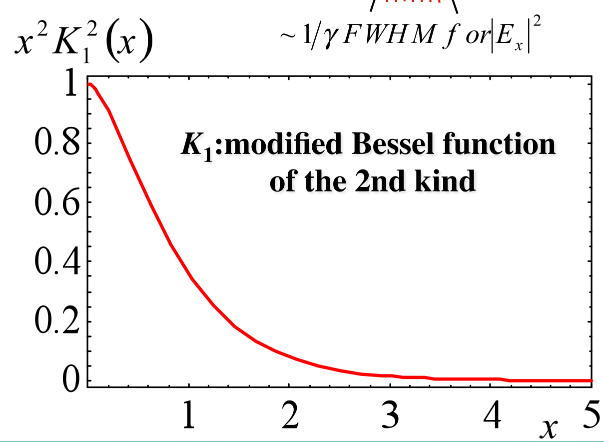
$$E_z = -\frac{\gamma qct}{\left[b^2 + (\gamma ct)^2\right]^{3/2}} \qquad E_x = \frac{\gamma qb}{\left[b^2 + (\gamma ct)^2\right]^{3/2}} \qquad E_y = 0$$



By Fourier transforming, the spectrum of the energy *W* per unit area due to the two terms is obtained:

$$\frac{dW_z(\omega,b)}{b\,db\,d\varphi\,d\omega} = \frac{c}{2\pi}\left|E_z(\omega,b)\right|^2 = \frac{q^2}{\pi^2 cb^2}\frac{1}{\gamma^2}\left(\frac{\omega b}{\gamma c}\right)^2 K_0^2\left(\frac{\omega b}{\gamma c}\right)$$

$$\frac{dW_x(\omega,b)}{b\,db\,d\varphi\,d\omega} = \frac{c}{2\pi}\left|E_x(\omega,b)\right|^2 = \frac{q^2}{\pi^2 cb^2}\left(\frac{\omega b}{\gamma c}\right)^2 K_1^2\left(\frac{\omega b}{\gamma c}\right)$$

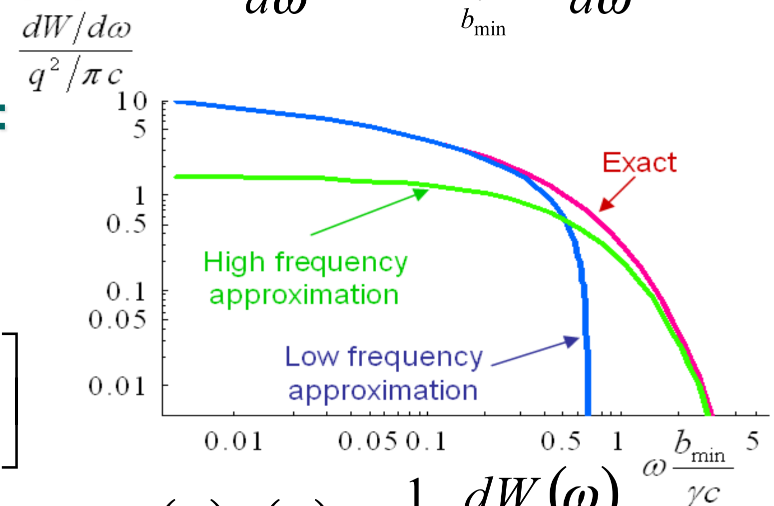


The Power Spectrum of the Virtual Photons

The total energy spectrum is obtained by integrating the previous spectrum over the possible values of b:

$$\frac{dW(\omega)}{d\omega} = 2\pi \int_{b_{\min}}^{\infty} \frac{dI(\omega,b)}{d\omega} b\, db$$

The complete analytical solution can be derived but the following approximations are very useful:



for $\omega \gg \gamma c / b_{\min}$

$$\frac{dW(\omega)}{d\omega} \approx \frac{q^2}{2c} \exp\left(-\frac{2b_{\min}}{\gamma c}\omega\right)$$

and for $\omega \ll \gamma c / b_{\min}$

$$\frac{dW(\omega)}{d\omega} \approx \frac{2}{\pi}\frac{q^2}{c}\left[\ln\left(\frac{1.123\gamma c}{\omega b_{\min}}\right) - \frac{1}{2}\right] \cong \frac{2}{\pi}\frac{q^2}{c}\left[\ln\left(\frac{\gamma c}{\omega b_{\min}}\right) - \frac{1}{2}\right]$$

The number of virtual photons per mode is given by:

$$n(\omega) d(\omega) = \frac{1}{\hbar\omega}\frac{dW(\omega)}{d\omega} d\omega$$

$$n(\omega) d\omega \approx \frac{2}{\pi}\alpha\left[\ln\left(\frac{\gamma c}{\omega b_{\min}}\right) - \frac{1}{2}\right]\frac{d\omega}{\omega}$$

Low frequency regime

$$with \quad \alpha = \frac{e^2}{\hbar c} \cong \frac{1}{137}$$

$$n(\omega) d\omega \approx \frac{\alpha}{2}\exp\left(-\frac{2b_{\min}}{\gamma c}\omega\right)\frac{d\omega}{\omega}$$

High frequency regime

The spectrum of the virtual photons associated with a particle extends up to about the critical wavelength ω_C

$$\omega_C = \frac{\gamma c}{b_{\min}}$$

The Calculation of b_{min}

The quantity b is the distance between the observation point and the particle trajectory (the *impact parameter* in collision terminology)

We already derived that for a particle $\beta\gamma\sigma_w\sigma'_w \geq \frac{\lambda_{Compton}}{4\pi} = \frac{h}{4\pi\, m_0 c} = \frac{\hbar}{2m_0 c} \quad w = x, y$

in our case $\beta \sim 1$ and $\sigma'_x \sim 1/2\gamma$ ➡ $\sigma_w \geq \sigma_{w\min} \sim \frac{\lambda_{Compton}}{2\pi} = \frac{\hbar}{m_0 c}$

The position of the particle cannot be defined within $\sigma_{w\ min}$, the **coherence length**. It is natural than to assume ➡

$$b^*_{\min} \sim \sigma_{w\min} \sim \frac{\lambda_{Compton}}{2\pi} = \frac{\hbar}{m_0 c}$$

$$b^*_{\min} \sim 4\times 10^{-3}\ \mathring{A} \quad f\, orelectrons$$

that used in a previous result for e^- ➡

$$n(\omega)d\omega \approx \frac{2}{\pi}\alpha\left[\ln\left(\frac{\gamma m_0 c^2}{\hbar\omega}\right) - \frac{1}{2}\right]\frac{d\omega}{\omega} \cong \frac{2}{\pi}\alpha\ln\left(\frac{\gamma m_0 c^2}{\hbar\omega}\right)\frac{d\omega}{\omega}$$

This expression shows how many virtual photons per mode are readily "available" for radiation!

The virtual photon spectrum is limited to ~ $\hbar\omega_C \sim \gamma m_0 c^2$

(The "log" term for typical cases ranges from few units to few tens)

The Radiation Divergence

We just showed that the quantity b_{min} represents the transverse coherence of the radiation at the critical wavelength.

$$\sigma_c \sim b_{\min} \qquad\qquad \omega_C \sim \frac{\gamma c}{b_{\min}}$$

We will see later in the talk that each radiation process is characterized by its own value of b_{min} (always > b^*_{min}). But before going into that, we can still extract some additional information common to all cases.

We previously found that: $\sigma_c \sigma_{\theta c} \sim \lambda / 4\pi$

so at the critical wavelength: $\sigma_{\theta c} \sim \dfrac{\lambda_C}{4\pi\sigma_c} \sim \dfrac{\lambda_C}{4\pi b_{\min}} = \dfrac{c}{2b_{\min}\omega_C} \sim \dfrac{c}{2b_{\min}} \dfrac{b_{\min}}{\gamma c} = \dfrac{1}{2\gamma}$

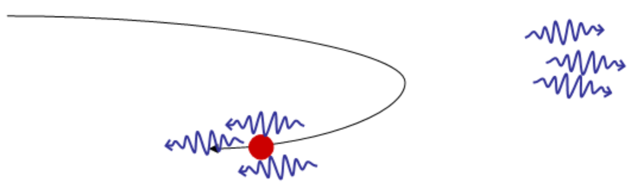
So independently from the radiating process, the angular width of the radiation at the critical wavelength is always:

$$\sigma_\theta \sim \sigma_{\theta c} \sim \frac{1}{2\gamma}$$

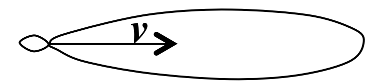
Virtual photons get real

We now know that a drifting particle can be considered as surrounded by a cloud of virtual photons responsible for the particle field.
Such photons cannot be distinguished from the particle itself but...

- If the charged particle receives a kick that delays it from its virtual photons the photons can be separated and become real
In vacuum when $>> 1$ the only practical way is by a transverse kick:



Synchrotron radiation
Edge Radiation
Bremsstrahlung, Beamstrahlung

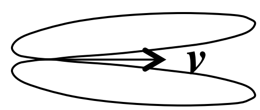


Synchrotron Radiation

- If in a media the speed of light at a given wavelength is smaller than the particle speed the photons lag behind the particle and separate.

$$v > c/n(\lambda)$$

Cerenkov radiation

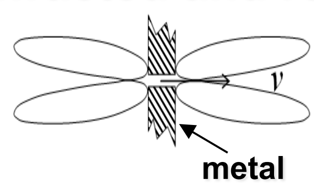


Radially polarized and hollow due to symmetry

- If a particle goes through an aperture with diameter $2b$ smaller than or comparable with the transverse coherence length of some of its virtual photons those photons will be diffracted and reflected.

$$2\sigma_{wc} = \frac{\lambda}{2\pi\sigma_{\theta w}} \sim \gamma \frac{\lambda}{2\pi} > b$$

Diffraction
Transition radiation
(Smith-Purcell)



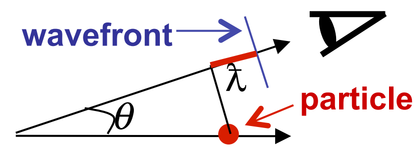
Radially polarized and hollow due to symmetry (not Smith-Purcell)

The Formation Length

*The **formation length** L_F is the trajectory length that a particle has to travel in order that the radiated wavefront advances one $\lambda/2\pi$ (one radian) ahead of the particle trajectory projection along the observation direction.*

Virtual photons become real after the parent particle travels for one L_F

Example: formation length for diffraction or transition radiation emitted during transition from media to vacuum:



$$L_F = \beta c t_F$$

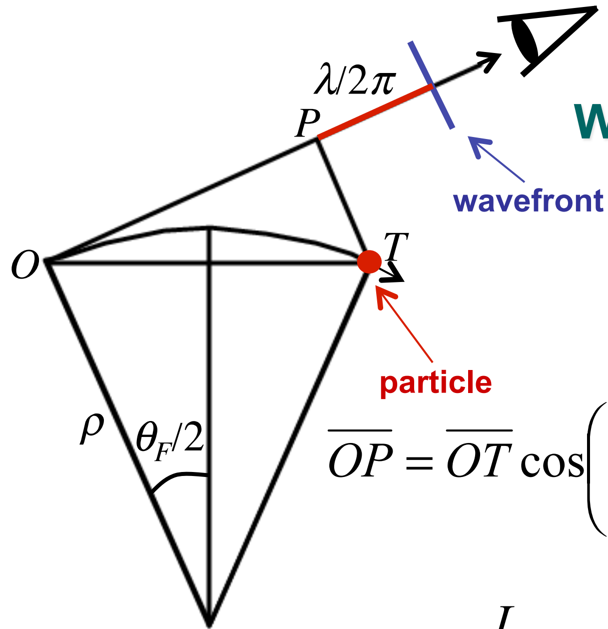
$$\lambda\!\!\!^{-} = c t_F - \beta c t_F \cos(\theta) = L_F\left[\frac{1}{\beta} - \cos(\theta)\right] \quad \Rightarrow \quad L_F = \frac{\lambda\!\!\!^{-}}{1/\beta - \cos(\theta)}$$

$$For\ \beta \sim 1,\ \theta << 1 \ \Rightarrow\ 1/\beta \cong 1 + 1/2\gamma^2 \quad and \quad \cos(\theta) \cong 1 - \theta^2/2$$

$$L_F \cong \frac{2\lambda\!\!\!^{-}}{1/\gamma^2 + \theta^2}$$

If we observe the radiation at $\lambda \sim \lambda_C$ at the peak for $\theta \sim 1/\gamma$: $\quad L_F \sim \gamma^2 \lambda\!\!\!^{-}$

Synchrotron Radiation Formation Length



With reference to the figure and using the definition of L_F:

$$\bar{\lambda} = ct_F - \overline{OP} \qquad and \quad L_F = \rho\theta_F \quad or \quad t_F = \frac{L_F}{\beta c}$$

$$\overline{OP} = \overline{OT}\cos\left(\frac{\theta_F}{2}\right) = 2\rho\sin\left(\frac{\theta_F}{2}\right)\cos\left(\frac{\theta_F}{2}\right) = \rho\sin\theta_F \qquad \boxed{\Rightarrow \bar{\lambda} = \frac{L_F}{\beta} - \rho\sin\frac{L_F}{\rho}}$$

$$\Rightarrow \bar{\lambda} \cong \frac{L_F}{\beta} - \rho\left(\frac{L_F}{\rho} - \frac{1}{6}\frac{L_F^3}{\rho^3}\right) = L_F\left(\frac{1}{\beta} - 1 + \frac{1}{6}\frac{L_F^2}{\rho^2}\right) \cong \frac{L_F}{2}\left(\frac{1}{\gamma^2} + \frac{1}{3}\frac{L_F^2}{\rho^2}\right)$$

$$If \ \frac{1}{3}\frac{L_F^2}{\rho^2} = \boxed{\frac{\theta_F^2}{3} >> \frac{1}{\gamma^2}} \Rightarrow \frac{\lambda}{2\pi} \sim \left(\frac{1}{6}\frac{L_F^3}{\rho^2}\right) \boxed{\Rightarrow L_F \sim \lambda^{1/3}\rho^{2/3}}$$

Low frequency regime

$$\boxed{or \quad L_F \sim \gamma^2\lambda/\pi}$$

High frequency regime

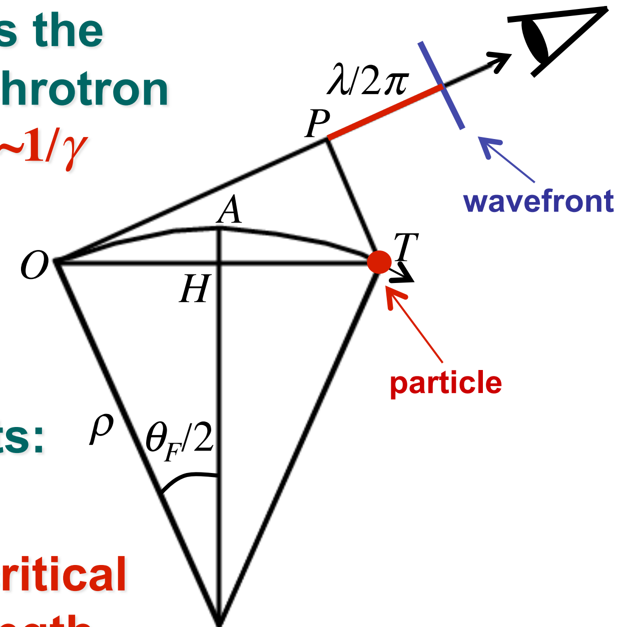
The angle $\theta_F = L_F/\rho$ also indicates the radiation angular width:

$$\boxed{\vartheta = \theta_F \sim \left(\frac{\lambda}{\rho}\right)^{1/3}}$$

Low frequency angular width

The Calculation of b_{min} for Synchrotron Radiation

As it was shown before, the parameter b_{min} represents the transverse coherence length for the radiation. For synchrotron radiation b_{min} is given by the segment AH when $\theta_F/2 \sim 1/\gamma$



$$\overline{AH} = \rho[1-\cos(\theta_F/2)] \cong \rho\left(1-1+\frac{1}{2}\frac{\theta_F^2}{4}\right) = \frac{\rho\theta_F^2}{8}$$

$$b_{\min} \cong \frac{\rho\theta_{F\min}^2}{8} \sim \frac{\rho}{8}\left(\frac{2}{\gamma}\right)^2 = \frac{\rho}{2\gamma^2}$$

And using previous results:

$$\omega_C = \frac{\gamma c}{b_{\min}} \sim 2\gamma^3 \frac{c}{\rho}$$

$$\lambda_C \sim \pi \frac{\rho}{\gamma^3}$$

Synchrotron radiation critical frequency and wavelength

$$n(\omega)d\omega \approx \frac{2}{\pi}\alpha\left[\ln\left(2\frac{\gamma^3 c}{\omega\rho}\right)-\frac{1}{2}\right]\frac{d\omega}{\omega} = \frac{2}{\pi}\alpha\left[\ln\left(\frac{\omega_C}{\omega}\right)-\frac{1}{2}\right]\frac{d\omega}{\omega}$$

$$\Rightarrow n(\omega)d\omega \sim \alpha\frac{d\omega}{\omega}$$

In the rest of the lecture, we will neglect the log and the -1/2 terms and the $2/\pi$ factor because for all radiation processes they are together of the order of 1.

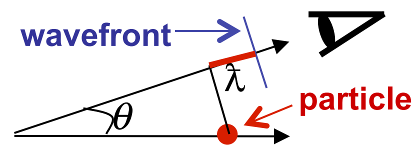
$$\frac{dP}{d\omega} = \hbar\omega\frac{c}{L_F}n(\omega) \sim \frac{e^2}{c}\omega_C^{2/3}\frac{\omega^{1/3}}{\gamma^2} \quad for\ \omega << \omega_C$$

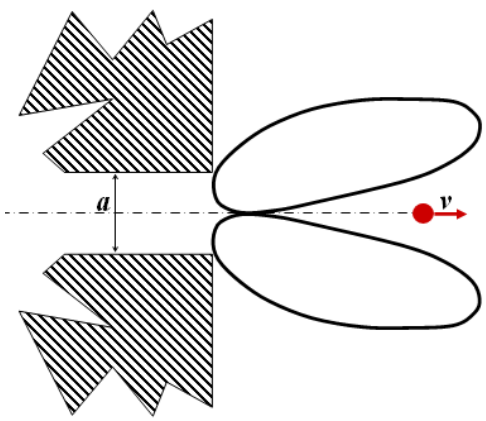
Low frequency power spectrum

Diffraction Radiation

We already calculated the formation length for this case:

$$L_F = \frac{\lambdabar}{1/\beta - \cos(\theta)} \sim \frac{2\lambdabar}{1/\gamma^2 + \theta^2}$$





$$b_{\min} \sim a$$

$$\Rightarrow \quad \omega_C = \gamma c / b_{\min} \sim \gamma c / a$$

So for $a = 1$ mm and a 1 GeV electron, the diffraction radiation spectrum extends to up $\sim \omega_C/2\pi \sim 100$ THz ($\lambda_C \sim 3$ μm).

The intensity peaks at $\theta \sim 1/\gamma$ where $L_F \sim \lambda\gamma^2/2\pi$ and the power spectrum is:

$$\frac{dP}{d\omega} = \hbar\omega \frac{c}{L_F} n(\omega) \sim \frac{e^2}{c}\frac{1}{\gamma^2}\omega$$

Low frequency power spectrum @ $\theta \sim 1/\gamma$

$$\frac{dP}{d\omega} \sim \frac{e^2}{c}\frac{1}{\gamma^2}\omega \exp\left(-2\frac{\omega}{\omega_C}\right)$$

High frequency power spectrum @ $\theta \sim 1/\gamma$

Transition Radiation

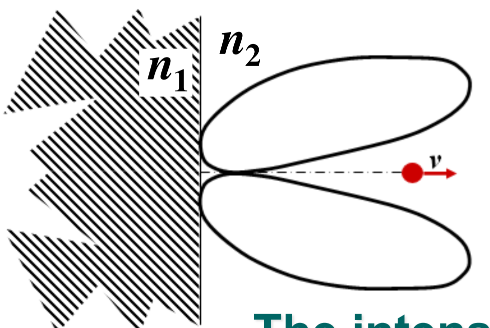
The fields of a relativistic particle crossing a media interact with the electrons of the media itself . Such electrons move under the action of the time varying electric field up to frequencies of the order of the *plasma frequency.* Above this frequency the electrons in the media cannot respond to the too fast excitation anymore and the media becomes transparent at these high frequency components.

$$\omega_P \sim 4\pi \frac{e^2}{m_0} n_e$$

$$n_e \equiv e^- \ density$$

cgs units

Transition radiation can be viewed as diffraction radiation through a hole of the size of ~ a plasma wavelength!



$$b_{\min} \sim \frac{\lambda_P}{2}$$

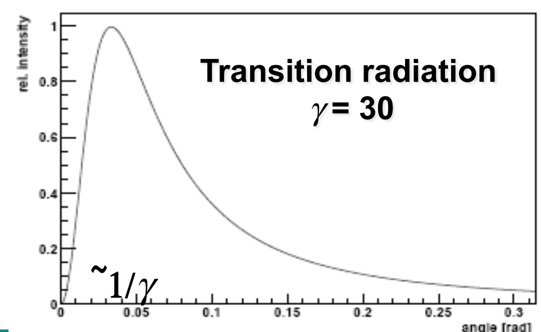
$$\Rightarrow \quad \omega_C = \gamma c / b_{\min} \sim 2\gamma c/\lambda_P = \gamma \omega_P / \pi$$

For a unity density material, $\omega_P \sim 3 \times 10^{16}$ s-1 and with a 1 GeV electron, the transition radiation spectrum extends to up $\sim \omega_C/2\pi \sim 3 \times 10^{18}$ Hz ($\lambda_C \sim 0.1$ nm - hard x-rays)!

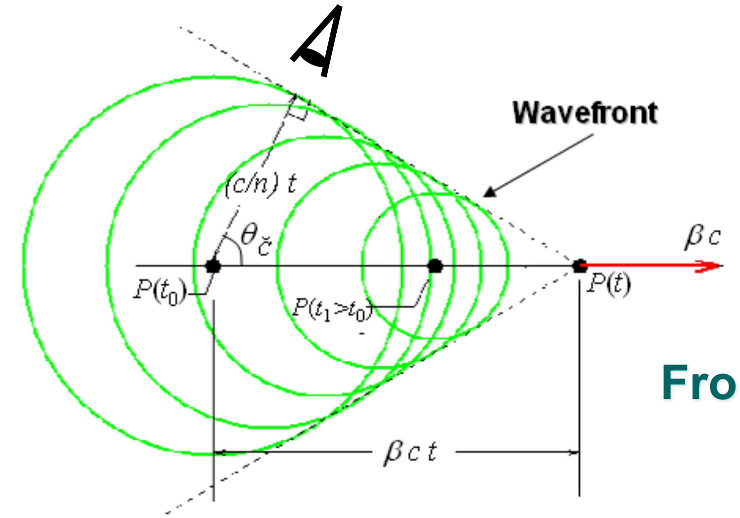
The intensity peaks at $\theta \sim 1/\gamma$ where $L_F \sim \lambda\gamma^2$ and the power spectrum becomes:

$$\frac{dP}{d\omega} = \hbar\omega \frac{c}{L_F} n(\omega) \sim \frac{e^2}{c}\frac{1}{\gamma^2}\omega$$

Low frequency power spectrum @ $\theta \sim 1/\gamma$



Cerenkov Radiation



For the emission of Cerenkov radiation:

$$\beta c >= \frac{c}{n(\omega)} = \frac{c}{\sqrt{\varepsilon_r(\omega)}}$$

From the figure:

$$\cos\theta_{\breve{C}} = \frac{(c/n)t}{\beta ct} = \frac{1}{\beta n}$$

$$\lambda\!\!\!^{-} = \beta c t_F - \frac{c}{n} t_F \cos\theta_{\breve{C}} = L_F\left(1 - \frac{1}{n\beta}\cos\theta_{\breve{C}}\right) = L_F\left(1 - \cos^2\theta_{\breve{C}}\right) = L_F \sin^2\theta_{\breve{C}}$$

$$\Rightarrow L_F = \frac{\lambda\!\!\!^{-}}{\sin^2\theta_{\breve{C}}}$$

As for the transition radiation case, in principle also for the Cerenkov $b_{min} \sim \lambda_P$. Nevertheless, the requirement $\beta c > c/n(w)$ imposes limitations to the bandwidth. Additionally, in order to extract the radiation from the media the latter must be transparent at that wavelength.

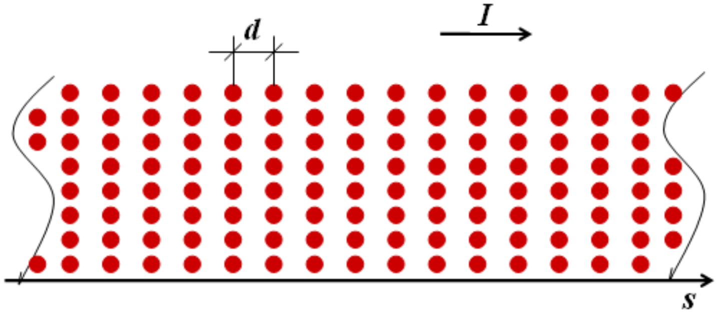
$$\frac{dP}{d\omega} = \hbar\omega \frac{c}{L_F} n(\omega) \sim \frac{e^2}{c}\omega \sin^2\theta_{\breve{C}} \sim \frac{e^2}{c}\omega\sin^2\theta_{\breve{C}}$$

Low frequency power spectrum

Radiation from a Beam of Charged Particles

We now want to investigate the case where many particles radiate together in a beam. We will show that for whatever radiation process (synchrotron radiation, Cerenkov radiation, transition radiation, etc.) **the incoherent component of the radiation is due to the random distribution of the particles along the beam.**

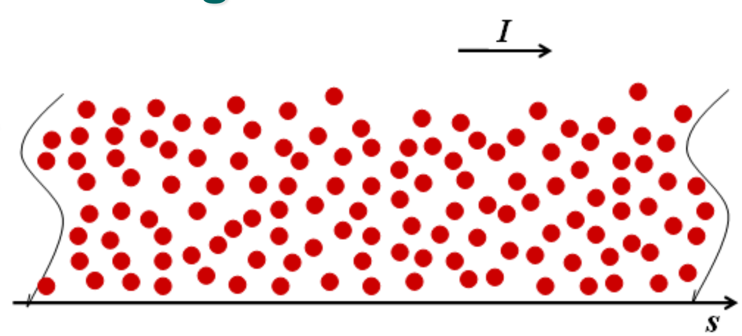
Example: "Ideal" coasting beam moving on a circular trajectory with the particles equally separated by a longitudinal distance d :



No synchrotron radiation emission for frequencies with $\lambda < \sim d$.

The interference between the radiation emitted by the evenly distributed electrons produces a vanishing net electric field.

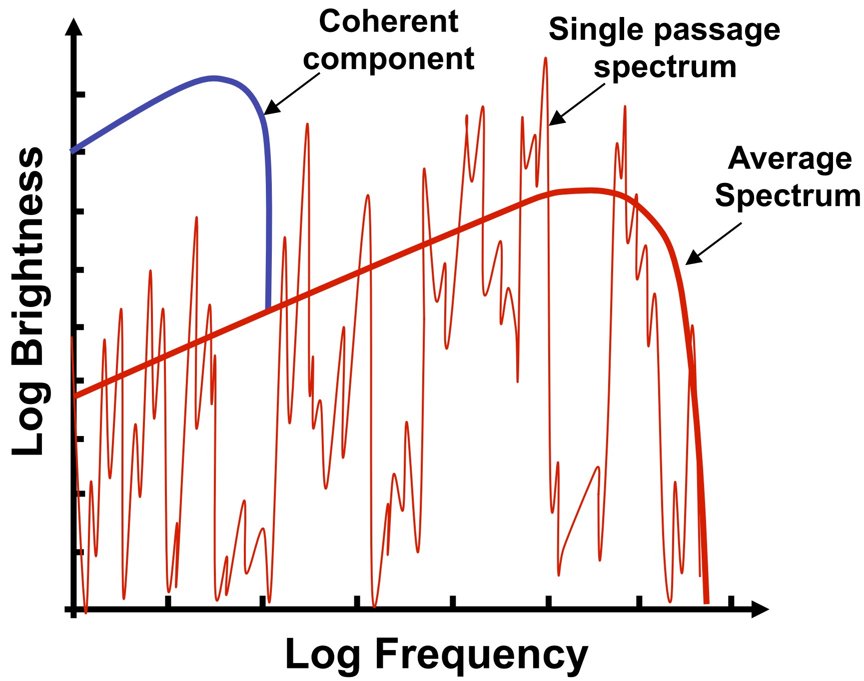
In a more realistic coasting beam, the particles are randomly distributed causing a small modulation of the beam current. The interference is not fully destructive anymore and the beam radiates also at longer wavelengths.



Radiation Fluctuations

If the particle turn by turn position along the beam changes (longitudinal dispersion, path length dependence on transverse position), the current modulation changes and the radiated energy and its spectrum <u>fluctuate</u> turn by turn.

By averaging over multiple passages, the measured spectrum converges to the characteristic incoherent spectrum of the radiation process under observation. (synchrotron radiation in the example).



In the case of bunched beams, a strong coherent component at those wavelengths comparable or longer than the bunch length shows up But the higher frequency part of the spectrum remains essentially unmodified.

More Quantitatively…

The electric field associated with the radiation emitted by the beam at the time t is:

$$E(t)=\sum_{k=1}^{N} e(t-t_k)$$

where e is the electric field of the electromagnetic pulse radiated by a single particle and t_k is the randomly distributed arrival time of the particle (Poisson process).

In the frequency domain:

$$\hat{E}(\omega)=\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt=\hat{e}(\omega)\sum_{k=1}^{N} e^{i\omega t_k}$$

And for the radiated power per passage:

$$P(\omega)\propto\left|\hat{E}(\omega)\right|^2=\left|\hat{e}(\omega)\right|^2\sum_{k=1}^{N}\sum_{l=1}^{N} e^{i\omega(t_k-t_l)}$$

The previous quantity fluctuates passage to passage, and the average radiated power from a beam with normalized distribution $f(t)$ is:

$$\langle P(\omega)\rangle\propto\left|\hat{e}(\omega)\right|^2\sum_{k,l=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt_k dt_l f(t_k)f(t_l)e^{i\omega(t_k-t_l)}=\left|\hat{e}(\omega)\right|^2\left[N+N(N-1)\left|\hat{f}(\omega)\right|^2\right]$$

Incoherent term: N; Coherent term: $N(N-1)\left|\hat{f}(\omega)\right|^2$

$$where\quad \hat{f}(\omega)=\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

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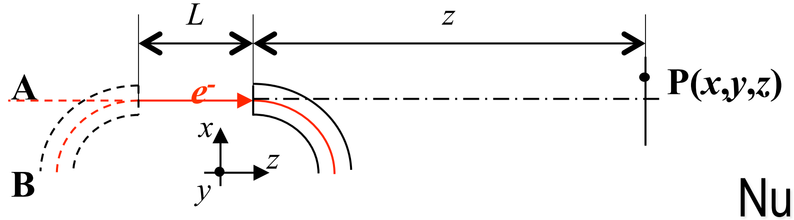
Edge Radiation (or TR) Approximate Analytical Formulae

$\lambda >> \lambda_c$

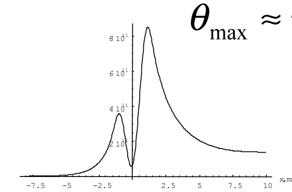


Numerical Illustrations

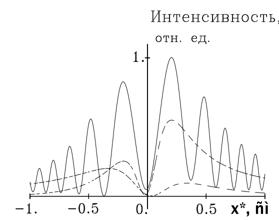
Far Field $z >> \lambda\gamma^2$

$\theta^2 \equiv (x^2+y^2)/z^2$

$\theta_{\max} \approx \gamma^{-1}$



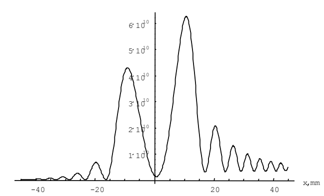
One BM Edge (A): $$\frac{dN}{dt\,dS\,(d\lambda/\lambda)} \approx \frac{\alpha I \gamma^2}{\pi^2 e\, z^2}\cdot\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^2}$$



Two BM Edges (B): $$\frac{dN}{dt\,dS\,(d\lambda/\lambda)} \approx \frac{4\alpha I \gamma^2}{\pi^2 e\, z^2}\cdot\frac{\gamma^2\theta^2 \sin^2[\pi L(1+\gamma^2\theta^2)/(2\lambda\gamma^2)]}{(1+\gamma^2\theta^2)^2}$$

Near Field $\lambda << z \leq \lambda\gamma^2$

$(x^2+y^2)^{1/2}{}_{\max} \approx (\lambda z)^{1/2}$

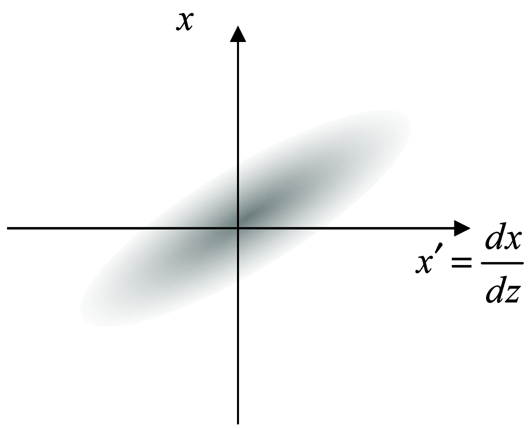


One BM Edge (A): $$\frac{dN}{dt\,dS\,(d\lambda/\lambda)} \approx \frac{4\alpha I}{\pi^2 e}\cdot\frac{\sin^2[\pi(x^2+y^2)/(2\lambda z)]}{x^2+y^2}$$

Two BM Edges (B): $$\frac{dN}{dt\,dS\,(d\lambda/\lambda)} \approx \frac{4\alpha I}{\pi^2 e}\cdot\frac{\sin^2[\pi(x^2+y^2)L/[2\lambda z(z+L)]]}{x^2+y^2}$$

By O. Chubar

The Concept of Emittance



Emittance: quantity proportional to the volume of the phase space occupied by the beam particles

$$\varepsilon_w = \frac{A_{ww'}}{\pi} \qquad w = x, y, z$$

The emittance is generally a 6D quantity but quite often, the planes can be decoupled and the 2D cases can be investigated independently.

Liouville Theorem: in a Hamiltonian system (non-dissipative system) the emittance is conserved

For most applications, smaller emittances are preferred.
It is very easy to increase this quantity, but very hard to preserve it!

Emittance and RMS Emittance

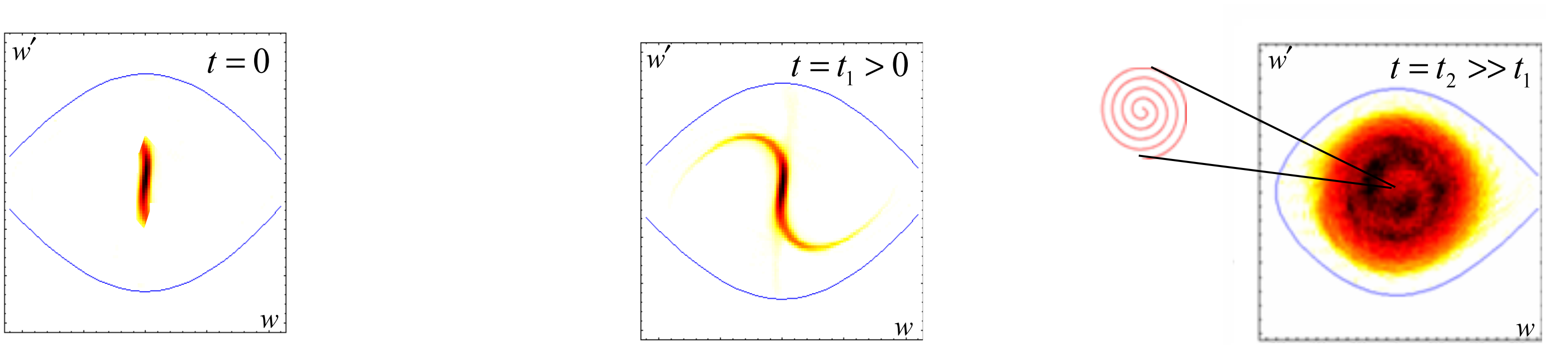
A statistical definition: the r.m.s. Emittance:

$$\varepsilon_{rms} = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle x\,x'\rangle^2}$$

• According to the Liouville theorem, in a Hamiltonian system the emittance is conserved. This is true even when the forces are nonlinear (space charge, nonlinear magnetic and/or electric fields, …)

• This is not true for the case of the rms emittance.
In the presence of nonlinear forces the rms emittance is not conserved

• Example: *filamentation*. Particles with different phase space coordinates, because of nonlinear forces can move with different phase space velocity



• The emittance according to Liouville is still conserved.
But the rms emittance increases with time.