Poynting vector

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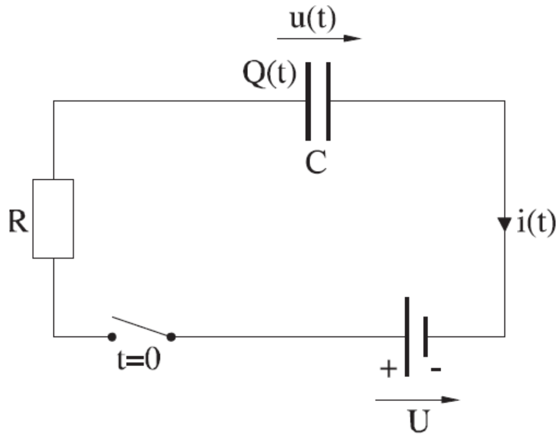
- Start with a simple example of the loading of a capacitor:



$$i(t) = \frac{d}{dt}(Q(t))$$

- Power:

$$p(t) = \frac{d}{dt}(W) = u(t) \cdot i(t) = u(t) \cdot \frac{d}{dt}(Q(t)) = u(t) \cdot \frac{d}{dt}((C \cdot u(t)))$$
$$Q(t) = C \cdot u(t)$$

- Energy accumulation:

$$W = \int_t \frac{d}{dt}(w)dt = C \cdot \int_t u(t)du(t) = \frac{1}{2} \cdot C \cdot U^2$$

- For an ideal plate capacitor it is: $C = \varepsilon \cdot \frac{A}{d}$

$$\begin{aligned} Q &= \oint \vec{D} d\vec{A} = D \cdot A \\ U &= \int \vec{E} d\vec{s} = E \cdot d \\ W &= \frac{1}{2} \cdot \varepsilon \cdot \frac{A}{d} \cdot E^2 \cdot d^2 = \frac{1}{2} \cdot \varepsilon \cdot E^2 \cdot \underbrace{A \cdot d}_{\text{Volume} V} \end{aligned}$$

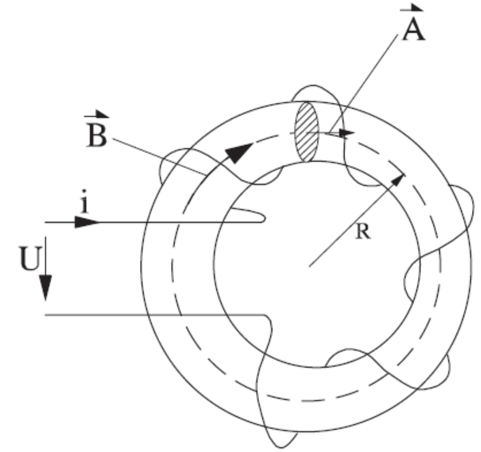
- Per volume the power density results

$$\frac{W_E}{V} = w_E = \frac{1}{2} \cdot D \cdot E = \frac{1}{2} \cdot \varepsilon \cdot E^2$$

- Voltage drop across the coils

$$U = n \cdot \frac{d}{dt}(\phi)$$

- Energy supplied at each instant:



$$\begin{aligned} dW_m &= U_i \cdot dt = n \cdot \frac{d}{dt}(\phi) \cdot i \cdot dt = n \cdot i \cdot d\phi \\ \phi &= \int \vec{B} \cdot d\vec{A} \\ d\phi &= \vec{B} \cdot d\vec{A} \end{aligned}$$

- Ampere's law:

$$\oint \vec{H} \cdot d\vec{s} = n \cdot i \quad \text{here :} \quad 2\pi \cdot R \cdot H = n \cdot i$$

$$dW_m = 2\pi \cdot \vec{A} \cdot H \cdot d\vec{B} = 2\pi \cdot A \cdot H \cdot dB, \text{ since } \vec{A} \text{ and } \vec{B} \text{ are in parallel.}$$

- The energy density (field energy per volume $V = 2\pi \cdot R \cdot A$ in the core is thus

$$\frac{dW_m}{V} = dw_m = H \cdot dB$$

- Energy density in a magnetic field of flux density $\vec{B}$

$$W_m = \int_{B=0}^{B=B_{max}} \vec{H} \cdot d\vec{B}$$

- The total energy is

$$W_m = \int_V \left[\int_{B=0}^{B=B_{max}} \vec{H} \cdot d\vec{B} \right] dV$$

- The energy density in magnetic fields with $\mu = const.$ and because of $\vec{H} = \frac{\vec{B}}{\mu} = const \cdot \vec{B}$, is $w_m = \frac{1}{2 \cdot \mu} \cdot B^2 = \frac{1}{2} \cdot \vec{H} \cdot \vec{B} = \frac{1}{2} \cdot \mu \cdot H^2$

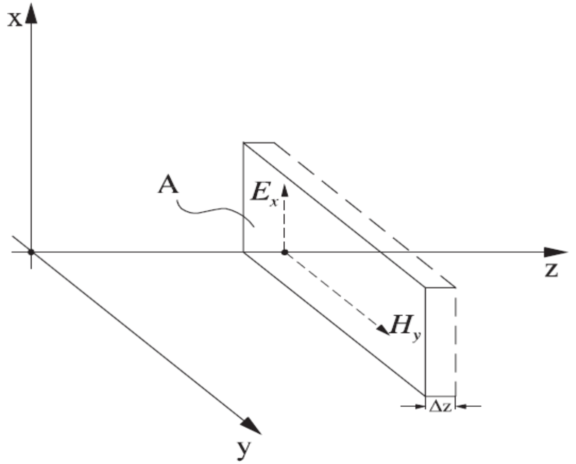
- If the inductance of an arrangement is known, then

$$
\begin{aligned}
U &= L \cdot \frac{d}{dt}(i) \\
dw_m &= U \cdot i \cdot dt = L \cdot i \cdot di \\
W &= \int L \cdot i \cdot di = \frac{1}{2} \cdot L \cdot i^2
\end{aligned}
$$

- The expressions for the energy density, which were given for static fields, are often applied to time varying fields as well.
- Energy density of EM fields:

$$
w = \frac{1}{2} \cdot \vec{E} \cdot \vec{D} + \frac{1}{2} \cdot \vec{H} \cdot \vec{B}
$$

- We are interested in the energy transmission by a plane wave: $E_x(z,t), H_y(z,t)$
- Consider a volume element with area A $\perp$ *z*-axis and with infinitely small thickness Δz. We will investigate the change of energy in this volume element.



- The energy $W(z,t)$ in the volume:

$$W(z,t) = A \cdot \Delta z \cdot \left(\frac{1}{2} \cdot \varepsilon \cdot E_x^2 + \frac{1}{2} \cdot \mu \cdot H_y^2 \right)$$

- Since we are interested in the change of energy, we differentiate wrt ∂t

$$\frac{\partial}{\partial t}W(z,t) = \frac{A\cdot\Delta z}{2}\cdot\left(\varepsilon\cdot\frac{\partial}{\partial t}(E_x^2)+\mu\cdot\frac{\partial}{\partial t}(H_y^2)\right)$$

$$\frac{\partial}{\partial t}E^2(z,t) = 2\cdot E(z,t)\cdot\frac{\partial}{\partial t}(E)$$

$$\frac{\partial}{\partial t}W(z,t) = A\cdot\Delta z\cdot\left(\varepsilon\cdot E_x\cdot\frac{\partial}{\partial t}(E_x)+\mu\cdot H_y\cdot\frac{\partial}{\partial t}(H_y)\right)$$

- Substitution

$$\frac{\partial}{\partial z}(H_y) = -\varepsilon\cdot\frac{\partial}{\partial t}(E_x)$$

$$\frac{\partial}{\partial z}(E_x) = -\mu\cdot\frac{\partial}{\partial t}(H_y)$$

$$\frac{\partial}{\partial t}(W) = -A\cdot\Delta z\cdot\left(E_x\cdot\frac{\partial}{\partial z}(H_y)+H_y\cdot\frac{\partial}{\partial z}(E_x)\right) = -A\cdot\Delta z\cdot\frac{\partial}{\partial t}((E_xH_y))$$

$$= -A\cdot\Delta z\cdot\left(\frac{(E_xH_y)_{z+\Delta z}-(E_xH_y)_z}{\Delta z}\right)$$

- The rate of the change of energy in the volume $A \cdot \Delta z$ is for a very small Δz

$$\begin{aligned}\frac{\partial}{\partial t}(W) &= A \cdot [E_x(z,t) \cdot H_y(z,t) - E_x(z+\Delta z,t) \cdot H_y(z+\Delta z,t)] \\ &= A \cdot [S_z(z,t) - S_z(z+\Delta z,t)]\end{aligned}$$

with $S_z(z,t) = E_x(z,t) \cdot H_y(z,t)$.

This can also be expressed as

$$S_z = \left(\vec{E} \times \vec{H}\right)_z$$

since

$$\left(\vec{E} \times \vec{H}\right)_z = \begin{bmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ E_x & 0 & 0 \\ 0 & H_y & 0 \end{bmatrix}$$

- For a plan wave E_x, H_y the energy flow is in the $+z$ -direction:

$$\vec{e}_z \cdot S_z = (E_x H_y) \cdot \vec{e}_z$$

- By similar derivations for the remaining directions we obtain the general expression:

$$\vec{S} = \vec{E} \times \vec{H} \left[\frac{\mathrm{Watt}}{\mathrm{m}^2}\right]$$