Resonators

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For a cubical resonator with *a* = *b* = *d*, we have

$$f_{101} = a^{-1}\sqrt{1/(2\mu\varepsilon)} \qquad \left(f_{101} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\frac{1}{a^2}+\frac{1}{d^2}} \right)$$

$$Q_{cube} = \frac{\pi\mu f_{101} a}{3R_s} = \frac{a\mu}{3\mu_m\delta} \qquad \left[\delta = (\pi f \mu_m \sigma)^{-1/2}\right]$$

Skin depth of the surrounding metallic walls, where μ_m is the permeability of the metallic walls.

We consider an air-filled cubical cavity designed to be resonant in TE_{101} mode at 10 GHz (free space wavelength λ=3cm)with silver-plated surfaces ($\sigma=6.14\times10^7$S-m^{-1}, $\mu_m=\mu_0$. Find the quality factor.

$$f_{101} = a^{-1}\sqrt{1/(2\mu\varepsilon)} \Rightarrow a = \frac{1}{f_{101}}\sqrt{\frac{1}{2\mu_\circ\varepsilon_\circ}} = \frac{c}{f_{101}\sqrt{2}} = \frac{\lambda}{2} \approx 2.12cm$$

At 10GHz, the skin depth for the silver is given by

$$\delta = \left(\pi\times10\times10^9\times4\pi\times10^{-7}\times6.14\times10^7\right)^{-1/2} \approx 0.642\mu m$$

and the quality factor is

$$Q = \frac{a}{3\delta} \cong \frac{2.12cm}{3\times0.642\mu m} \cong 11{,}000$$

Previous example showed that very large quality factors can be achieved with normal conducting metallic resonant cavities. The Q evaluated for a cubical cavity is in fact representative of cavities of other simple shapes. Slightly higher Q values may be possible in resonators with other simple shapes, such as an elongated cylinder or a sphere, but the Q values are generally on the order of magnitude of the volume-to-surface ratio divided by the skin depth.

$$Q = \omega_\circ \frac{\overline{W}_{str}}{P_{wall}} = \frac{\omega_\circ 2\overline{W}_m}{P_{wall}} = \frac{(2\pi f_\circ)\frac{\mu}{2}\int_V H^2 dv}{\frac{R_s}{2}\oint_s H_t^2 ds} \cong \frac{2}{\delta}\frac{V_{cavity}}{S_{cavity}}$$

Where S_{cavity} is the cavity surface enclosing the cavity volume V_{cavity}.

Although very large Q values are possible in cavity resonators, disturbances caused by the coupling system (loop or aperture coupling), surface irregularities, and other perturbations (e.g. dents on the walls) in practice act to increase losses and reduce Q.

Dielectric losses and radiation losses from small holes may be especially important in reducing Q. The resonant frequency of a cavity may also vary due to the presence of a coupling connection. It may also vary with changing temperature due to dimensional variations (as determined by the thermal expansion coefficient). In addition, for an air-filled cavity, if the cavity is not sealed, there are changes in the resonant frequency because of the varying dielectric constant of air with changing temperature and humidity.

Additional losses in a cavity occur due to the fact that at microwave frequencies for which resonant cavities are used most dielectrics have a complex dielectric constant $\varepsilon = \varepsilon' - j\varepsilon''$. A dielectric material with complex permittivity draws an effective current $J_{eff} = \omega_\circ \varepsilon'' E$, leading to losses that occur effectively due to $E \cdot J_{eff}^*$

The power dissipated in the dielectric filling is

$$P_{dielectric} = \frac{1}{2}\int_V E \cdot J_{eff}^* dv = \frac{1}{2}\int_V E \cdot \omega\varepsilon'' E^* dv$$

$$= \frac{\omega_\circ \varepsilon''}{2}\int_0^a \int_0^b \int_0^d \left|E_y\right|^2 dydxdz$$

Using the expression for E_y for the TE_{101} mode, we have

$$P_{dielectric} = \frac{\varepsilon''}{\varepsilon'} \omega_\circ \frac{\mu H_\circ^2 abd}{8} \left[\frac{a^2}{d^2} + 1 \right]$$

$$Q_d = \omega_\circ \frac{\overline{W}_{str}}{P_d} = \frac{\varepsilon'}{\varepsilon''}$$

$$\overline{W}_{str} = 2\overline{W}_m = \frac{\varepsilon'}{2} \int_V \left| E_y \right|^2 dv$$

$$and \quad P_{dielectric} = \frac{\omega_\circ \varepsilon''}{2} \int_V \left| E_y \right|^2 dv$$

The total quality factor due to dielectric losses is

$$\frac{1}{Q} = \frac{1}{Q_d} + \frac{1}{Q_c}$$

Teflon-filled cavity

We found that an air-filled cubical shape cavity resonating at 10 GHz has a Q_c of 11,000, for silver-plated walls. Now consider a Teflon-filled cavity, with $\varepsilon = \varepsilon_0(2.05 - j0.0006)$. Find the total quality factor Q of this cavity.

$$f_\circ = [f_{101}]_{a=d} \cong \frac{1}{2\sqrt{\mu\varepsilon'}}\sqrt{\frac{2}{a^2}} = \frac{c}{a\sqrt{2\mu_r\varepsilon_r}} \Rightarrow a = \frac{c}{\sqrt{2}f_\circ\sqrt{\varepsilon'_r}}$$

μ_r=1 for Teflon. This shows that the the cavity is $\sqrt{\varepsilon'_r}$ smaller, or a=b=d=1.48 cm. Thus we have

$$Q_c = \frac{a}{3\delta} \cong 7684$$

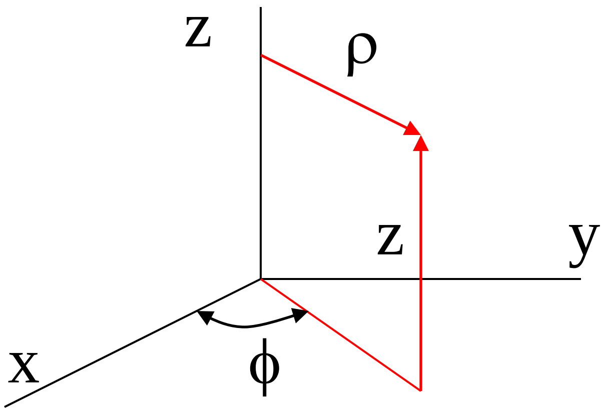
Or $\sqrt{\varepsilon'_r}$ times lower than that of the air-filled cavity. The quality factor Q_d due to the dielectric losses is given by

$$Q = \frac{Q_d Q_c}{Q_d + Q_c} \cong 2365$$

Thus, the presence of the Teflon dielectric substantially reduces the quality factor of the resonator.

The Helmholtz equation in cylindrical coordinates is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0$$



The method of separation of variables gives the solution of the form

$$\frac{1}{\rho R}\frac{d}{d\rho}\frac{\rho dR}{d\rho}+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{\partial\phi^2}+\frac{1}{Z}\frac{d^2Z}{\partial z^2}+k^2=0$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho\, dR}{d\rho} + \frac{1}{\Phi}\frac{d^2\Phi}{\partial\phi^2} + \left(k^2 - k_z^2\right)\rho^2 = 0$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -n^2$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho\, dR}{d\rho} - n^2 + \left(k^2 - k_z^2\right)\rho^2 = 0$$

Define k_{ρ} *to satisfy*

$$k_{\rho}^{2} + k_{z}^{2} = k^{2}$$

$$\frac{\rho}{R}\frac{d}{d\rho}\frac{\rho dR}{d\rho} + \left[(k_{\rho}\rho)^{2} - k_{z}^{2}\right]R = 0$$

$$\frac{d^{2}\psi}{\partial\phi^{2}} + n^{2}\Phi = 0$$

$$\frac{d^{2}Z}{\partial z^{2}} + K_{z}^{2}Z = 0$$

These are harmonic equations. Any solution to the harmonic equation we call harmonic functions and here is denoted by $h(n\phi)$ and $h(k_z z)$. Commonly used cylindrical harmonic functions are:

$$B_n(k_\rho \rho) \sim J_n(k_\rho \rho), N_n(k_\rho \rho), H_n^1(k_\rho \rho), H_n^2(k_\rho \rho)$$

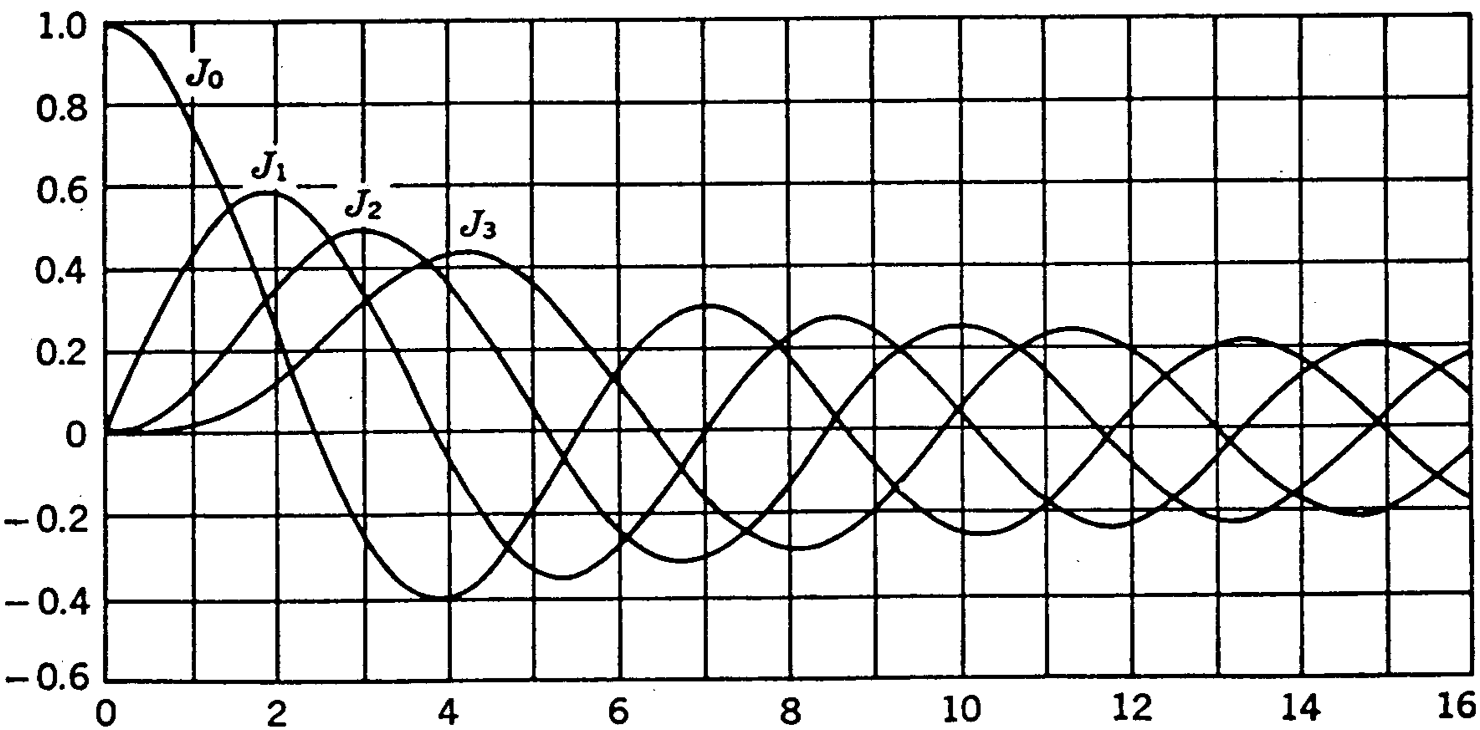
Where $J_n(k_\rho \rho)$ is the Bessel function of the first kind, $N_n(k_\rho \rho)$ Is the Bessel function of the second kind, $H_n^{(1)}(k_\rho \rho)$ is the Hankel function of the first kind, and $H_n^{(2)}(k_\rho \rho)$ is the Hankel function of the second kind.

- Any two of these are linearly independent.
- A constant times a harmonic function is still a harmonic function
- Sum of harmonic functions is still a harmonic function

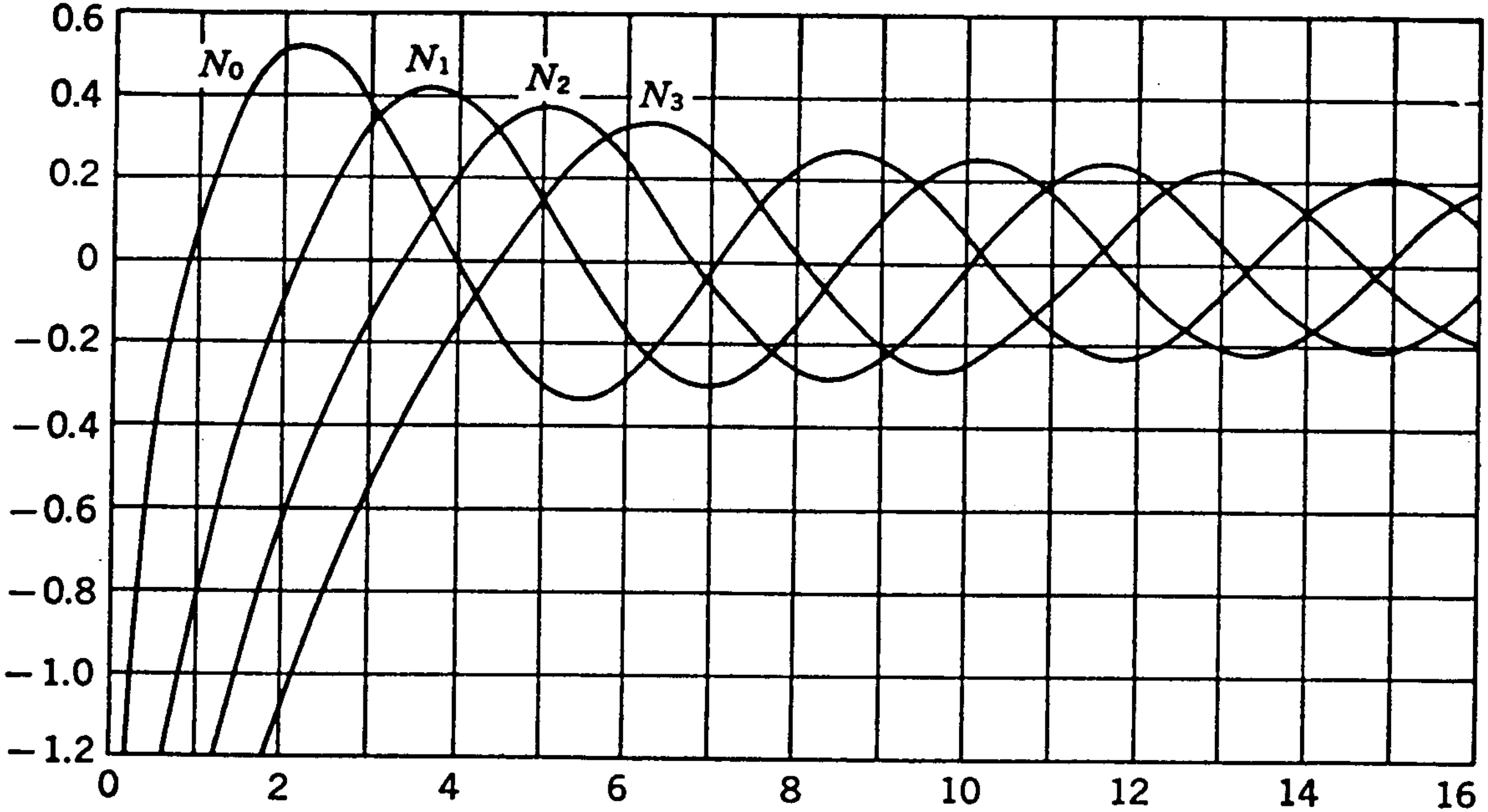
We can write the solution as :

$$\psi_{k_\rho, n, k_z} = B_n(k_\rho \rho) h(n\phi) h(k_z z)$$

Bessel functions of 1st kind



Bessel functions of 2nd kind



The $J_n(k_\rho \rho)$ are nonsingular at ρ=0. Therefore, if a field is finite at ρ=0, $B_n(k_\rho \rho)$ must be $J_n(k_\rho \rho)$ and the wave functions are

$$\psi_{k_\rho, n, k_z} = J_n(k_\rho \rho) e^{jn\phi} e^{jk_z z}$$

The $H_n^{(2)}(k_\rho \rho)$ are the only solutions which vanish for large ρ. They represent outward-traveling waves if k_ρ is real. Thus $B_n(k_\rho \rho)$ must be $H_n^{(2)}(k_\rho \rho)$ if there are no sources at $\rho \to \infty$. The wave functions are

$$\psi_{k_\rho, n, k_z} = H_n^{(2)}(k_\rho \rho) e^{jn\phi} e^{jk_z z}$$

$$J_n(k_\rho \rho) \quad \textit{analogous to} \quad \cos k\rho$$

$$N_n(k_\rho \rho) \quad \textit{analogous to} \quad \sin k\rho$$

$$H_n^{(1)}(k_\rho \rho) \quad \textit{analogous to} \quad e^{jk_\rho}$$

$$H_n^{(2)}(k_\rho \rho) \quad \textit{analogous to} \quad e^{-jk_\rho}$$

The $J_n(k_\rho \rho)$ and $N_n(k_\rho \rho)$ functions represent cylindrical standing waves for real k as do the sinusoidal functions. The

$H_n^{(1)}(k_\rho \rho)$and $H_n^{(2)}(k_\rho \rho)$ functions represent traveling waves for

real k as do the exponential functions. When k is imaginary (k = -jα)it is conventional to use the modified Bessel functions:

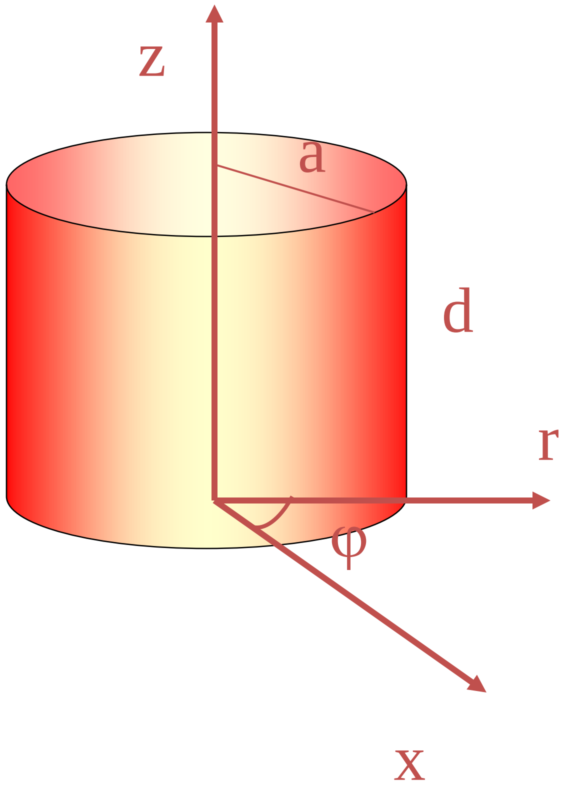
$$I_n(\alpha\rho) = j^n J_n(-j\alpha\rho)$$

$$\mathrm{K}_n(\alpha\rho) = \frac{\pi}{2}(-j)^{n+1} H_n^{(2)}(-\alpha\rho)$$

$I_n(\alpha\rho)$ *analogous to* $e^{\alpha\rho}$

$\mathrm{K}_n(\alpha\rho)$ *analogous to* $e^{-\alpha\rho}$

As in the case of rectangular cavities, a circular cavity resonator can be constructed by closing a section of a circular wave guide at both ends with conducting walls.



The resonator mode in an actual case depends on the way the cavity is excited and the application for which it is used. Here we consider TE_{011} mode, which has particularly high Q.

$$\psi_{mnq}^{TM} = J_n\left(\frac{x_{mn}\rho}{a}\right)\begin{cases}\sin m\phi \\ \cos m\phi\end{cases}\cos\left(\frac{q\pi z}{d}\right)$$

where $m = 0,1,2,3,\ldots;n = 1,2,3,\ldots;q = 0,1,2,3,\ldots$

$$\psi_{mnq}^{TE} = J_n\left(\frac{x'_{mn}\rho}{a}\right)\begin{cases}\sin m\phi \\ \cos m\phi\end{cases}\sin\left(\frac{q\pi z}{d}\right)$$

where $m = 0,1,2,3,\ldots;n = 1,2,3,\ldots;q = 1,2,3,\ldots$

The separation constant equation becomes

$$\left(\frac{x_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 = k^2$$

$$\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2 = k^2$$

For the TM and TE modes, respectively. Setting $k = 2\pi f\sqrt{\mu\varepsilon}$, we can solve for the resonant frequencies

$$\frac{f_{r_{mnq}}}{f_{r_{dominant}}}$$ for the circular cavity of radius a and length d

d/a	TM_{010}	TE_{111}	TM_{110}	TM_{011}	TE_{211}	TM_{111} TE_{011}	TE_{112}	TM_{210}	TM_{020}
0.00	1.00	∞	1.59	∞	∞	∞	∞	2.13	2.29
.50	1.00	2.72	1.59	2.80	2.90	3.06	5.27	2.13	2.29
1.00	1.00	1.50	1.59	1.63	1.80	2.05	2.72	2.13	2.29
2.00	1.00	1.00	1.59	1.19	1.42	1.72	1.50	2.13	2.29
3.00	1.13	1.00	1.80	1.24	1.52	1.87	1.32	2.41	2.60
4.00	1.20	1.00	1.91	1.27	1.57	1.96	1.20	2.56	3.00
∞	1.31	1.00	2.08	1.31	1.66	2.08	1.00	2.78	3.00

Ordered zeros X_{mn} of $J_n(X)$

m / n	0	1	2	3	4	5
1	2.405	3.832	5.136	6.380	7.588	8.771
2	5.520	7.016	8.417	9.761	11.065	12.339
3	8.654	10.173	11.620	13.015	14.372	
4	11.792	13.324	14.796			

Ordered zeros X'_{mn} $J'_n(X)$

m / n	0	1	2	3	4	5
1	3.832	1.841	3.054	4.201	5.317	6.416
2	7.016	5.331	6.706	8.015	9.282	10.520
3	10.173	8.536	9.969	11.346	12.682	13.987
4	13.324	11.706	13.170			

Cylindrical cavities are often used for microwave frequency meters. The cavity is constructed with movable top wall to allow mechanical tuning of the resonant frequency, and the cavity is loosely coupled to a wave guide with a small aperture.

The transverse electric fields (E_ρ, E_ϕ) of the TE_{mn} or TM_{mn} circular wave guide mode can be written as

$$\bar{E}_t(\rho,\phi,z) = \bar{\mathcal{E}}(\rho,\phi)\left[A^+ e^{-j\beta_{mn}z} + A^- e^{j\beta_{mn}z}\right]$$

The propagation constant of the TE_{nm} mode is

$$\beta_{mn} = \sqrt{\kappa^2 - \left(\frac{x'_{mn}}{a}\right)^2}$$

While the propagation constant of the TM_{nm} mode is

$$\beta_{mn} = \sqrt{\kappa^2 - \left(\frac{x_{mn}}{a}\right)^2}$$

Now in order to have $E_t = 0$ at z=0, d, we must have $A^+ = -A^-$, and $A^+ \sin \beta_{nm} d = 0$ *or*

$\beta_{mn} d = l\pi$, *for* $l = 0,1,2,3,\ldots$, which implies that the wave guide must be an integer number of half-guide wavelengths long. Thus, the resonant frequency of the TE_{mnl} mode is

$$f_{mnq} = \frac{c}{2\pi\sqrt{\mu_r \varepsilon_r}} \sqrt{\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2}$$

And for TM_{nml} mode is

$$f_{mnq} = \frac{c}{2\pi\sqrt{\mu_r \varepsilon_r}} \sqrt{\left(\frac{x_{nm}}{a}\right)^2 + \left(\frac{q\pi}{d}\right)^2}$$

Then the dominant TE mode is the TE_{111} mode, while the dominant TM mode is the TM_{110} mode. The fields of the TE_{nml} mode can be written as

$$H_z = H_\circ J_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\phi \sin\frac{q\pi z}{d}$$

$$H_\rho = \frac{\beta a H_\circ}{x'_{mn}} J'_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\varphi \cos\frac{q\pi z}{d}$$

$$H_\phi = \frac{-\beta a^2 m H_\circ}{(x'_{mn})^2\rho} J_n\left(\frac{x'_{mn}\rho}{a}\right)\sin m\varphi \cos\frac{q\pi z}{d}$$

$$E_\rho = \frac{j\kappa\eta a^2 m H_\circ}{(x'_{mn})^2\rho} J_n\left(\frac{x'_{mn}\rho}{a}\right)\sin m\varphi \sin\frac{q\pi z}{d}$$

$$E_\phi = \frac{j\kappa\eta a H_\circ}{x'_{mn}} J'_n\left(\frac{x'_{mn}\rho}{a}\right)\cos m\varphi \sin\frac{q\pi z}{d}$$

$$\boxed{\eta = \sqrt{\mu/\varepsilon} \ \ and \ \ H_\circ = -2jA^+}$$

$$E_z = 0$$

Since the time-average stored electric and magnetic energies are equal, the total stored energy is

$$W = 2W_e = \frac{\varepsilon}{2}\int_0^d \int_0^{2\pi} \int_0^a \left(\left|E_\rho\right|^2 + \left|E_\phi\right|^2\right)\rho\, d\rho\, d\phi\, dz$$

$$= \frac{\varepsilon\kappa^2\eta^2 a^2 \pi d H_\circ^2}{4\left(x'_{mn}\right)^2}\int_{\rho=0}^{a}\left[J_n'^2\left(\frac{x'_{mn}\rho}{a}\right) + \left(\frac{ma}{x'_{mn}}\right)^2 J_n^2\left(\frac{x'_{mn}\rho}{a}\right)\right]\rho\, d\rho$$

$$= \frac{\varepsilon\kappa^2\eta^2 a^4 \pi d H_\circ^2}{8\left(x'_{mn}\right)^2}\left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] J_n^2\left(x'_{mn}\right)$$

The power loss in the conducting walls is

$$P_c = \frac{R_s}{2}\int_S |H_t|^2 ds = \frac{R_s}{2}\left\{\int_{z=0}^{d}\int_{\phi=0}^{2\pi}\left[|H_\phi(\rho=a)|^2 + |H_z(\rho=a)|^2\right]a\,d\phi\,dz\right.$$

$$\left. + 2\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\left[|H_\rho(z=0)|^2 + |H_\phi(z=0)|^2\right]\rho\,d\rho\,d\phi\right\}$$

$$= \frac{R_s}{2}\pi H_\circ^2 J_n^2(x'_{mn})\left\{\frac{da}{2}\left[1+\left(\frac{\beta a m}{(x'_{mn})^2}\right)^2\right] + \left(\frac{\beta a^2}{x'_{nm}}\right)^2\left(1-\frac{m^2}{(x'_{mn})^2}\right)\right\}$$

$$Q_c = \frac{\omega W}{P_c} = \frac{(\kappa a)^3 \eta a d}{4(x'_{mn})^2 R_s}\frac{1-\left(\frac{m}{x'_{mn}}\right)^2}{\left\{\frac{ad}{2}\left[1+\left(\frac{\beta a m}{(x'_{mn})^2}\right)^2\right] + \left(\frac{\beta a^2}{x'_{mn}}\right)^2\left(1-\frac{m^2}{(x'_{mn})^2}\right)\right\}}$$

To compute the Q due to dielectric loss, we must compute the power dissipated in the dielectric. Thus,

$$P_d = \frac{1}{2}\int_V J\cdot E^* dv = \frac{\omega\varepsilon''}{2}\int_V \left[\left|E_\rho\right|^2 + \left|E_\phi\right|^2\right] dv$$

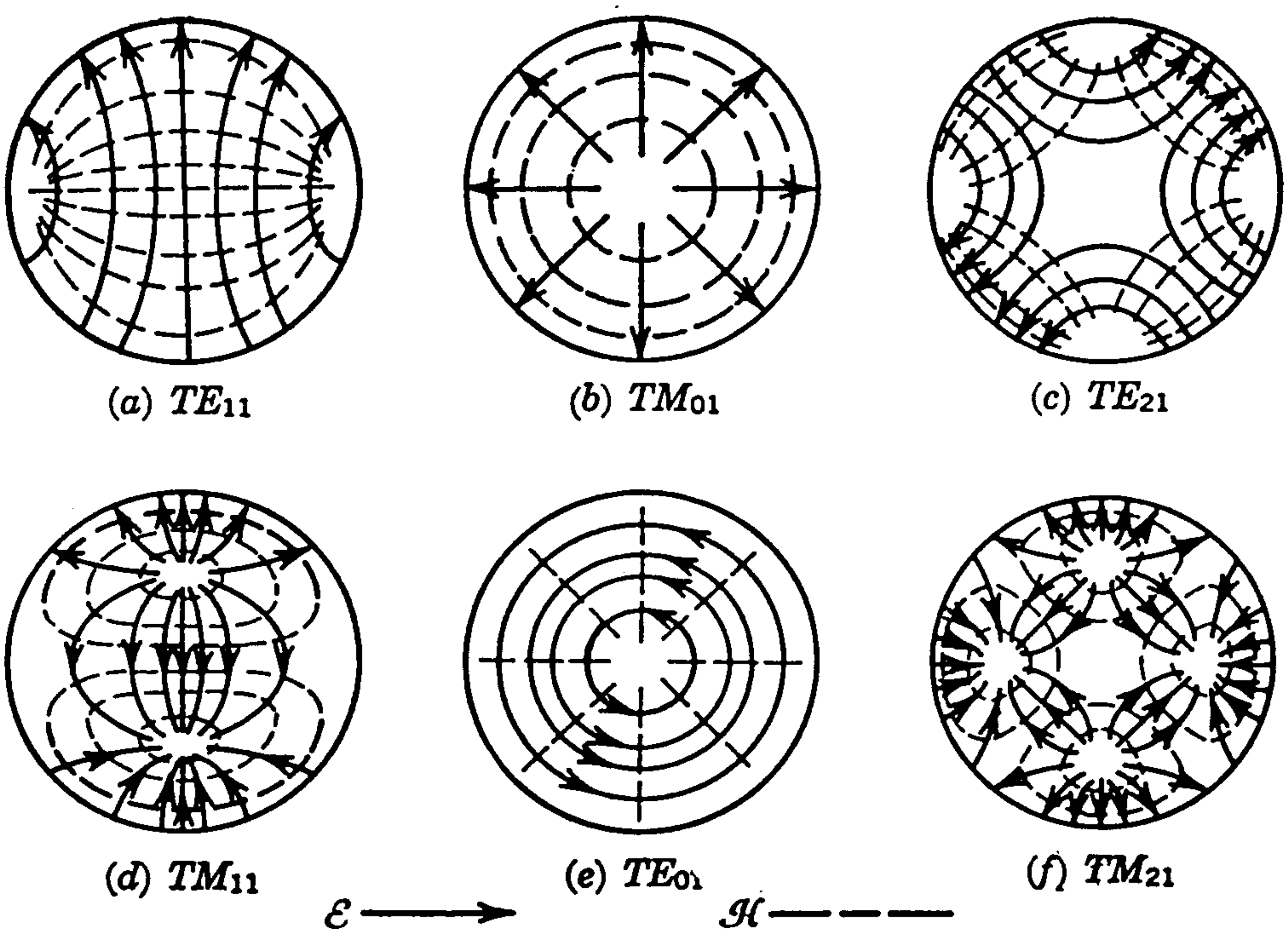
$$= \frac{\omega\varepsilon''\kappa^2\eta^2 a^2 H_\circ^2 \pi d}{4(x'_{mn})^2}\int_{\rho=0}^{a}\left[\left(\frac{ma}{x'_{mn}\rho}\right)^2 J_n^2\left(\frac{x'_{mn}\rho}{a}\right) + J_n'^2\left(\frac{x'_{mn}\rho}{a}\right)\right]\rho\, d\rho$$

$$= \frac{\omega\varepsilon''\kappa^2\eta^2 a^4 H_\circ^2}{8(x'_{mn})^2}\left[1-\left(\frac{m}{x'_{mn}}\right)^2\right] J_n^2(x'_{mn})$$

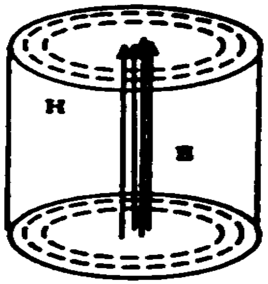
$$Q_d = \frac{\omega W}{P_d} = \frac{\varepsilon}{\varepsilon''} = \frac{1}{\tan\delta}$$

Where $\tan\delta$ is the loss tangent of the dielectric. This is the same as the result of Q_d for the rectangular cavity.

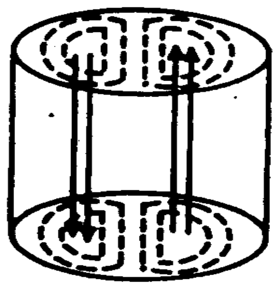
Cavity wave guide mode patterns

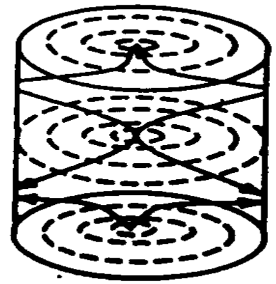


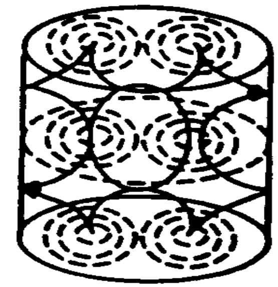
Cylindrical Cavity mode patterns

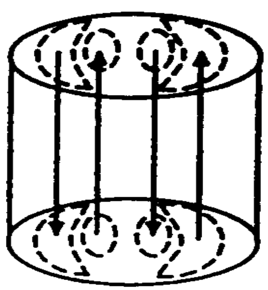


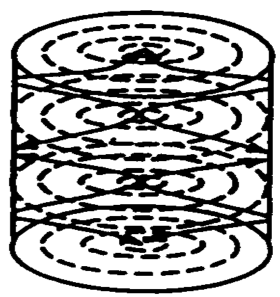
(a) TM 010 mode

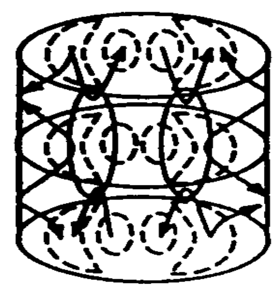

(b) TM110 mode


(c) TM012 mode


(d) TM112 mode


(e) TM120 mode


(f) TM013 mode


(g) TM122 mode

Circular TE_{011} mode

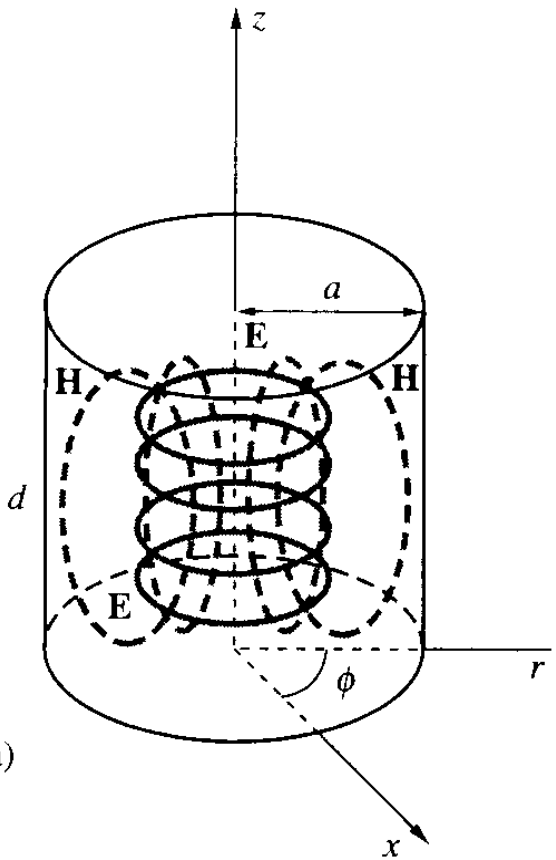
$$H_z = H_\circ J_\circ\left(\frac{3.832\rho}{a}\right) sin\left(\frac{\pi z}{d}\right)$$

$$H_\rho = \frac{\pi a H_\circ}{3.832 d} J_1\left(\frac{3.832\rho}{a}\right) cos\left(\frac{\pi z}{d}\right)$$

$$E_\phi = \frac{-j\omega\mu a H_\circ}{3.832 d} J_1\left(\frac{3.832\rho}{a}\right) sin\left(\frac{\pi z}{d}\right)$$

$$NOTE: \ J_1(h\rho) = -J_\circ'(h\rho)$$

Circular TE_{011} mode



(a)

- The electric field lines form closed circular loops centered around the cylinder axis.
- The electric field lines are threaded with closed loops of magnetic filed lines in the radial planes.
- No surface charges appear on any of the cavity walls, since the normal electric field is zero everywhere on the walls
- However surface currents $\bar{J}_s = \hat{n} \times \bar{H}$ do flow in the walls due to tangential magnetic fields.

Circular TE_{011} mode

- On the curved surface of the cylinder we have $J_{s\varphi}$ due to H_z given by

$$J_{s_\varphi} = H_\circ J_\circ(3.832) sin\left(\frac{\pi z}{d}\right) \approx -0.403 H_\circ \, sin\left(\frac{\pi z}{d}\right) \quad at \;\; r = a$$

- On the flat end surfaces we have $J_{s\varphi}$ due to H_r given by

$$J_{s_\phi} = \pm \frac{\pi a H_\circ}{3.832 d} J_1\left(\frac{3.832 r}{a}\right) \qquad at \;\; z = 0 \quad and \quad z = a$$

- It is interesting to note that the surface currents are entirely circumferential. No surface current flows between the flat walls and the curved walls.

- Hence, if one end of the cavity is mounted on micrometers and moved to change the length of the cavity, the TE_{011} can still be fully supported, since the current flow is not interrupted.

Circular TE_{011} mode

- Movable construction of the end faces also suppress other modes, particularly TM_{111}, which has the same resonant frequency but lower Q.

- The currents that are required to support TE_{111} are interrupted by the space between the movable ends and the side walls.

- As in the case of of the rectangular cavity resonator, the resonant frequency of the TE_{011}mode can be found by substituting the expressions for any one of the field components into the wave equation.

$$\omega_{011} = \frac{1}{\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{\pi}{d}\right)^2 + \left(\frac{x_{01}}{a}\right)^2}$$

- The resonant free-space wavelength of the cavity corresponding to the resonant frequency is given by

$$\lambda_{011} = \frac{c}{f_{011}} = \frac{2\sqrt{\mu_r \varepsilon_r}}{\sqrt{\left(\frac{1}{d}\right)^2 + \left(\frac{x_{01}}{\pi a}\right)^2}}$$

- For the most general case of Te_{mnp} mode, the resonant free-space wavelength λ_{mnp} is given by

$$\lambda_{mnp} = \frac{c}{f_{mnp}} = \frac{2\sqrt{\mu_r \varepsilon_r}}{\sqrt{\left(\frac{p}{d}\right)^2 + \left(\frac{x_{mn}}{\pi a}\right)^2}}$$

Circular TE_{011} mode

- Using the expression derived for the Q of the circular Te_{mnp} and using x_{01}=3.832, we have

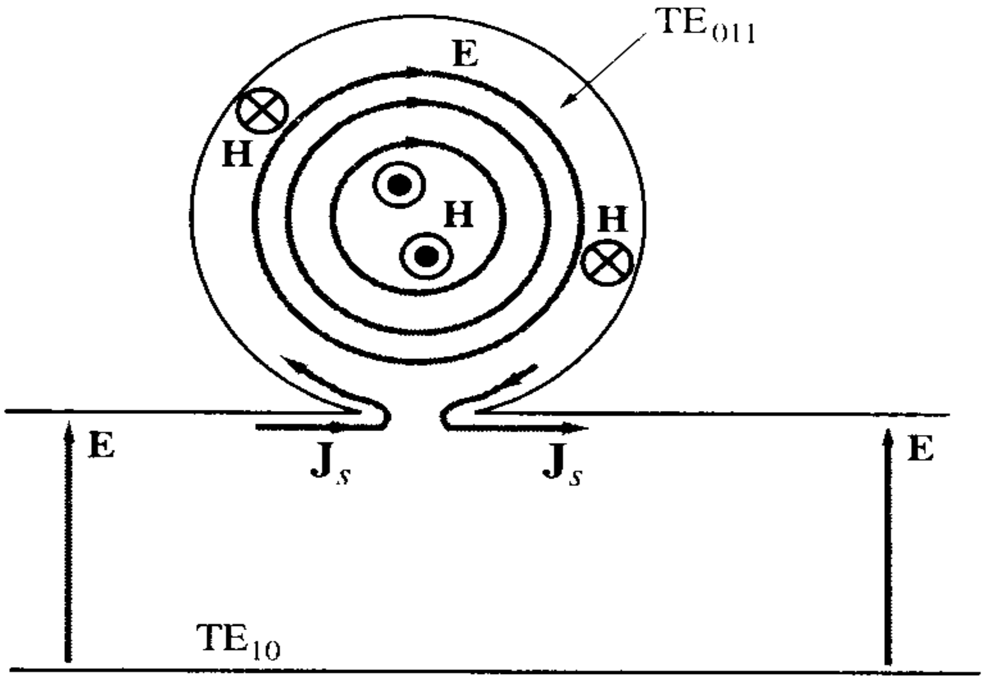
$$Q \approx \frac{0.61\lambda}{\delta}\sqrt{1+0.168(2a/d)^2}\left[\frac{1+0.168(2a/d)^2}{1+0.168(2a/d)^3}\right]$$

- X-band circular cavity: d=2a such that its TE_{011} resonates at 10 GHz. What is Q?

$$\lambda_{011} = \frac{c}{f_{011}} \approx \frac{3\times 10^8\, m-s^{-1}}{10\times 10^9\, Hz} = \frac{2}{\sqrt{\left(\frac{1}{2a}\right)^2+\left(\frac{3.832}{\pi a}\right)^2}} \rightarrow a = 1.98\ cm$$

$$\delta = \frac{1}{\sqrt{\pi f \mu_\circ \sigma}} \approx 6.42\times 10^{-7}\, m$$

$$Q = \frac{0.61\lambda_\circ}{\delta}\sqrt{1+0.168} \approx 30792!$$



Excitation of the TE$_{011}$ mode in a circular cavity via coupling from a TE$_{01}$mode in a rectangular wave guide.

Loop or Probe Coupling

For a probe coupler the electric flux arriving on the probe tip furnishes the current induced by a cavity mode:

$$I = \omega\varepsilon SE$$

where E is the electric field from a mode averaged over probe tip and S is the antenna area. The external Q of this simple coupler terminated on a resistive load R for a mode with stored energy W is

$$Q_{ext} = \frac{2W}{R\omega\varepsilon^2 S^2 E^2}$$

In the same way for a loop coupler the magnetic flux going through the loop furnishes the voltage induced in the loop by a cavity mode:

$$V = \omega\mu SH$$

Problem 1

A WR-1500 rectangular air wave guide has inner dimensions 38.1cm ×19.05 cm. Find (a) the cutoff wavelength for the dominant mode; (b) the phase velocity, guide wavelength and wave impedance for the dominant mode at a wavelength of 0.8 times the cutoff wavelength;(c) the modes that will propagate in the wave guide at a wavelength of 30 cm.

$$\lambda_{c_{mn}} = \frac{v_p}{f_{c_{mn}}} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} = \frac{2ab}{\sqrt{m^2 b^2 + n^2 a^2}}$$

So the cutoff wavelength of the dominant TE_{10} mode is

$$\lambda_{c_{10}} = 2a = 76.2cm$$

The phase velocity, guide wavelength and the wave impedance can be calculates as

$$v_{p_{mn}} = \frac{\omega}{\beta_{mn}} = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$v_p \cong \frac{3\times10^8}{\sqrt{1-(.8)^2}} \cong 5\times10^8\, m-s^{-1}$$

$$\lambda_{mn} = \frac{2\pi}{\beta_{mn}} = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2}} = \frac{\lambda}{\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\lambda_{10} \cong \frac{0.8\times76.2}{\sqrt{1-(0.8)^2}} \approx 1.02 \quad m$$

$$Z_{TE_{10}} = \frac{\eta}{\sqrt{1-\left(f_{c_{10}}/f\right)^2}} = \frac{2a\mu f}{\sqrt{4a^2\mu\varepsilon f^2 - 1}}$$

$$Z_{TE_{10}} \approx \frac{377}{\sqrt{1-(0.8)^2}} \approx 628.3\Omega$$

(c) At λ_{air}=40 cm, only the dominate mode TE_{10} mode propagates since the next higher mode TE_{20} (or TE_{11}) has $\lambda_{c20} = \lambda_{c10}/2$ = 38.1 cm < 40 cm.

(d) At λ_{air}=30 cm, the propagating modes are TE_{10}, TE_{20}, and TM_{20} (λ_{c20}=38.1 cm), TE_{01}(λ_{c01}=38.1 cm), and TE_{11} and TM_{11}

$$\lambda_{c_{11}} = \frac{2ab}{\sqrt{a^2+b^2}} \approx 34.1cm$$

Problem 2

Design a rectangular cavity resonator that will resonant in the TE_{101} mode at 10GHz and resonant in the TM_{110} mode at 20 GHz.

Assuming *a=2b*. The resonant frequencies of the TE_{101} and TM_{110} modes are given by

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}+\frac{p^2\pi^2}{d^2}}$$

$$f_{101} \approx \frac{3\times10^{10}}{2}\sqrt{\frac{1}{a^2}+\frac{1}{d^2}} \approx 10GHz$$

$$f_{110} \approx \frac{3\times10^{10}}{2}\sqrt{\frac{1}{a^2}+\frac{4}{a^2}} \approx 20GHz \rightarrow a \approx 1.677cm$$

$b = a/2 \approx 0.839cm$ from f_{101} we find $d \approx 3.354cm$

Dielectric Wave guide

- We have shown that it is possible to propagate electromagnetic wave s down a hollow conductor. However, other types of guiding structures are also possible.

- The general requirement for a guide of electromagnetic waves is that there be a flow of energy along the axis of the guiding structure but not perpendicular to it.

- This implies that the electromagnetic fields are appreciable only in the immediate neighborhood of the guiding structure.

- Consider an axisymmetric tube of arbitrary cross section made of some dielectric material and surrounded by a vacuum. This structure can serve as a wave guide provided that dielectric constant of the material is sufficiently large.

- The boundary conditions satisfied by the electromagnetic fields are significantly different to those of a conventional wave guide. The transverse fields are governed by two equations; one for the region inside the dielectric, and the other for the vacuum regions.

- Inside the dielectric we have

$$\left[\nabla_s^2 + \left(\varepsilon_1 \frac{\omega^2}{c^2} - k_g^2\right)\right]\psi = 0$$

- In the vacuum region we have

$$\left[\nabla_s^2 + \left(\frac{\omega^2}{c^2} - k_g^2\right)\right]\psi = 0$$

- Here, $\psi(x,y)e^{ik_g z}$ stands for either E_z or H_z, ε_1 is the relative permittivity of the dielectric material, and k_g is the guide propagation constant.
- The guide propagation constant must be the same both inside and outside dielectric in order to satisfy the electromagnetic boundary conditions at all points on the surface of the tube.
- Inside the dielectric the transverse Laplacian must be negative, so that the constant

$$k_s^2 = \varepsilon_1 \frac{\omega^2}{c^2} - k_g^2$$

is positive. Outside the cylinder the requirement of no transverse flow of energy can only be satisfied if the fields fall off exponentially (instead of oscillating).

- Thus $$k_t^2 = k_g^2 - \frac{\omega^2}{c^2}$$

- The oscillatory solutions (inside) must be matched to the exponentially solutions (outside). The boundary conditions are the continuity of normal B and D and tangential E and H on the surface of the tube.

- The boundary conditions are far more complicated than those in a conventional wave guide. For this reason, the normal modes cannot usually be classified as either pure TE or TM modes.

- In general, the normal modes possess both electric and magnetic field components in the transverse plane. However, for the special case of a cylindrical tube of dielectric material the normal modes can have either pure TE or pure TM characteristics.

- Consider a dielectric cylinder of radius a and dielectric constant ε_1. For the sake of simplicity, let us only search for normal modes whose electromagnetic fields have no azimuthal variation. We can write

$$\left(r^2\frac{d^2}{dr^2}+r\frac{d}{dr}+r^2k_s^2\right)\psi=0 \quad (for\ r<a)$$

$$\left[\nabla_s^2+\left(\varepsilon_1\frac{\omega^2}{c^2}-k_g^2\right)\right]\psi=0$$

$$k_s^2=\varepsilon_1\frac{\omega^2}{c^2}-k_g^2$$

- The general solution to this equation is some linear combination of the Bessel functions $J_0(k_s r)$ and $Y_0(k_s r)$. However, since $Y_0(k_s r)$ is "badly" behaved at the origin(r=0) the physical solution is $\psi \propto J_\circ(k_s r)$

- We can write

$$\left(r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} - r^2 k_t^2 \right)\psi = 0$$

- This can be rewritten

$$\left(z^2 \frac{d^2}{dz^2} + z\frac{d}{dz} - z^2 \right)\psi = 0 \quad where\ z = k_t r$$

- This type of *modified Bessel's equation*, whose most general form is

$$\left(z^2 \frac{d^2}{dz^2} + z\frac{d}{dz} - \left(z^2 + m^2\right) \right)\psi = 0$$

- The two linearly independent solutions are denoted $I_m(z)$ and $K_m(z)$. The asymptotic behavior of these solutions at small $|z|$ is as follows:

$$I_m(z)=\left(\frac{z}{2}\right)^m \sum_{k=0}^{\infty}\frac{\left(z^2/4\right)^k}{k!\,(k+m)!}$$

$$K_m(z)=\frac{1}{2}\left(\frac{z}{2}\right)^{-m}\sum_{k=0}^{\infty}\frac{(m-k-1)!}{k!}\left(-z^2/4\right)^k+(-1)^{m+1}\ln(z/2)I_m(z)$$

$$+(-1)^m\frac{1}{2}\left(\frac{z}{2}\right)^m\sum_{k=0}^{\infty}\left[\psi(k+1)+\psi(m+k+1)\right]\frac{\left(z^2/4\right)^m}{k!\,(m+k)!}$$

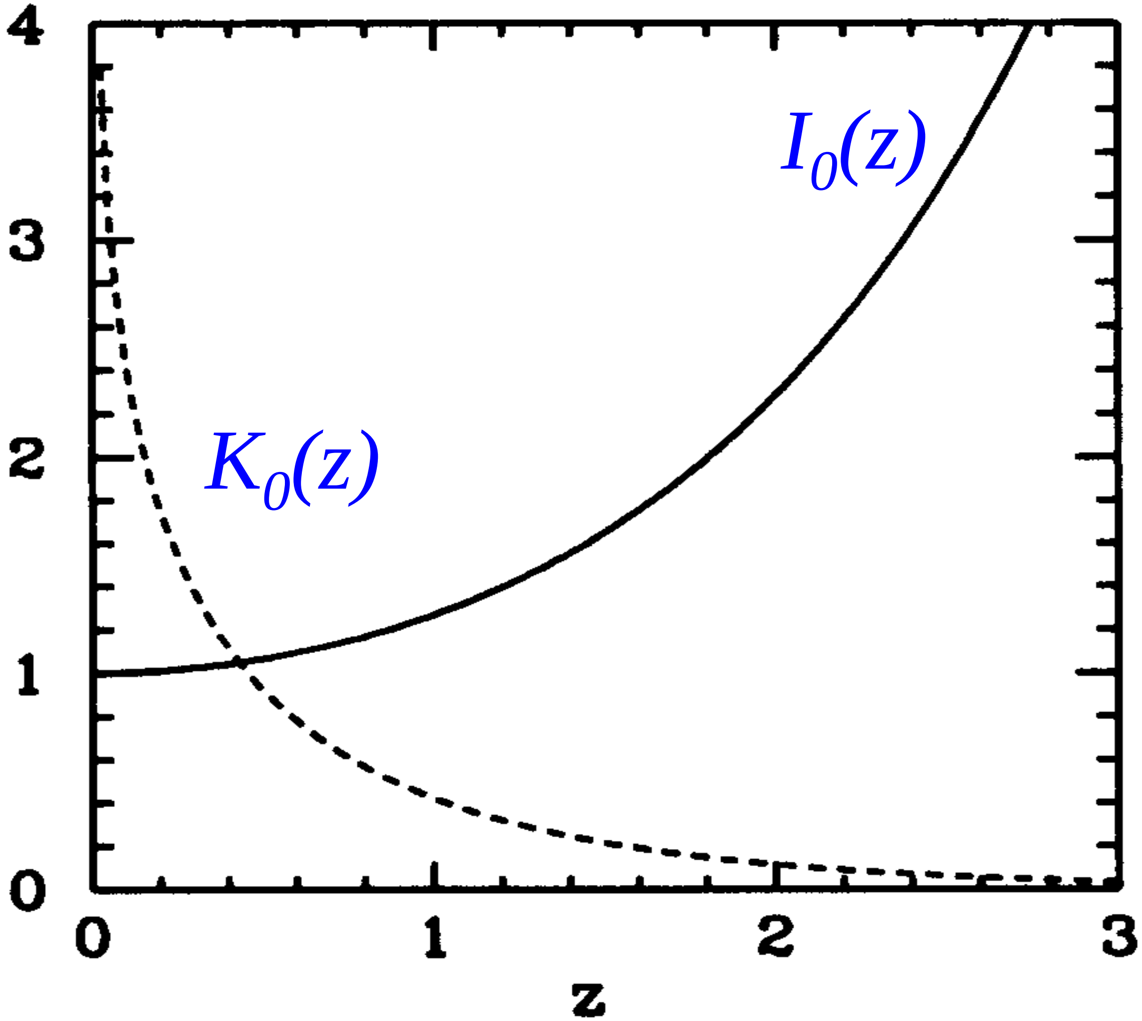
- Hence $I_m(z)$ is well behaved in the limit $|z| \to 0,$ whereas $K_m(z)$ is badly behaved. The asymptotic behavior at large $|z|$ is

$$I_m(z) \cong \frac{e^z}{\sqrt{2\pi z}}\left[1 + o\left(\frac{1}{z}\right)\right],$$

$$K_m(z) \cong \sqrt{\frac{\pi}{2z}} e^{-z}\left[1 + o\left(\frac{1}{z}\right)\right].$$

- Hence, $I_m(z)$ is badly behaved in the limit $|z| \to \infty,$ whereas $K_m(z)$ is well behaved.

- The behavior of $I_0(z)$ and $K_0(z)$



- Is it clear that the physical solution (I.e., the one which

Decays as $|r| \to \infty$ is $\psi \propto K_\circ(k_t r)$.

- The physical solution is

$$\psi = J_\circ(k_t r)$$

$$\textit{for } r \le a, \textit{ and}$$

$$\psi = A\, K_\circ(k_t r)$$

$$\textit{for } r > a.$$

- A is an arbitrary constant, and $\psi(r)e^{ik_g z}$ stands for either E_z or H_z.

- We can now write

$$H_r = i\frac{k_g}{k_s^2}\frac{\partial H_z}{\partial r}, \quad E_\theta = -\frac{\omega\mu_\circ}{k_g}\frac{\partial E_z}{\partial r},$$

$$E_r = \frac{k_g}{\omega\varepsilon_\circ\varepsilon_r}H_\theta \qquad for\ r \le a$$

- There are analogous set of relationships for $for\ r \ge a$. The fact that the field components form two groups;(H_r,E_θ), which depends on H_z and (H_θ,E_r), which depend on E_z; means that the normal modes takes the form of either pure TE modes or pure TM modes.

- For a TM mode (E_z=0) we find that

$$H_z = J_\circ(k_s r)$$

$$H_r = -i\frac{k_g}{k_s} J_1(k_s r)$$

$$E_\theta = i\frac{\omega\mu_\circ}{k_s} J_1(k_s r)$$

$$for\ r \le a, and$$

$$H_z = AK_\circ(k_t r)$$

$$H_r = iA\frac{k_g}{k_t} K_1(k_t r)$$

$$E_\theta = -iA\frac{\omega\mu_\circ}{k_t} K_1(k_t r)$$

$$for\ r \ge a$$

- Here we have used

$$J_\circ'(z) = -J_1(z),$$
$$K'(z) = -K_1(z).$$

- The boundary conditions require H_z,H_r, and E_θ to be continuous across r = a. Thus, it follows that

$$AK_\circ(k_t a) = J_\circ(k_s a),$$

$$-A\frac{K_1(k_t a)}{k_t} = \frac{J_1(k_s a)}{k_s}.$$

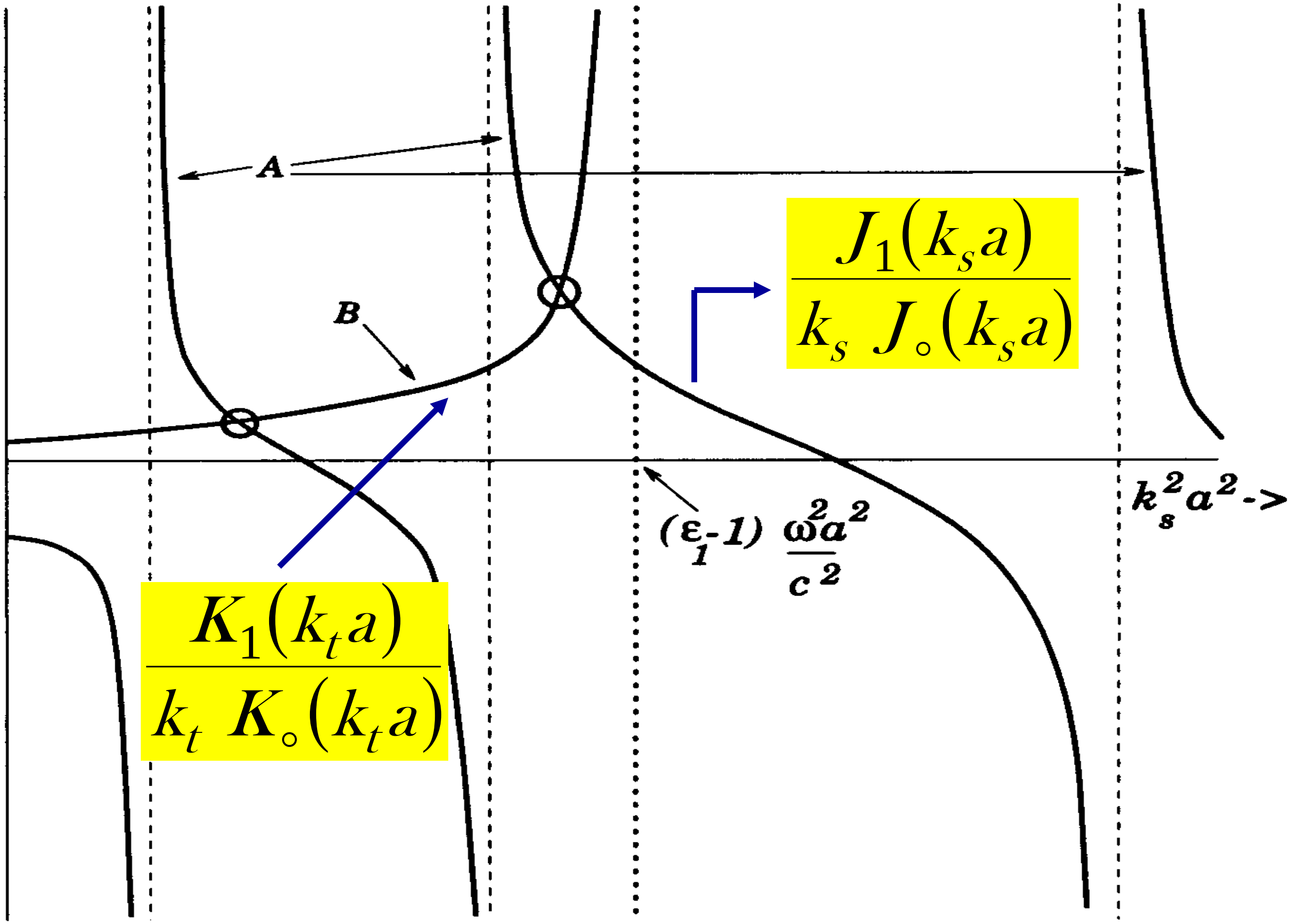
- Eliminating the arbitrary constant A, will yield

$$\frac{J_1(k_s a)}{k_s\ J_\circ(k_s a)} + \frac{K_1(k_t a)}{k_t\ K_\circ(k_t a)} = 0$$

where

$$k_t^2 + k_s^2 = (\varepsilon_1 - 1)\frac{\omega^2}{c^2}.$$

Graphical solution of the dispersion relation



- Since the first root of $J_\circ(z)$ occurs at z=2.4048 the condition condition for the existence of propagating modes can be written

$$\omega > \omega_{01} = \frac{2.4048c}{\sqrt{\varepsilon_1 - 1}a}$$

- In other words, the mode frequency must lie above the cutoff frequency ω_{01} for the TE_{01} mode (here, the 0 corresponds to the number of nodes in the azimuthal direction, and 1 refers to the 1^{st} root of $J_0(z)=0$.

- The cutoff frequency for the TE_{0p} mode is given by

$$\omega_{0p} = \frac{j_{0p}c}{\sqrt{\varepsilon_1 - 1}a}$$

- At the cutoff frequency for a particular k_t=0, which implies that

$$k_g = \frac{\omega}{c} \qquad \left[k_t^2 = k_g^2 - \frac{\omega^2}{c^2} \right]$$

- The mode propagates along the guide at the velocity of light in vacuum. Immediately below this cutoff frequency the system no longer acts as a guide but as an antenna, with energy being radiated radially. For the frequencies well above the cutoff, k_t and k_g are of the same order of magnitude, and are large compared to k_s. This implies that the fields do not extend appreciably outside the dielectric cylinder.

- For the TM mode ($H_z=0$) we find that

$$E_z = AJ_\circ(k_s r)$$

$$H_\theta = -i\frac{\omega\varepsilon_\circ\varepsilon_1}{k_s}J_1(k_s r)$$

$$E_r = -i\frac{k_g}{k_s}J_1(k_s r)$$

$$for\ \ r \le a$$

$$E_z = AK_\circ(k_t r)$$

$$H_\theta = iA\frac{\omega\varepsilon_\circ}{k_t}K_1(k_t r)$$

$$E_r = iA\frac{k_g}{k_t}K_1(k_t r)$$

$$for\ \ r > a$$

- The boundary conditions require E_z, H_θ, and D_r to be continuous across r = a. Thus, it follows that

$$AK_\circ(k_t a) = J_\circ(k_s a),$$

$$-A\frac{K_1(k_t a)}{k_t} = \varepsilon_1 \frac{J_1(k_s a)}{k_s}.$$

- Again, eliminating constant A between the two equations gives the dispersion relation

$$\boxed{\varepsilon_1 \frac{J_1(k_s a)}{k_s\ J_\circ(k_s a)} + \frac{K_1(k_t a)}{k_t\ K_\circ(k_t a)} = 0}$$

- It is clear from this dispersion relation that the cutoff frequency For the TM_{0p} mode is exactly the same as that for the TE_{0p} mode.

- It is also clear that in the limit $\varepsilon_1 >> 1$ the propagation constants are determined by the roots of $J_1(k_s a) \approx 0$. However, this is exactly The same as the determining equation for TE modes in a metallic Wave guide of circular cross section (filled with dielectric of relative Permittivity ε_1).

- Modes with azimuthal dependence (*i.e.*, $m > 0$) have longitudinal Components of both E and H. This makes the math somewhat more Complicated. However, the basic results are the same as for $m=0$ Modes: for frequencies well above the cutoff frequency the modes are localized in the immediate vicinity of the cylinder.