4. Space Charge Effects in RF Linear Accelerators

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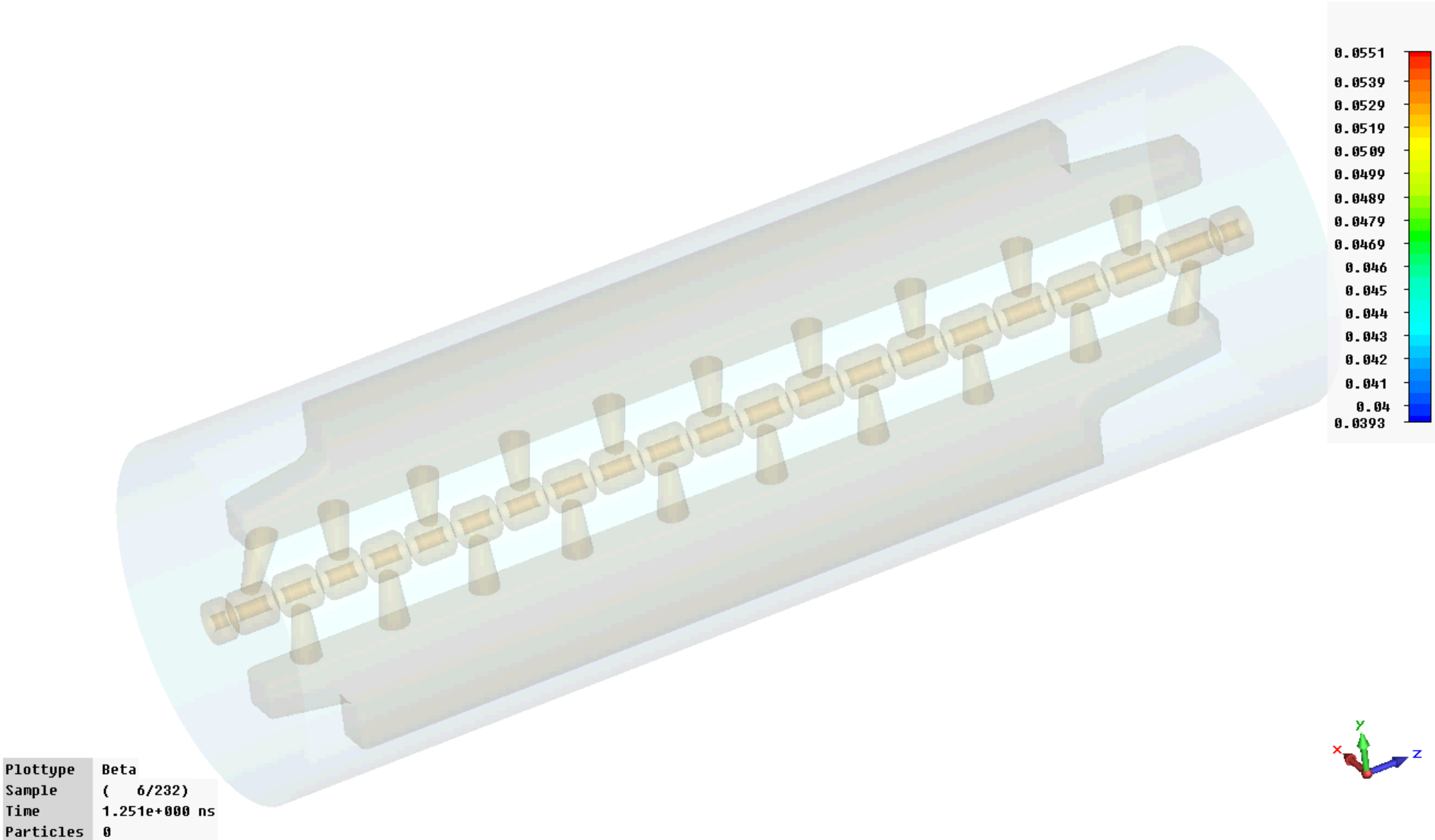
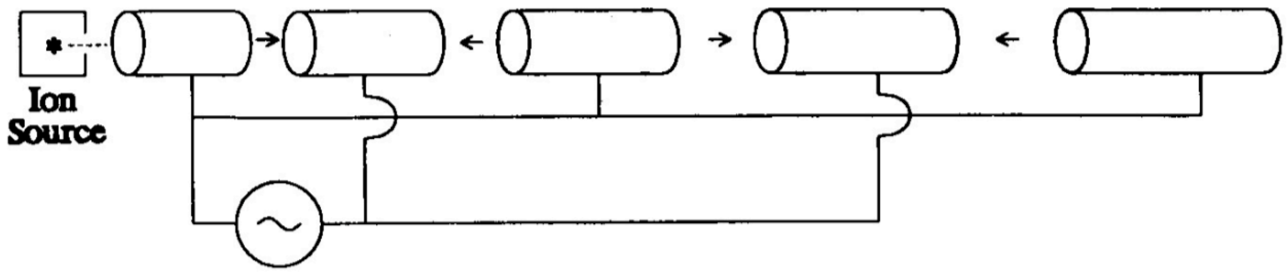
4.1. Principles of linear resonance acceleration



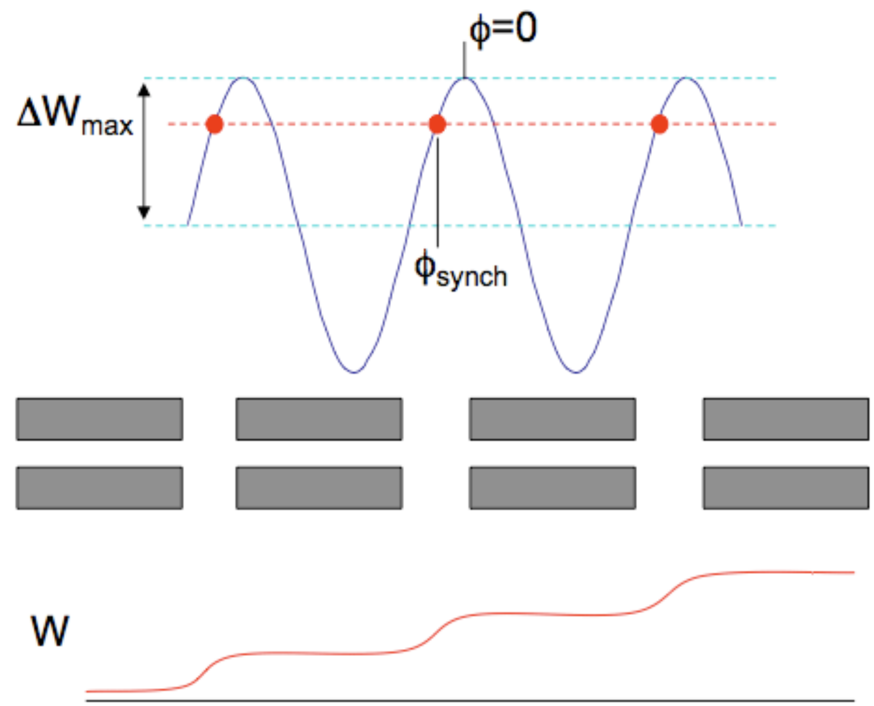
Layout of RF linear accelerator. (Courtesy of Sergey Kurennoy).



Layout of ion linear resonance accelerator.

Particle remains synchronized with accelerating field, i.e. arrives at the center of each accelerating gap at a specified phase ϕ_s wrt RF

At each gap the particle gets the required "kick" to reach the next gap at ϕ_s and remain synchronized



Energy gain in RF linear accelerator. (Courtesy of Larry Rybarcyk.)

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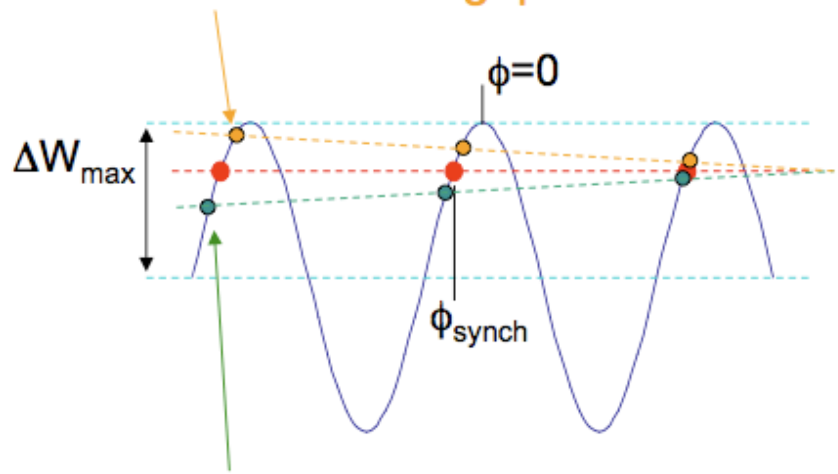
Choose ϕ_s To Produce Stable Motion About Synchronous Particle

- Need a longitudinal restoring (focusing) force to ensure that non-synchronous particles also get accelerated
- By choosing $\phi_s < 0$ particles make stable, oscillatory motion about the synchronous particle
- Bunching action is a natural result

- Slower particle arrives late and gets a larger kick which makes it arrive sooner at the next gap



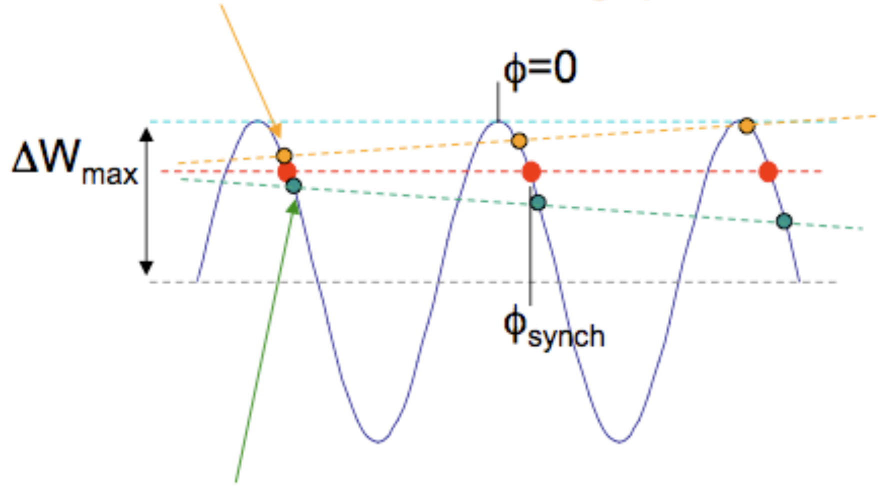
- Faster particle arrives early and gets a smaller kick which makes it arrive later at the next gap

(Courtesy of Larry Rybarcyk)

The Other Choice For ϕ_S Will Produce Unstable Motion About Synchronous Particle

- **Does not produce the needed longitudinal restoring (focusing) force to ensure that non-synchronous particles also get accelerated and remain bunched**
- **By choosing $\phi_s > 0$ particles move away from synchronous particle and are lost**

- Faster particle arrives early and gets a larger kick which makes it arrive even earlier at the next gap



- Slower particle arrives late and gets a smaller kick which makes it arrive even later at the next gap

(Courtesy of Larry Rybarcyk)

4.2. Electromagnetic wave equations

To obtain the electromagnetic wave equation in a vacuum using the modern method, we begin with the modern 'Heaviside' form of Maxwell's equations. In a vacuum and charge free space, these equations are:

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Taking the curl of the curl equations gives:

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla \times \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E} = -\mu_o \varepsilon_o \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

By using the vector identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$$

where $\mathbf{V}$ is any vector function of space, it turns into the wave equations:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} - {c_0}^2 \cdot \nabla^2 \mathbf{E} = 0$$

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} - {c_0}^2 \cdot \nabla^2 \mathbf{B} = 0$$

where $c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \times 10^8$ m/s is the speed of light in free space.

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

The solution is usually given in the form of a product of functions of one variable:

$$E_z\ (z,r,t) = Z(z)\ R(r)\ T(t)$$

Knowing E_z, one can compute the other electromagnetic field components: E_r with the divergence theorem

$$\operatorname{div}\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z}\ ,$$

giving

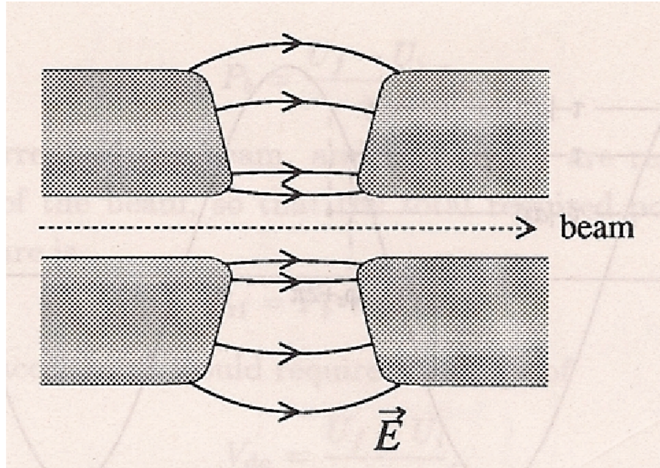
$$E_r(r) = -\frac{1}{r}\int_0^r \frac{\partial E_z}{\partial z} \mathrm{r}'\,\mathrm{dr}'\ ,$$

and B_ϑ via

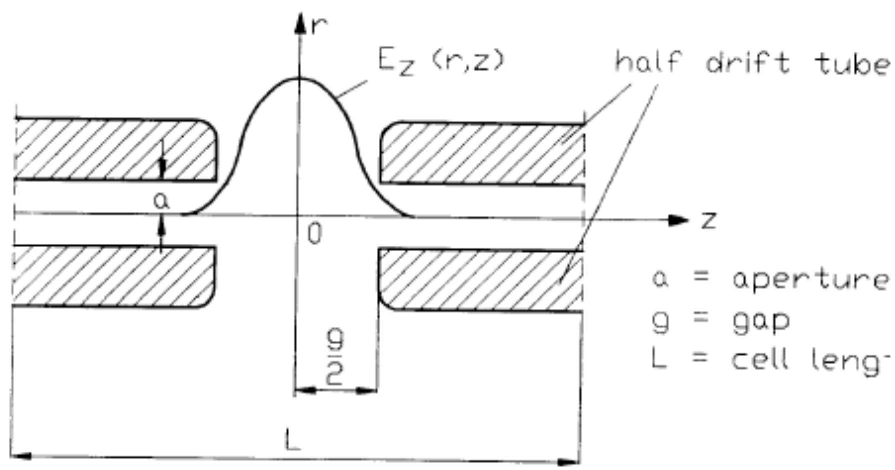
$$\operatorname{rot}\ \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\ ,$$

giving

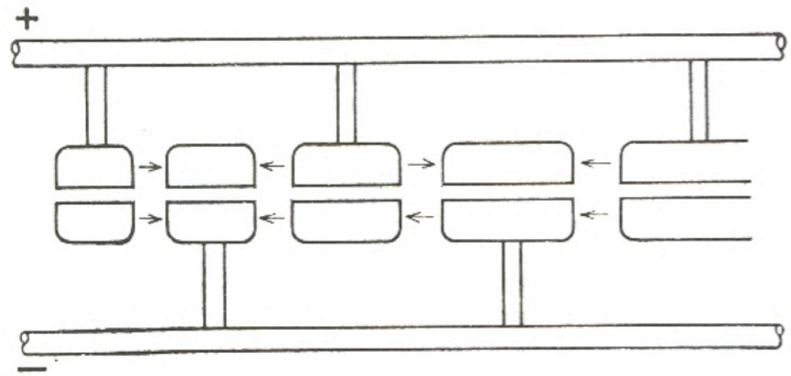
$$\frac{\partial B_\vartheta}{\partial z} = -\frac{1}{c^2}\frac{\partial E_r}{\partial t}\ .$$



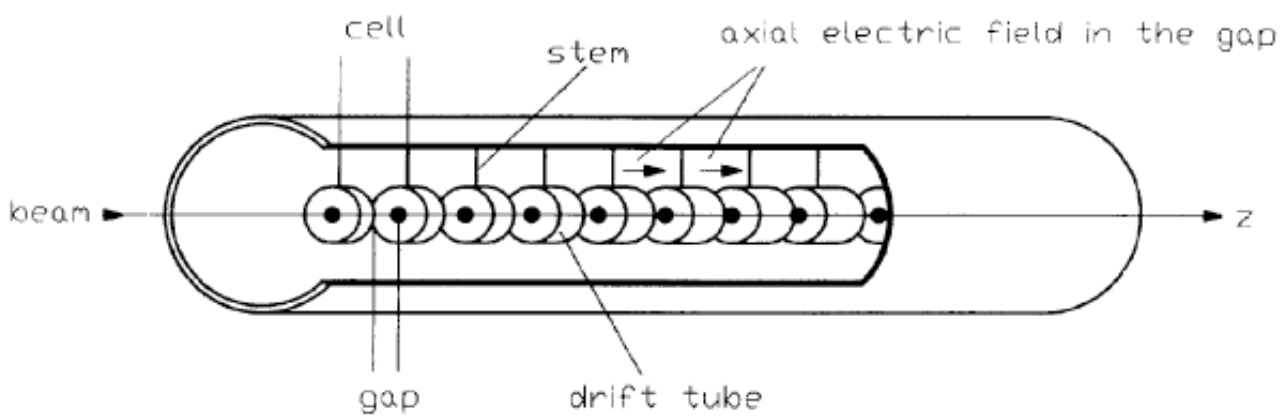
Electric field lines between the ends drift tubes (from M.Konte, W.MacKay, 1991)



Cell of Drift Tube Accelerator (from M.Weiss, CERN 96-02, p.39)



Accelerating structure with a π-mode standing wave.



Alvarez accelerating structure (from M.Weiss, CERN 96-02, p.39)

Standing wave: $$Z(z)T(t) = E_o \cos(k_z z)\cos(\omega t) = \frac{E_o}{2}[\cos(\omega t - k_z z) \quad + \quad \cos(\omega t + k_z z)]$$

accelerating wave — opposite wave

Cyclic frequency of RF field $$\omega = \frac{2\pi c}{\lambda} = 2\pi f_{RF}$$

Wave number $$k_z = \frac{2\pi}{L} = \frac{2\pi}{\beta\lambda}$$

Equivalent traveling wave $$E_z(z,\ r,\ t) = E\cos(\omega t - k_z z)R(r)$$

Substitution into wave equation gives for radial field component: $$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) - R(k_z^2 - \frac{\omega^2}{c^2}) = 0$$

$$k_z^2 - \frac{\omega^2}{c^2} = k_z^2(1 - \frac{\omega^2}{k_z^2 c^2}) = k_z^2\ (1-\beta^2) = \frac{k_z^2}{\gamma^2}$$

Solution for radial field component: $$R(r) = I_o(\frac{k_z r}{\gamma})$$

where $I_o(x)$ is the modified Bessel function

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Finally, equivalent traveling wave is

$$E_z = E\, I_o(\frac{k_z r}{\gamma})\, cos\, (\omega t\ -\ k_z z), \tag{5.1}$$

$$E_r = -\ \gamma E I_1(\frac{k_z r}{\gamma})\, sin\, (\omega t\ -\ k_z z), \tag{5.2}$$

$$B_\theta = -\ \frac{1}{c}\, \beta\gamma\, E I_1(\frac{k_z r}{\gamma})\, sin\, (\omega t\ -\ k_z z) \tag{5.3}$$

Effective traveling wave can be represented in Hamiltonian by a potential function

$$U_a = \frac{E}{k_z}\, I_o(\frac{k_z\, r}{\gamma})\, sin\, (\omega t\ -\ k_z z). \tag{5.4}$$

Particle, which velocity coincides with the velocity of the accelerating wave, is called synchronous particle. Dynamics of the synchronous particle is described by the integration of equation for synchronous particle momentum, P_s, and position, z_s:

$$\frac{dP_s}{dt} = qE\, cos\, \varphi_s$$

$$\frac{d\, z_s}{dt} = \frac{P_s}{m\gamma_s},$$

where $\varphi_s = \omega t\ -\ k_z z_s$ is the synchronous phase.

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4.3. Hamiltonian of particle motion in RF field

Consider Hamiltonian in a focusing channel with RF field:

$$K = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} + qU_a + qU_{el} + qU_b, \quad (5.9)$$

where U_{el} is the potential of electrostatic focusing lenses, and U_b is the scalar potential of field of the beam. For the further analysis, let us introduce new variables

$$p_z = P_z - P_s, \qquad \zeta = z - z_s, \quad (5.10)$$

which define deviation from synchronous particle. Generating function of the transformation is

$$F_3(\zeta, P_z, t) = -(\zeta + z_s)(P_z - P_s), \quad (5.11)$$

which can be easily verified by differentiation:

$$p_z = -\frac{\partial F_3}{\partial \zeta}, \qquad z = -\frac{\partial F_3}{\partial P_z}. \quad (5.12)$$

New Hamiltonian is given by

$$T = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_s + p_z - qA_z)^2} + qU_a + qU_{el} + qU_b + \frac{\partial F_3}{\partial t}. \quad (5.13)$$

Consider separately expression for square root in Hamiltonian:

$$s\,(p_x, p_y, p_\eta) = \sqrt{m^2c^2 + p_x^2 + p_y^2 + (P_s + p_\eta)^2}, \qquad (5.14)$$

where for simplification, the components of canonical momentum are substituted by that of mechanical momentum, $p_x = P_x - q\,A_x$, $p_y = P_y - q\,A_y$, and an additional variable is $p_\eta = p_z - q\,A_z$. Typically, momentum of the synchronous particle is much larger than transverse particle momentum and longitudinal momentum spread, $P_s >> p_x, p_y, p_\eta$. Let us expand expression for square root in the vicinity of $s(p_x, p_y, p_\eta)$ up to the order of p_x^2, p_y^2, p_η^2:

$$s = \sqrt{m^2c^2 + P_s^2} + \frac{\partial s}{\partial p_x}p_x + \frac{\partial s}{\partial p_y}p_y + \frac{\partial s}{\partial p_\eta}p_\eta + \frac{1}{2}\frac{\partial^2 s}{\partial p_x^2}p_x^2 + \frac{1}{2}\frac{\partial^2 s}{\partial p_y^2}p_y^2 +$$

$$+ \frac{1}{2}\frac{\partial^2 s}{\partial p_\eta^2}p_\eta^2 + \frac{1}{2}\frac{\partial^2 s}{\partial p_x \partial p_y}p_x p_y + \frac{1}{2}\frac{\partial^2 s}{\partial p_x \partial p_\eta}p_x p_\eta + \frac{1}{2}\frac{\partial^2 s}{\partial p_y \partial p_\eta}p_y p_\eta, \qquad (5.15)$$

where all derivatives are taken at $p_x = 0$, $p_y = 0$, $p_\eta = 0$. Calculations of expansion gives:

$$c\sqrt{m^2c^2 + p_x^2 + p_y^2 + (P_s + p_\eta)^2} = mc^2\gamma + \frac{p_x^2}{2m\gamma} + \frac{p_y^2}{2m\gamma} + \frac{P_s p_\eta}{m\gamma} + \frac{p_\eta^2}{2m\gamma^3}, \qquad (5.16)$$

where reduced energy is

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$$\gamma = \sqrt{1 + \left(\frac{P_s}{mc}\right)^2}. \qquad (5.17)$$

Time derivative of the generating function, Eq. (5.11), is:

$$\frac{\partial F_3}{\partial t} = \zeta \dot{P}_s - \dot{z}_s P_z + \dot{z}_s P_s + z_s \dot{P}_s. \tag{5.18}$$

where dot means derivative over time. Taking into account that the particle velocity is $\dot{z}_s = \frac{P_s}{m\gamma}$, the following expressions in time derivative, Eq. (5.11), are:

$$\dot{z}_s P_z = \frac{P_s}{m\gamma}(P_s + p_z), \qquad \dot{z}_s P_s = \frac{P_s^2}{m\gamma}, \tag{5.19}$$

and the time derivative of the generating function is therefore

$$\frac{\partial F_3}{\partial t} = \zeta \dot{P}_s - \frac{P_s p_z}{m\gamma} + z_s \dot{P}_s. \tag{5.20}$$

Substitution of expansions, Eqs. (5.16), (5.20), into Eq. (5.13) gives for the new Hamiltonian, $H = T - m^2c^2\gamma$:

$$H = \frac{(P_x - qA_x)^2}{2m\gamma} + \frac{(P_y - qA_y)^2}{2m\gamma} + \frac{(p_z - qA_z)^2}{2m\gamma^3} + qU_a + qU_{el} + qU_b - \frac{qP_sA_z}{m\gamma} + \dot{P}_s(z_s+\zeta). \quad (5.21)$$

The term $\dot{P}_s z_s$ can be excluded, because it does not depend on canonical variables and does not contribute to equations of particle motion. The acceleration of synchronous particle according to Eq. (5.7) is $\dot{P}_s = qE\cos\varphi_s$. The term $\dot{P}_s\zeta$ can be combined with the accelerating potential:

$$qU_a + \dot{P}_s\zeta = q\frac{E}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s]. \quad (5.22)$$

Finally, the new Hamiltonian is

$$H = \frac{(P_x - q\,A_x)^2}{2m\gamma} + \frac{(P_y - q\,A_y)^2}{2m\gamma} + \frac{(p_z - q\,A_z)^2}{2m\gamma^3} +$$

$$+ q\frac{E}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s] + qU_{el} + qU_b - \frac{qP_sA_z}{m\gamma}. \quad (5.23)$$

Consider the following terms in the Hamiltonian:

$$\frac{(p_z - q\,A_z)^2}{2m\gamma^3} - \frac{qP_sA_z}{m\gamma} = \frac{p_z^2}{2m\gamma^3} - \frac{qP_sA_z}{m\gamma}\left(1 + \frac{p_z}{P_s\,\gamma^2} - \frac{qA_z}{2P_s\gamma^2}\right). \tag{5.24}$$

As soon as $p_z << P_s$, $qA_z << P_s$, the second and the third terms in parentheses in Eq. (5.24) can be omitted:

$$\frac{qP_sA_z}{m\gamma}\left(1 + \frac{p_z}{P_s\,\gamma^2} - \frac{q\,A_z}{2P_s\gamma^2}\right) \approx \frac{qP_sA_z}{m\gamma} = q\beta cA_z. \tag{5.25}$$

The vector - potential is $A_z = A_{z\ magn} + \frac{\beta}{c}U_b$. Therefore, in the adopted assumptions, the Hamiltonian becomes:

$$H = \frac{(P_x - q\,A_x)^2}{2m\gamma} + \frac{(P_y - q\,A_y)^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} +$$

$$+q\frac{E}{k_z}\left[I_o\left(\frac{k_z r}{\gamma}\right)\sin\left(\varphi_s - k_z\zeta\right) + k_z\zeta\cos\varphi_s\right] + q(U_{el} - \beta cA_{z\ magn}) + q\frac{U_b}{\gamma^2}. \tag{5.26}$$

Consider separately structures with quadrupole focusing and with longitudinal magnetic focusing. In the absence of longitudinal magnetic field, transverse components of the vector potential are $A_x = 0,\ A_y = 0$, therefore, the transverse components of canonical momentum coincide with that of mechanical momentum: $p_x = P_x$, $p_y = P_y$. The term $U_{el} - \beta c A_{z,\ magn}$ is the focusing potential of the structure. Averaged potential of quadrupole structure is given by

$$U_{el} - \beta c A_{z\ magn} = G_t \frac{(x^2 + y^2)}{2}, \qquad (5.27)$$

where G_t is the gradient of averaged focusing potential. The Hamiltonian for particle motion in RF field with quadrupole focusing is

$$H = \frac{p_x^2}{2m\gamma} + \frac{p_y^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}[I_o(\frac{k_z r}{\gamma})sin(\varphi_s - k_z\zeta) + k_z\zeta cos\varphi_s] + qG_t\frac{(x^2+y^2)}{2} + q\frac{U_b}{\gamma^2}. \qquad (5.28)$$

In presence of longitudinal magnetic field, the Hamiltonian, Eq. (5.26), is

$$H = \frac{(P_x - qA_x)^2}{2m\gamma} + \frac{(P_y - qA_y)^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s] + q\frac{U_b}{\gamma^2} \qquad (5.29)$$

where transverse components of vector-potential are given by

$$A_{x\ magn} = -A_{\theta\ magn}\, sin\theta = -B\frac{y}{2} \qquad (5.30)$$

$$A_{y\ magn} = A_{\theta\ magn}\, cos\theta = B\frac{x}{2} \qquad (5.31)$$

Transformation to Larmor system is given by

$$\hat{x} = x \cos \omega_L t - y \sin \omega_L t ,$$

$$\hat{y} = x \sin \omega_L t + y \cos \omega_L t ,$$

$$\hat{P}_x = P_x \cos \omega_L t - P_y \sin \omega_L t ,$$

$$\hat{P}_y = P_y \cos \omega_L t + P_x \sin \omega_L t .$$

where Larmor frequency $\omega_L = \dfrac{qB}{2m\gamma}$

Hamiltonian of particle motion in magnetic field :

$$H = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s] + m\gamma\omega_L^2\frac{r^2}{2} + q\frac{U_b}{\gamma^2}$$

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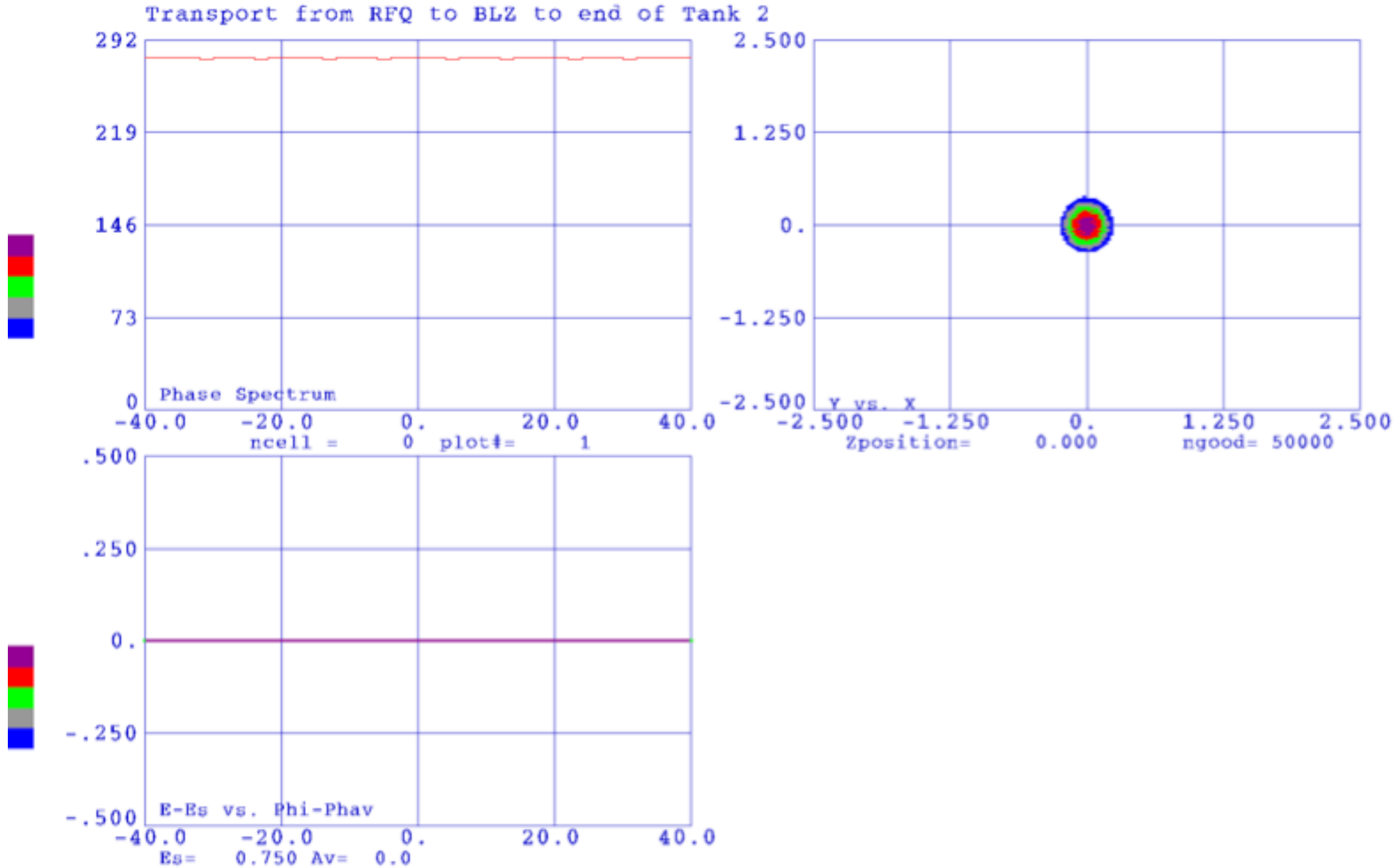


Example of beam dynamics in accelerating structure. (Courtesy of Larry Rybarcyk.)

4.4. Longitudinal particle motion in RF field

Consider longitudinal particle motion neglecting space charge forces. From Hamiltonian, Eq. (5.26), equations of particle motion around synchronous particle are:

$$\frac{dp_z}{dt} = qE\,[I_o(\frac{k_z r}{\gamma})\,cos(\varphi_s - k_z\zeta) - cos\varphi_s], \qquad (5.33)$$

$$\frac{d\zeta}{dt} = \frac{p_z}{m\gamma^3}. \qquad (5.34)$$

In most of accelerators particles perform radial oscillation close to axis in the way that $k_z r << 1$ and the value of Bessel function in Eq. (5.33) is close to 1. Radial dependence of longitudinal motion can be usually neglected and longitudinal motion can be consider only for on-axis particles, given by the Hamiltonian:

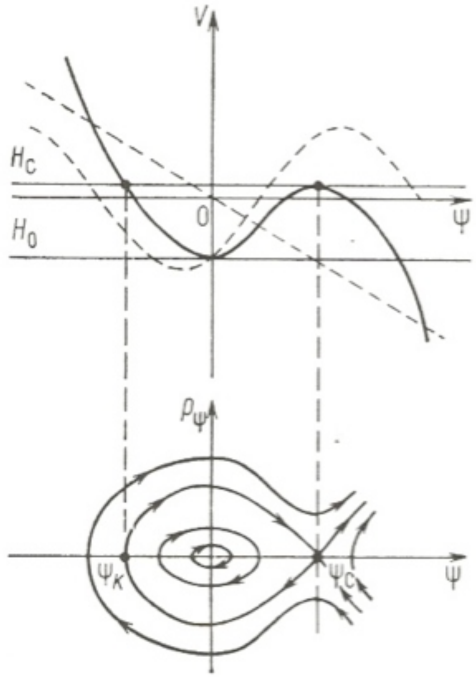
$$T = \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}\,[sin\,(\varphi_s - k_z\zeta) + k_z\zeta cos\varphi_s]. \qquad (5.35)$$

Hamiltonian. Eq. (5.35), describes particle oscillations around synchronous particle. Let us assume that parameters γ, E, k_z, are changing slowly during particle oscillations. Hamiltonian, Eq. (5.35), with constant values of γ, E, k_z, is a constant of motion.

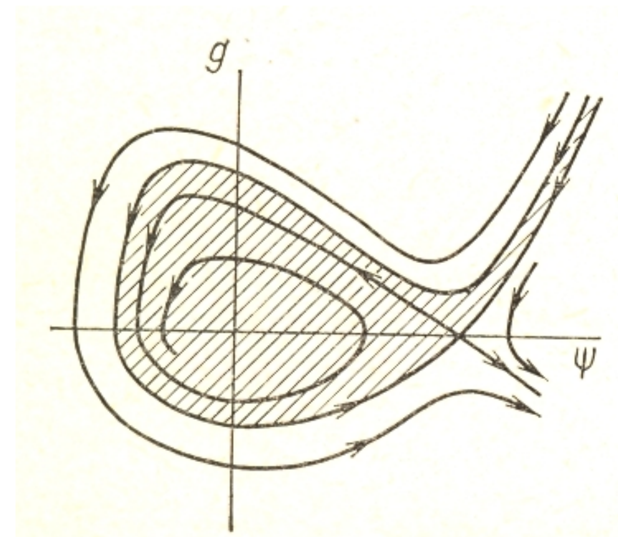
Figure below illustrates phase space trajectories corresponding to constant values of Hamiltonian. Phase space trajectory, corresponding to maximum stable oscillations is called separatrix. Particle trajectories inside separatrix are closed and, therefore, stable. Outside separatrix, particle trajectories are open, which corresponds to unstable particle motion. Consider separately potential function

$$V(\psi) = q\,\frac{E}{k_z}\,[\sin(\varphi_s + \psi) - \psi\cos\varphi_s], \tag{5.36}$$

where ψ is the phase deviation from synchronous particle $\psi = k_z\zeta$ (5.37)



Relief of potential function and a family of phase trajectories (from Kapchinsky, 1985), $p_\psi = W_s - W$.



Separatrix of longitudinal phase space oscillations including acceleration.

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Derivative of potential function
$$\frac{dV}{d\psi} = q\,\frac{E}{k_z}\,[cos\,(\varphi_s + \psi) - cos\varphi_s] = 0, \qquad (5.38)$$

determines two extremum points $\psi = 0$ and $\psi = -2\varphi_s$. First point corresponds to stable position, while the second one is unstable. To be stable, potential function must have minimum in extremum point $\psi = 0$, which means that the second derivative hast to be positive:

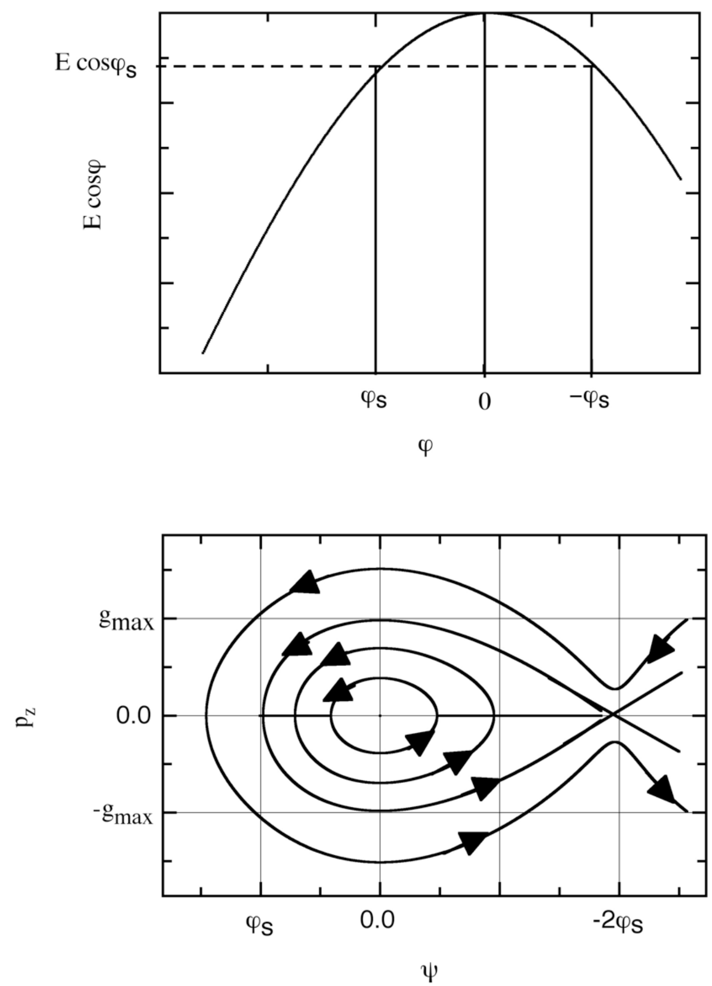
$$\left.\frac{d^2V}{d\psi^2}\right|_{\psi = 0} = -\,q\,\frac{E}{k_z}\,sin\,(\varphi_s) > 0. \qquad (5.39)$$

Eq. (5.39) defines stability condition, i. e. $sin\varphi_s < 0$, or synchronous phase must be negative, $\varphi_s < 0$. Taking in Hamiltonian, Eq. (5.35), $p_z = 0$, $\psi = -2\varphi_s$, the value of Hamiltonian, corresponding to separatrix, is

$$T\,|_{p_z = 0,\ \psi = -2\varphi_s} = q\,\frac{E}{k_z}\,(-sin\varphi_s + 2\varphi_s cos\varphi_s). \qquad (5.40)$$

Substitution of this value into Eq. (5.35) gives the equation for separatrix:

$$\frac{p_z^2}{2m\gamma^3} + q\,\frac{E}{k_z}\,[cos\varphi_s(sin\psi - \psi - 2\varphi_s) + sin\varphi_s(1 + cos\psi)] = 0. \qquad (5.41)$$



(Up) accelerating field and (bottom) longitudinal phase space trajectories.

Phase length of separatrix is determined from Eq. (5.41) assuming $p_z = 0$:

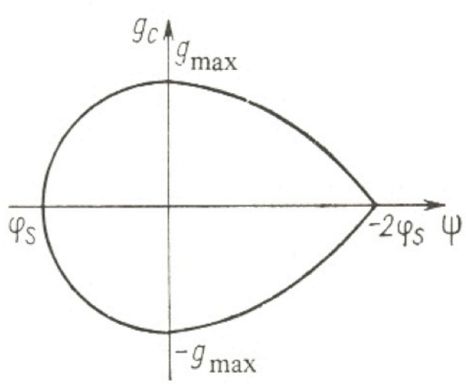
$$\cos\varphi_s(\sin\psi - \psi - 2\varphi_s) + \sin\varphi_s(1 + \cos\psi) = 0. \qquad (5.42)$$

The first root is $\psi = -2\varphi_s$. The second root, $\psi \approx \varphi_s$, can be obtained approximately via expanding trigonometric functions in Eq. (5.42) in Taylor series. Finally, the phase length of the separatrix is approximately $3\varphi_s$.

The width of separatrix in momentum is determined from Eq. (5.41) assuming $\psi = 0$. For practical applications, the relative momentum spread is important:

$$g = \frac{P_z - P_s}{P_s} = \frac{p_z}{P_s}. \qquad (5.43)$$

Taking into account that $P_s = mc\beta\gamma$, the half-width of the separatrix in relative momentum is



A separatrix on the phase plane of longitudinal oscillations.

$$g_{max} = \sqrt{\frac{\gamma}{\pi\beta}\left(\frac{qE\lambda}{mc^2}\right) 2|\sin\varphi_s| \left(1 - \frac{\varphi_s}{tg\varphi_s}\right)}. \qquad (5.44)$$

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From Eqs. (5.33), (5.34) the equation for longitudinal oscillations is

$$\frac{d^2\zeta}{dt^2} = \frac{qE}{m\gamma^3}[cos\,(\varphi_s - k_z\zeta) - cos\varphi_s] \qquad (5.45)$$

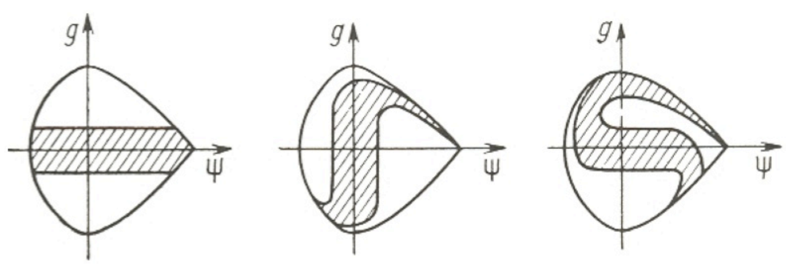
Eq (5.45) describes nonlinear oscillations around synchronous particle. Expanding of trigonometric function in Eq. (5.45) $cos\,(\varphi_s - k_z\zeta) \approx cos\varphi_s + (k_z\zeta)sin\varphi_s$ results in equation for linear oscillations

$$\frac{d^2\zeta}{dt^2} + (\frac{qE\,k_z\,|sin\varphi_s|}{m\gamma^3})\zeta = 0. \qquad (5.46)$$

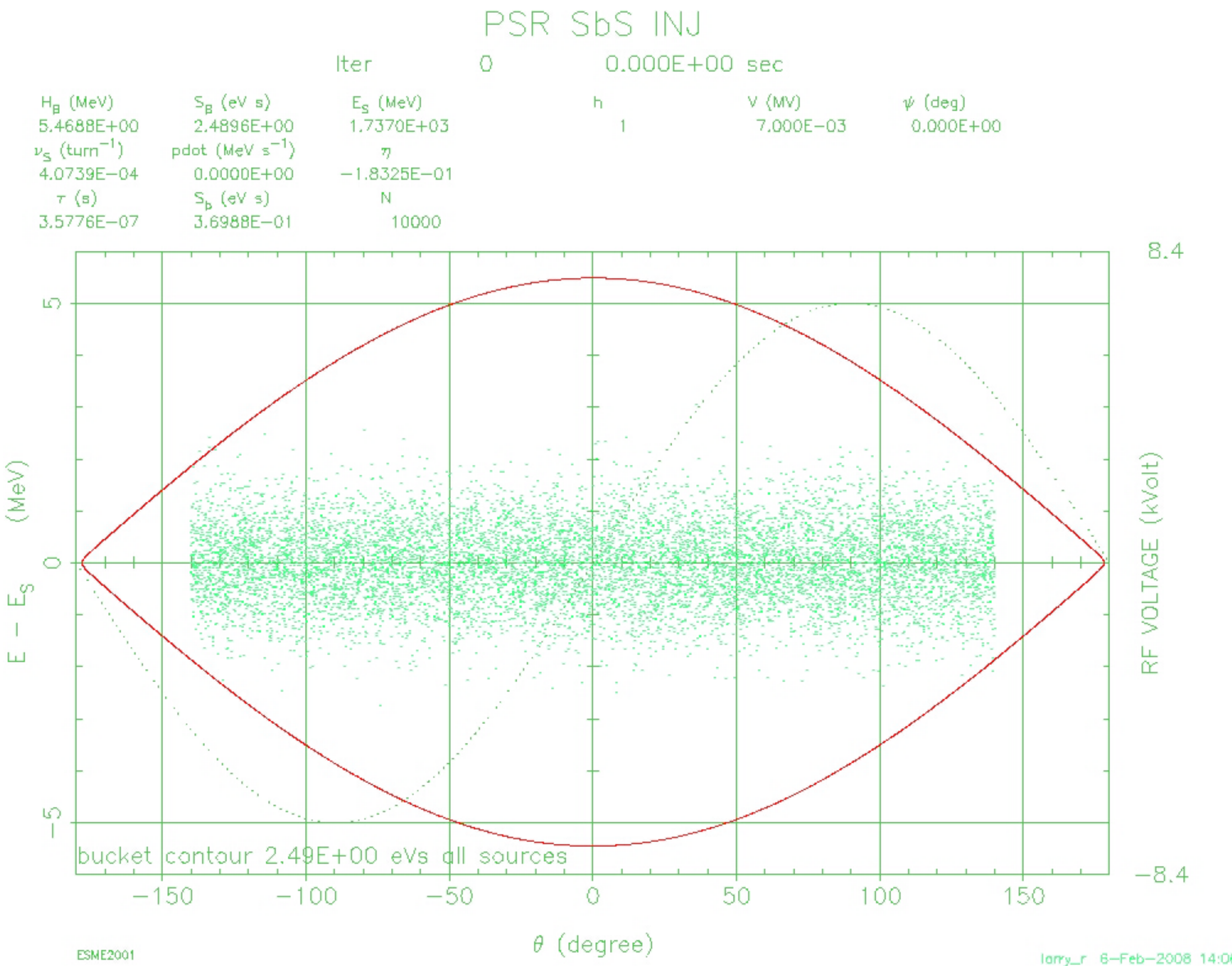
Expression in brackets in Eq. (5.46) is a square of longitudinal oscillations. It can be rewritten as dimensionless value

$$(\frac{\Omega}{\omega})^2 = (\frac{qE\lambda}{mc^2})\frac{|sin\varphi_s|}{2\pi\beta\gamma^3}. \qquad (5.47)$$

Frequency of longitudinal particle oscillations drops with particle energy.

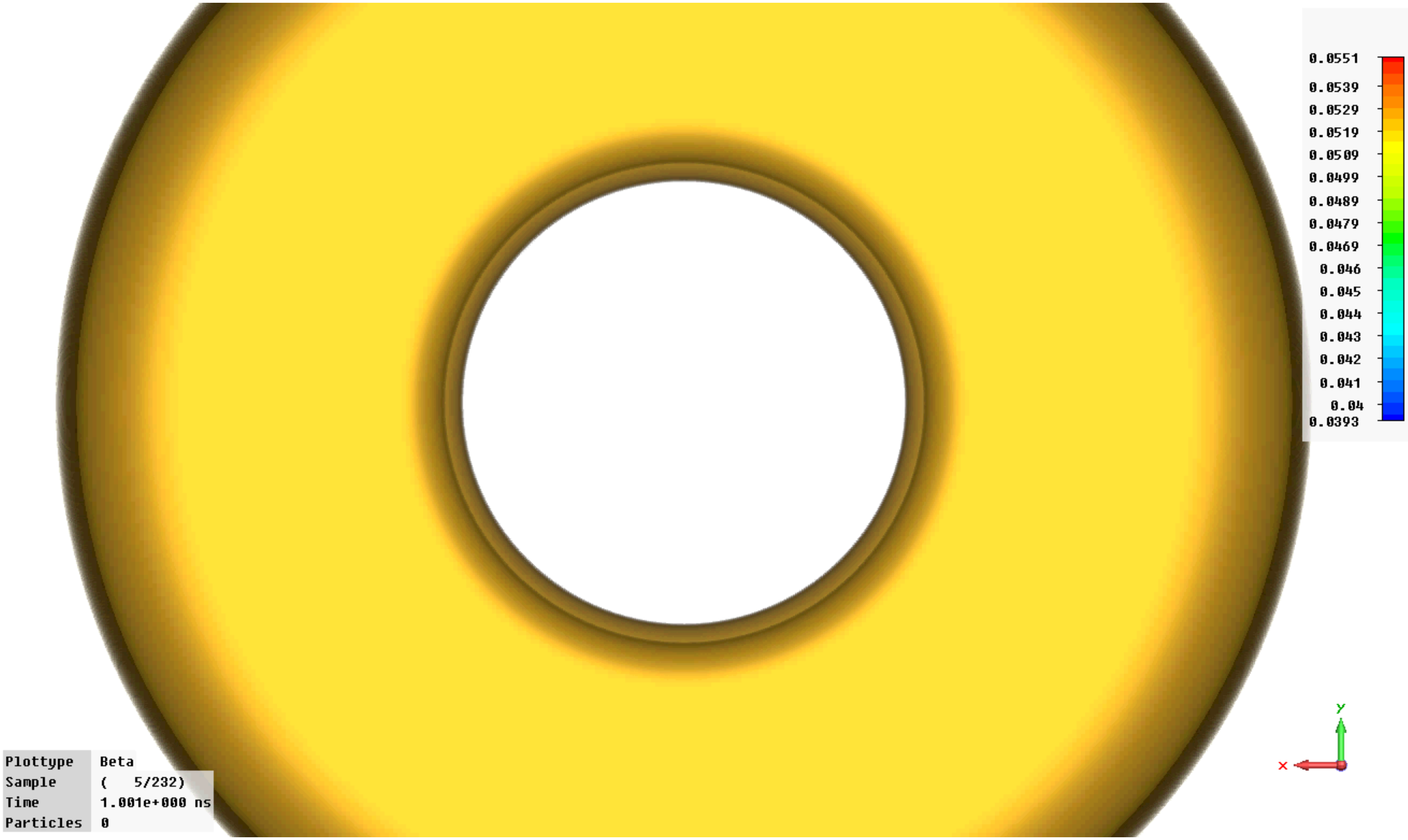


Distortion of the longitudinal phase space due to nonlineartity of longitudinal forces.



Longitudinal oscillations in RF field with φ_s = - 90^o. (Courtesy of Larry Rybarcyk.)

4.5. Transverse particle motion in RF field



Quadrupole beam focusing in RF linear accelerator. (Courtesy of Sergey Kurennoy).

Hamiltonian of particle motion in RF field:

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s] + m\gamma\Omega_r^2\frac{r^2}{2} + q\frac{U_b}{\gamma^2}$$

Near-axis approximation: $$I_o(\frac{k_z r}{\gamma}) \approx 1 + \frac{1}{4}(\frac{k_z r}{\gamma})^2$$

Hamiltonian of transverse motion: $$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{qE}{4k_z}(\frac{k_z r}{\gamma})^2 \sin(\varphi_s - k_z\zeta) + m\gamma\Omega_r^2\frac{r^2}{2} + q\frac{U_b}{\gamma^2}$$

$$\frac{qE}{4k_z}(\frac{k_z r}{\gamma})^2 = \frac{qE\pi}{2\beta\gamma^2\lambda}$$

Expansion near synchronous particle:

$$\sin(\varphi_s - k_z\zeta) \approx \sin\varphi_s - k_z\zeta\cos\varphi_s = \sin\varphi_s(1 - \psi\,\mathrm{ctg}\,\varphi_s)$$

Phase deviation from synchronous particle $$\psi = k_z\zeta$$

Hamiltonian of near-axis, near synchronous particle motion, with $U_b = 0$:

$$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{qE\pi}{2\beta\gamma^2\lambda}\sin\varphi_s(1-\psi\,\mathrm{ctg}\varphi_s)r^2 + m\gamma\,\Omega_r^2\frac{r^2}{2}$$

Frequency of longitudinal oscillations: $$\Omega^2 = \frac{2\pi}{\lambda}\frac{qE}{m}\frac{|\sin\varphi_s|}{\beta\gamma^3}$$

Hamiltonian becomes: $$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} - \frac{m\gamma}{4}\Omega^2(1-\psi\,\mathrm{ctg}\varphi_s)r^2 + m\gamma\,\Omega_r^2\frac{r^2}{2}$$

$$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{m\gamma}{2}r^2[\Omega_r^2 - \frac{\Omega^2}{2}(1-\psi\,\mathrm{ctg}\varphi_s)]$$

Transverse oscillation frequency of synchronous particle in presence of RF field: $$\Omega_{rs}^2 = \Omega_r^2 - \frac{\Omega^2}{2}$$

4.6. Parametric resonance and beam emittance growth in RF field

Hamiltonian becomes:

$$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{m\gamma}{2} r^2 (\Omega_{rs}^2 + \frac{\Omega^2}{2} \psi \operatorname{ctg} \varphi_s)$$

Longitudinal particle oscillations with amplitude Φ and frequency Ω :

$$\psi = -\Phi \sin(\Omega t + \psi_o)$$

Finally, Hamiltonian is:

$$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{m\gamma}{2} r^2 [\Omega_{rs}^2 - \frac{\Omega^2}{2} \operatorname{ctg} \varphi_s \Phi \sin(\Omega t + \psi_o)]$$

Transversal equation of motion:

$$\frac{d^2 x}{dt^2} + x[\Omega_{rs}^2 - \frac{\Omega^2}{2} \operatorname{ctg} \varphi_s \Phi \sin(\Omega t + \psi_o)] = 0$$

Parametric resonance occurs when

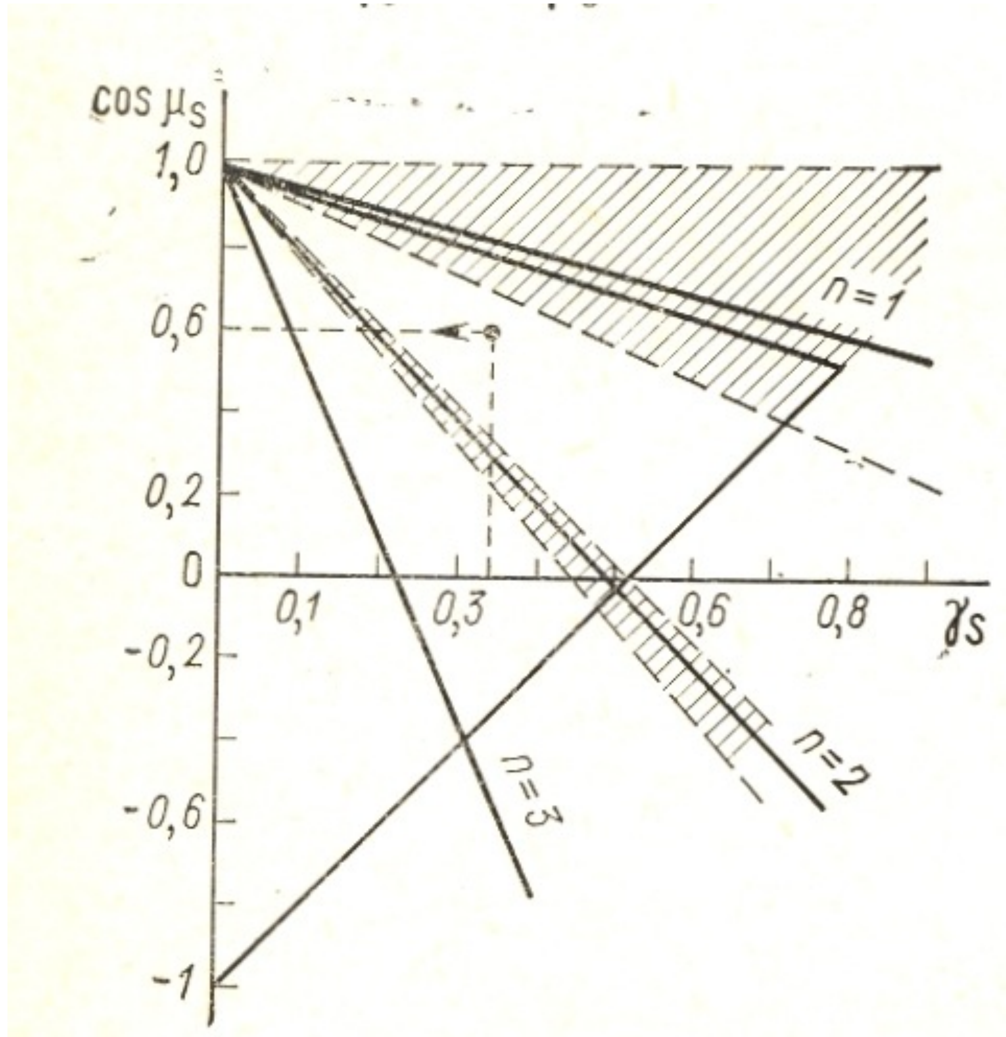
$$\Omega_{rs} = \frac{n}{2}\Omega, \quad n = 1, 2, 3$$

Let us introduce phase advance for synchronous particle in RF field

$$\mu_s = \Omega_{rs} \frac{L}{\beta_z c}$$

and defocusing factor

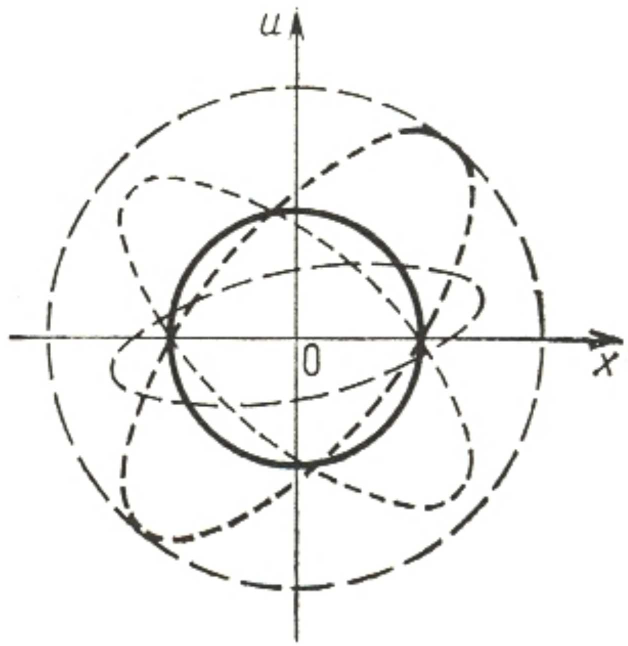
$$\gamma_s = \frac{1}{4}\Omega^2 (\frac{L}{\beta_z c})^2$$



Parametric resonance regions (from Kapchinsky, 1985).

Effective beam emittance growth outside of parametric resonance:

$$\frac{\varepsilon_{eff}}{\varepsilon} = 1 + \Phi \operatorname{ctg}\varphi_s \frac{\Omega^2}{4\Omega_{rs}^2 - \Omega^2}$$



Phase space of transverse oscillations in presence of RF field (from Kapchinky, 1985).

Required transverse focusing in presence of RF field

Hamiltonian of particle motion in RF field with solenoid focusing

$$H = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m\gamma} + m\gamma \frac{r^2}{2}(\omega_L^2 - \frac{\Omega^2}{2}\frac{\sin\varphi}{\sin\varphi_s}) + q\frac{U_b}{\gamma^2}$$

Transverse oscillation frequency in presence of RF field

$$\Omega_r^2 = \omega_L^2 - \frac{\Omega^2}{2}\frac{\sin\varphi}{\sin\varphi_s}$$

Envelope equation

$$\frac{d^2R}{dz^2} - \frac{\text{э}^2}{R^3} + \frac{\Omega_r^2}{(\beta c)^2}R - \frac{2I}{I_c(\beta\gamma)^3 R} = 0$$

Beam equilibrium condition $\frac{d^2R_e}{dz^2} = 0$

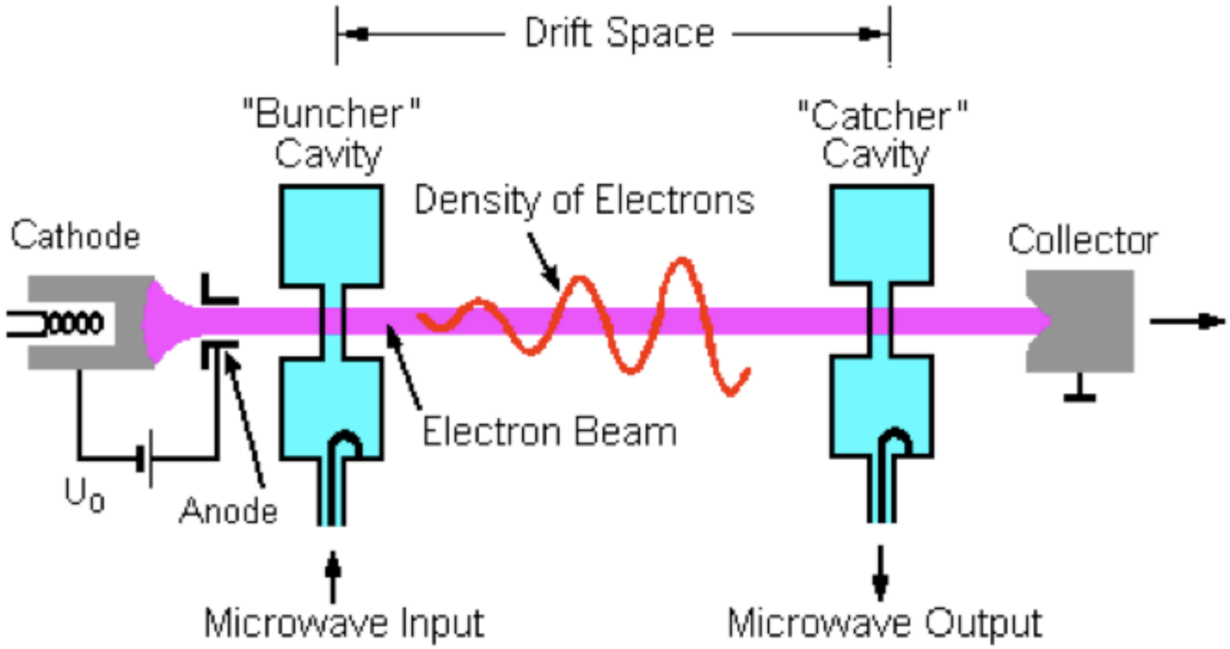
$$\frac{\Omega_r^2}{(\beta c)^2}R_e + \frac{\text{э}^2}{R_e^3} - \frac{2I}{I_c(\beta\gamma)^3 R_e} = 0$$

$$\Omega_r^2 = (\frac{\beta c}{R_e})^2(\frac{\text{э}^2}{R_e^2} + \frac{2I}{I_c(\beta\gamma)^3})$$

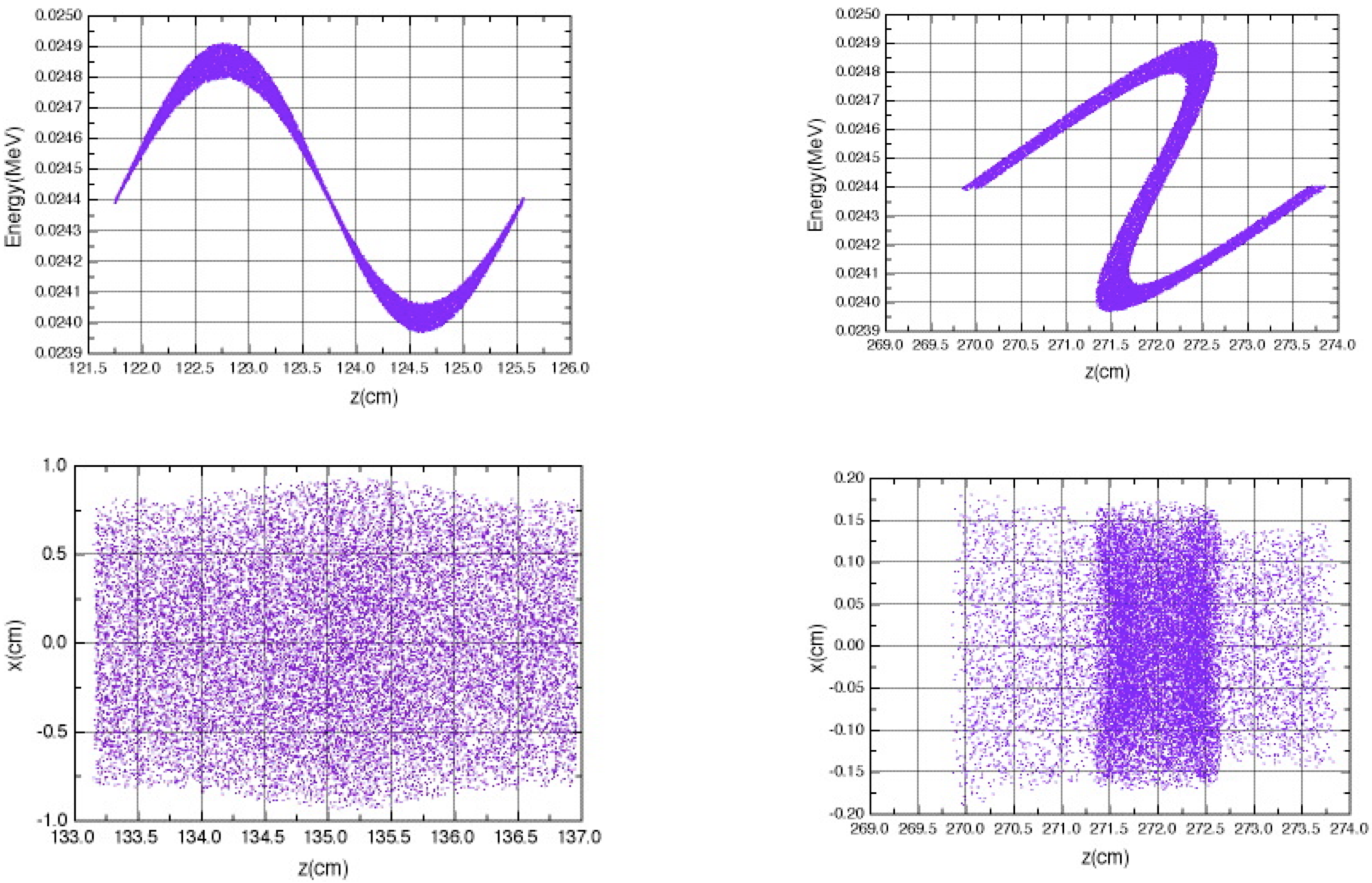
Required magnetic field

$$B = \frac{2mc\beta\gamma}{qR_e}\sqrt{(\frac{\text{э}}{R_e})^2 + \frac{2I}{I_c(\beta\gamma)^3} + \pi(\frac{qE\lambda}{mc^2})\frac{\sin\varphi}{(\beta\gamma)^3}(\frac{R_e}{\lambda})^2}$$

4.7. Beam bunching in RF field



Layout of klystron beam bunching scheme (from http://en.wikipedia.org/wiki/Klystron)



4. RF beam bunching scheme: (left) initial beam modulation in longitudinal momentum, (right) final beam modulation in density.

Equation of motion in RF gap of width *d* and applied voltage U_1

$$\frac{dv}{dt} = \frac{q}{m}\frac{U_1}{d}\sin\omega t$$

Longitudinal particle velocity in RF gap

$$v = v_o + \frac{q}{m}\frac{U_1}{d}\int_{t_{in}}^{t_{out}} \sin\omega t\, dt$$

Initial particle velocity after extraction voltage U_o

$$v_o = \sqrt{\frac{2qU_o}{m}}$$

Longitudinal particle velocity after RF gap

$$v = v_o + \frac{q}{m}\frac{U_1}{\omega d} 2\sin\left(\frac{\varphi_{in} + \varphi_{out}}{2}\right)\sin\left(\frac{\varphi_{out} - \varphi_{in}}{2}\right)$$

RF phase in the center of the gap

$$\frac{\varphi_{in} + \varphi_{out}}{2} = \omega t_1$$

Transit time angle through the gap

$$\theta_1 = \frac{\omega d}{v_o} \qquad \frac{\varphi_{out} - \varphi_{in}}{2} = \frac{\theta_1}{2}$$

Longitudinal particle velocity after RF gap

$$v = v_o + v_1 \sin\omega t_1$$

Amplitude of modulation of longitudinal velocity

$$v_1 = v_o \frac{U_1}{2U_0} M_1$$

Transit time factor of RF gap

$$M_1 = \frac{\sin\frac{\theta_1}{2}}{\frac{\theta_1}{2}}$$

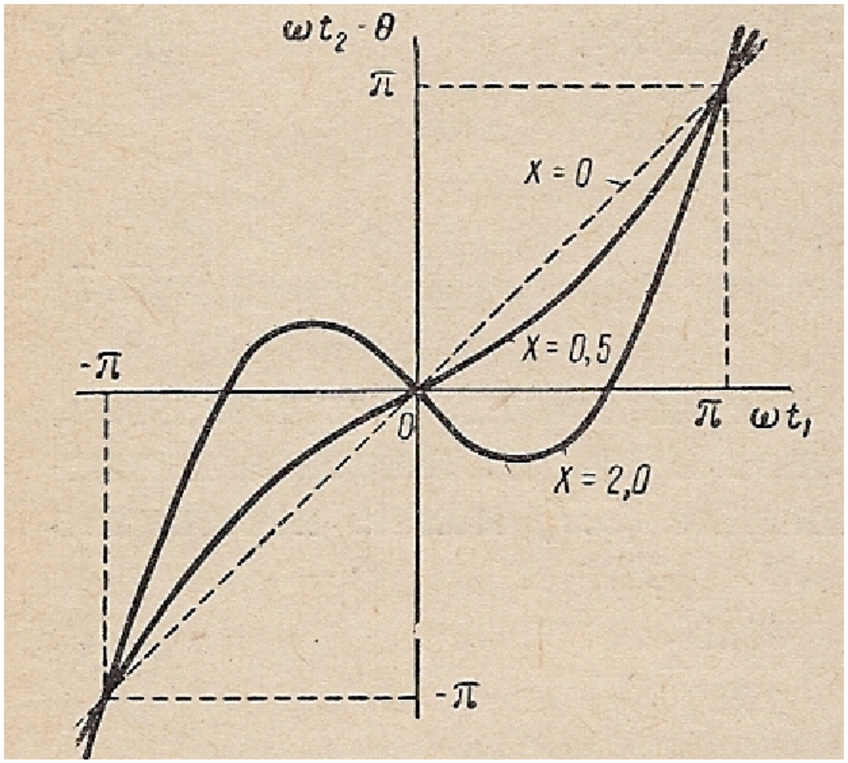
Time of arrival of particle to the second gap

$$t_2 = t_1 + \frac{z}{v_o + v_1 \sin \omega t_1} \approx t_1 + \frac{z}{v_o}(1 - \frac{v_1}{v_o} \sin \omega t_1)$$

Phase of arrival of particle into the second gap

$$\omega t_2 - \omega \frac{z}{v_o} = \omega t_1 - \omega \frac{z v_1}{v_o^2} \sin \omega t_1$$

$$\omega t_2 - \theta = \omega t_1 - X \sin \omega t_1$$



Phase of arrival of particle into second gap as a function phase of the same particle in the first gap.

Transit angle between gaps $\theta = \omega \frac{z}{v_o}$

Bunching parameter $X = \omega \frac{z v_1}{v_o^2} = \frac{U_1 M_1}{2U_o} \frac{\omega z}{v_o}$

Conservation of charge

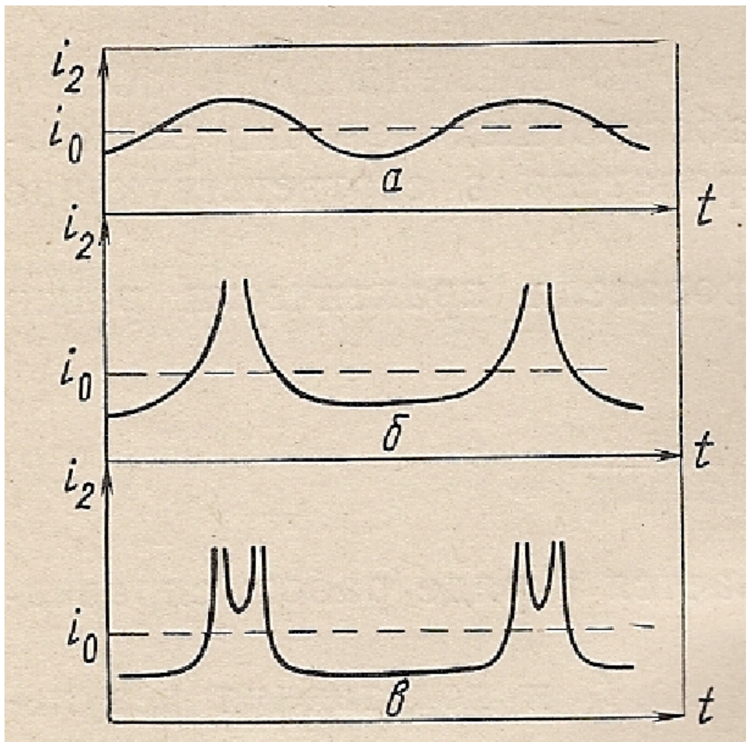
$$i_1 dt_1 = i_2 dt_2$$

Beam current in the second gap

$$i_2 = i_1 \frac{dt_1}{dt_2} = \frac{I}{\frac{dt_2}{dt_1}}$$

Beam current in the second gap as a function of RF phase in the first gap and bunching parameter

$$i_2 = \frac{I}{1 - X\cos\omega t_1}$$



$X < 1$

$X = 1$

$X > 1$

Current in the second gap as a function of time.

4.

Phase of arrival of particle into second gap $$x=\omega t_2-\theta=\omega t_1-X\sin\omega t_1$$

Expansion of the current in the second gap in Fourier series $$i_2(x)=A_o+\sum_{n=1}^{\infty}A_n\cos nx$$

Fourier coefficients $$A_o=\frac{1}{\pi}\int_o^{\pi}i_2(x)dx \qquad A_n=\frac{2}{\pi}\int_o^{\pi}i_2(x)\cos nx\,dx$$

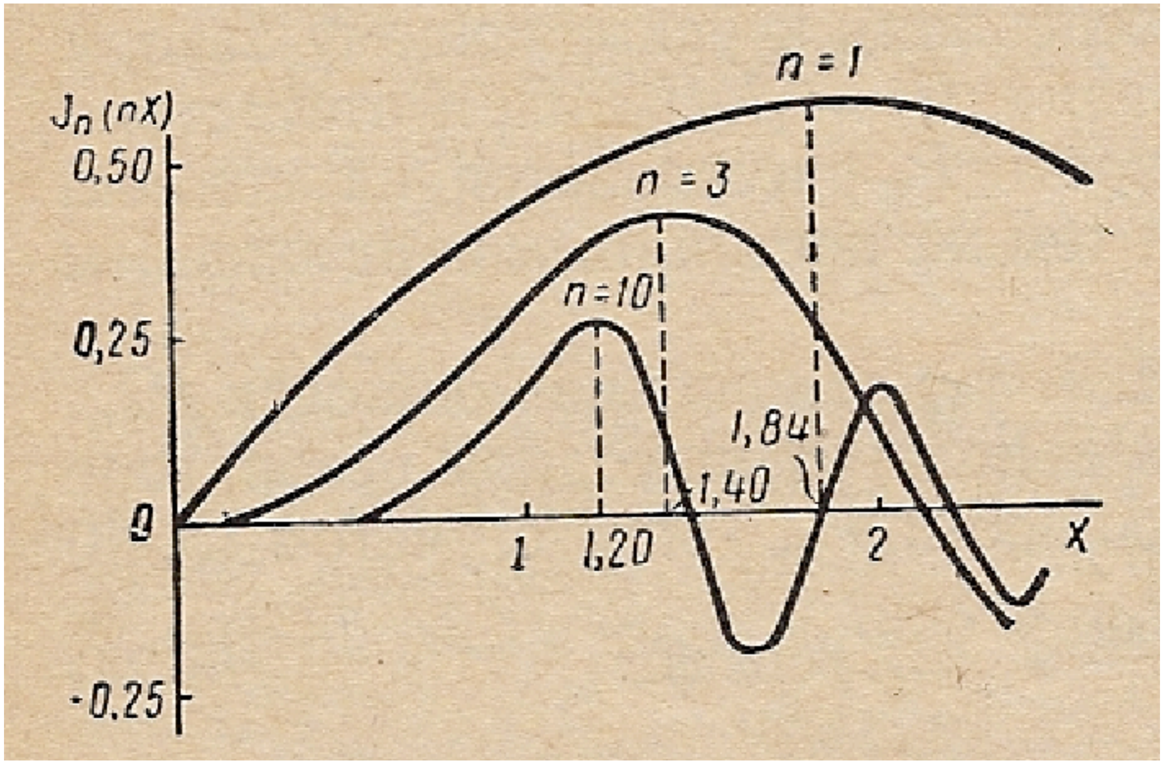
Differentiation of RF phase $$dx=\omega dt_2$$

Constant in Fourier series $$A_o=\frac{1}{\pi}\int_o^{\pi}I\frac{dt_1}{dt_2}\omega\,dt_2=I$$

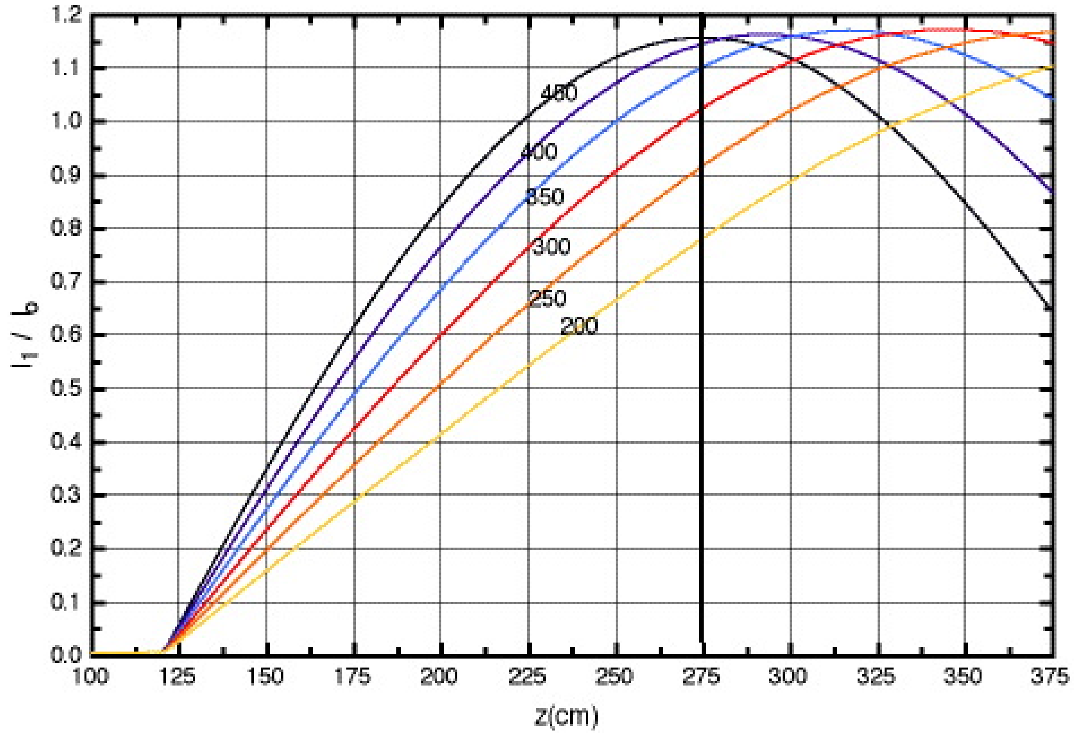
Other coefficients in Fourier series $$A_n=\frac{2I}{\pi}\int_o^{\pi}\cos(n\omega t_1-nX\sin\omega t_1)d\omega t_1=2IJ_n(nX)$$

Bessel function (integral representation) $$J_n(z)=\frac{1}{\pi}\int_o^{\pi}\cos(n\varphi-z\sin\varphi)d\varphi$$

Beam current in the second gap $$i_2(x)=I+2I\sum_{n=1}^{\infty}J_n(nX)\cos nx$$



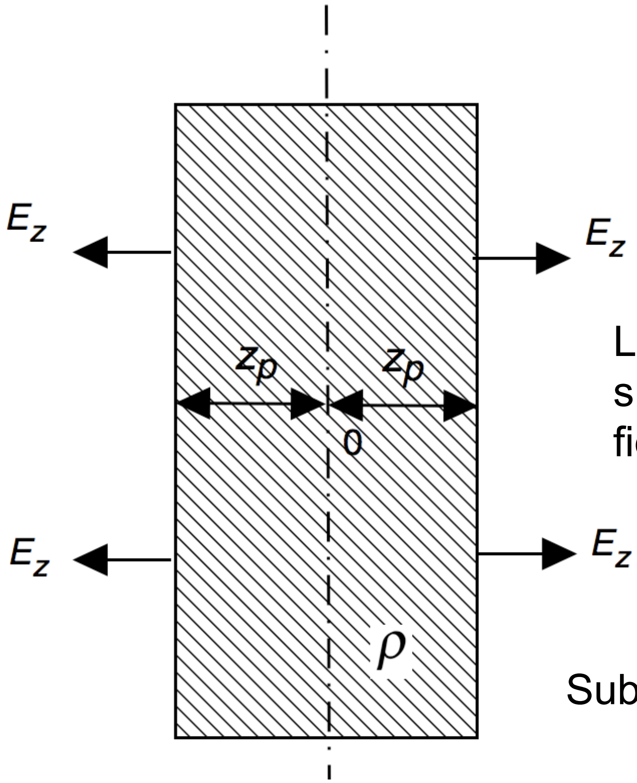
Bessel functions determine amplitude of the fist, third and tenth harmonics of induced current in two-resonator buncher.



The first harmonic of the induced beam current in the second gap as a function of z for different values of voltage at first gap. $\frac{I_1}{I} = 2J_1(X)$

The optimal value of bunching parameter is $X_{opt} = 1.84$.

Beam bunching in presence of space charge forces*



Gauss theorem $$2E_z = \frac{\rho}{\varepsilon_o} 2z_p$$

1D longitudinal space charge field $$E_z = \frac{\rho}{\varepsilon_o} z_p$$

Longitudinal oscillation in presence of space charge field, E_z, and external field E_{ext} $$m\frac{d^2 z_p}{dt^2} = q(E_{ext} - E_z)$$

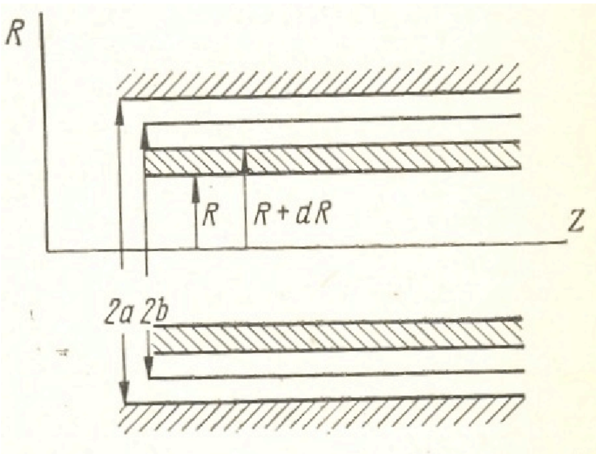
Substitution of space charge field gives: $$\frac{d^2 z_p}{dt^2} + \omega_p^2 z_p = \frac{q}{m} E_{ext}$$

Plasma frequency $$\omega_p = \sqrt{\frac{q\rho}{m\varepsilon_o}} = \frac{2c}{R}\sqrt{\frac{I}{I_c \beta}}$$

Space charge density of the beam $$\rho = \frac{I}{\pi R^2 \beta c}$$

4. * From Yu.A.Katsman, Microwave Devices, Moscow, 1973 (in Russian).

Reduction of beam plasma frequency in presence of conducting tube



Reduced plasma frequency of the beam of radius *R* in the tube of radius *a*

$$\omega_q = \sqrt{F_p}\,\omega_p$$

Plasma frequency reduction factor $F_p = 2.56 \dfrac{J_1^2(2.4\frac{R}{a})}{1 + \dfrac{5.76}{(\frac{\omega a}{v_o})^2}}$

Longitudinal plasma oscillations in tube

$$\frac{d^2 z_p}{dt^2} + \omega_q^2 z_p = 0$$

Longitudinal particle oscillations under space charge forces

$$z_p = B_o \sin\omega_q (t - t_1)$$

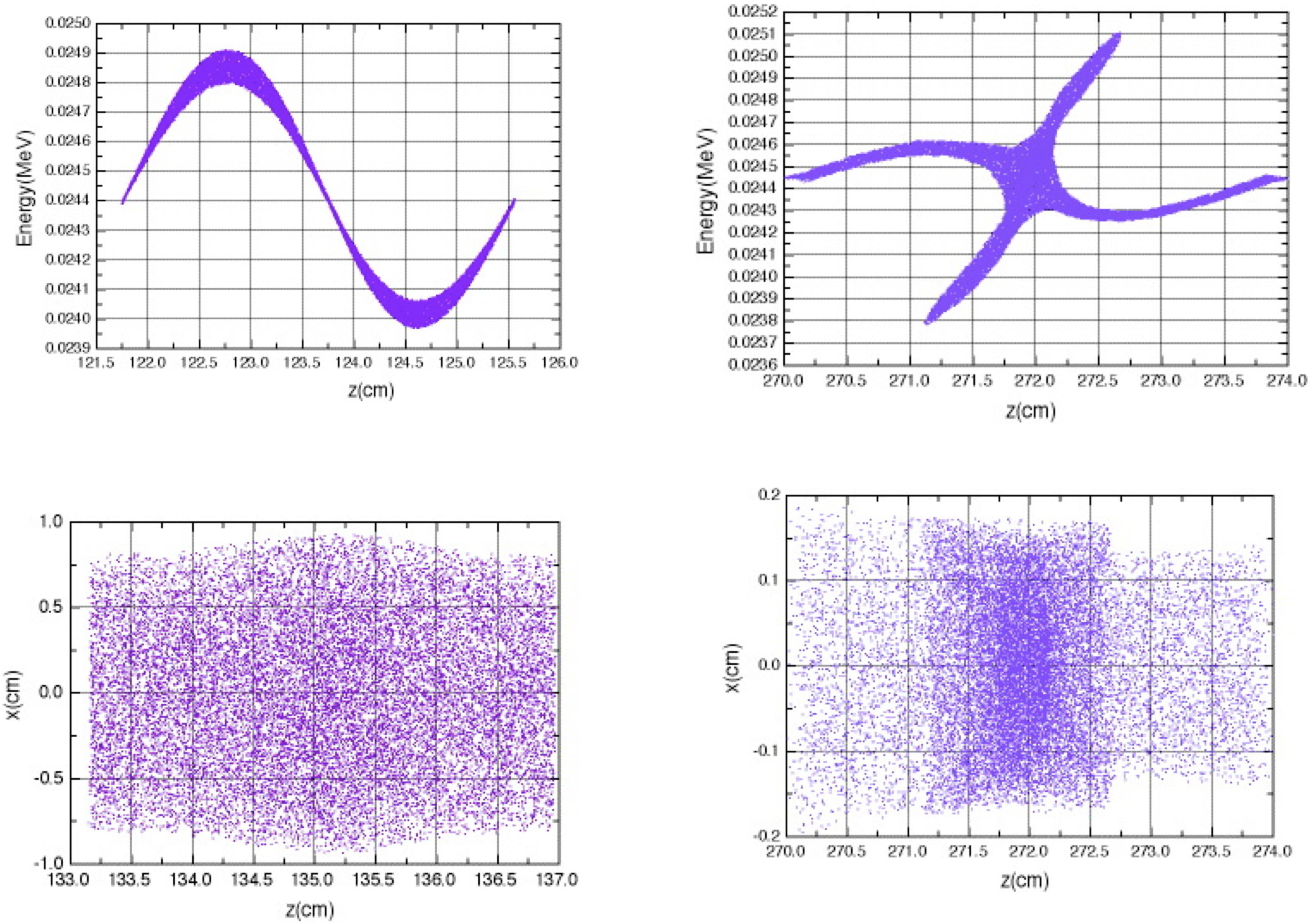
Longitudinal velocity of particle oscillations under space charge forces:

$$\frac{dz_p}{dt} = B_o \omega_q \cos\omega_q (t - t_1)$$

Constant B_o is defined from initial conditions for particle velocity after first RF gap:

$$\frac{dz_p}{dt}(t_1) = B_o \omega_q = v_1 \sin\omega t_1$$

$$B_o = \frac{v_1}{\omega_q} \sin\omega t_1$$



Effect of space charge repulsion on beam bunching.

Finally, particle oscillations under space charge forces in the moving system

$$z_p = \frac{v_1}{\omega_q} \sin \omega_q (t - t_1) \sin \omega t_1$$

Particle drift

$$z = v_o (t_2 - t_1) + z_p$$

$$z = v_o (t_2 - t_1) + \frac{v_1}{\omega_q} \sin \omega_q (t_2 - t_1) \sin \omega t_1$$

Multiply by ω

$$\frac{\omega z}{v_o} = \omega t_2 - \omega t_1 + \frac{\omega v_1}{\omega_q v_o} \sin \omega_q (t_2 - t_1) \sin \omega t_1$$

RF phase in the second gap

$$\omega t_2 - \theta = \omega t_1 - X \sin \omega t_1$$

Modified bunching parameter in presence of space charge forces

$$X = \frac{\omega v_1}{\omega_q v_o} \sin \omega_q (t_2 - t_1)$$

$$X = \frac{U_1 M_1}{2U_o} \left(\frac{\omega z}{v_o}\right) \frac{\sin(\omega_q \frac{z}{v_o})}{\omega_q \frac{z}{v_o}}$$

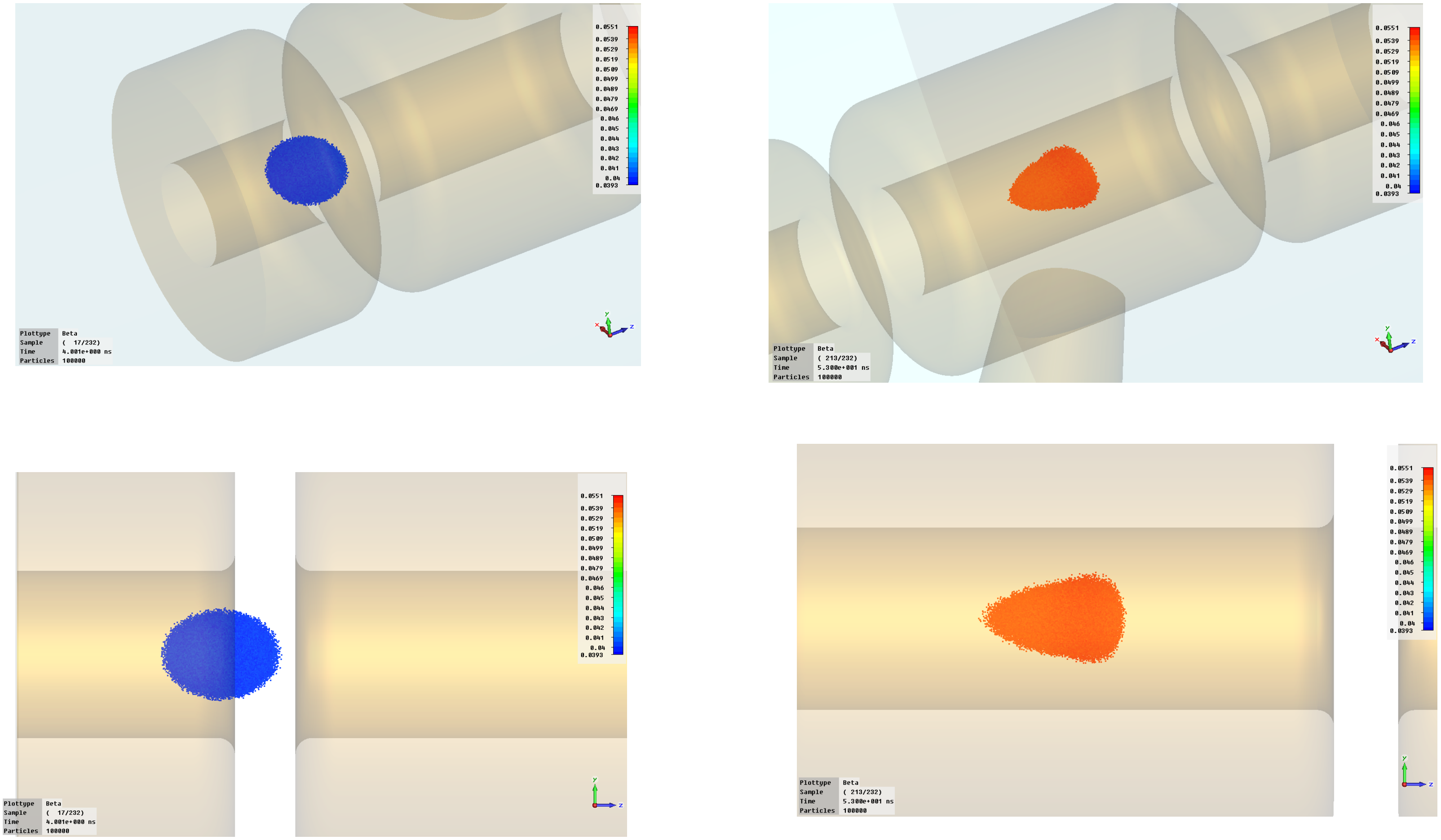
Condition for maximum bunching:

$$\sin(\omega_q \frac{z}{v_o}) = 1 \qquad \omega_q \frac{z}{v_o} = \frac{\pi}{2}$$

4.

$$X_{opt} = \frac{U_1 M_1}{2U_o} \left(\frac{\omega}{\omega_q}\right) \qquad \frac{I_1}{I} = 2J_1(X_{opt})$$

4.8. Space charge dominated bunched beam in RF field*



(Left) initial and (right) final beam distribution in RF field. (Courtesy of Sergey Kurennoy.)

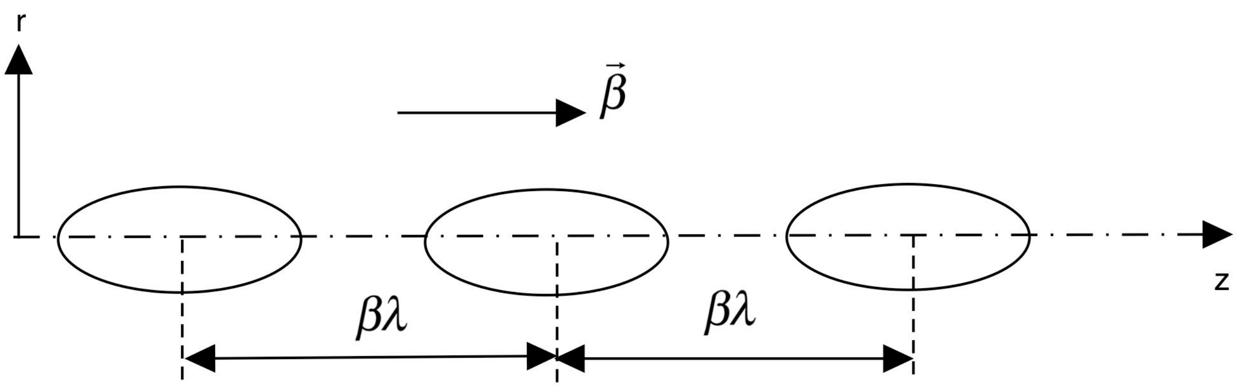
* From Y.B., NIM-A 483 (2002), 611-628.

Consider an intense bunched beam of particles with charge q and mass m, propagating in a continuous focusing channel with an applied accelerating RF field. Beam is bunched at the RF frequency $\omega = 2\pi c/\lambda$. Particle motion is governed by the single-particle Hamiltonian:

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\, U_{ext} + q\, \frac{U_b}{\gamma^2}\,, \tag{5.48}$$

$$U_{ext} = \frac{E}{k_z}\, [\, I_o(\frac{k_z r}{\gamma})\, sin(\varphi_s - k_z \zeta) - sin\varphi_s + k_z \zeta\, cos\varphi_s] + G_t\, \frac{r^2}{2}, \tag{5.49}$$

where U_{ext} is the potential of external field, U_b is the space charge potential of the beam, G_t is the gradient of the focusing field, and r is the particle radius.



4. Sequence of bunches in RF field.

The space charge density distribution of a moving bunched beam has the form $\rho = \rho\ (x, y, z - v_s t)$. The moving bunch creates an electromagnetic field with a scalar potential $U_b = U_b\ (x, y, z - v_s t)$ and a vector potential $\vec{A}_b = \vec{A}_b\ (x, y, z - v_s t)$, which obey the wave equations:

$$\Delta U_b - \frac{1}{c^2}\frac{\partial^2 U_b}{\partial t^2} = -\frac{\rho}{\varepsilon_o}, \qquad (5.50)$$

$$\Delta \vec{A}_b - \frac{1}{c^2}\frac{\partial^2 \vec{A}_b}{\partial t^2} = -\mu_o \vec{j}, \qquad (5.51)$$

where $\vec{j} = \rho \vec{v}_s$ is the current density of the beam. The current density has only longitudinal component

$$j_x = j_y = 0, \qquad j_z = v_s\ \rho\ (x, y, z - v_s t), \qquad (5.52)$$

and, therefore, the vector potential has only a longitudinal component A_z.

In a moving coordinate system where particles are static, the vector potential of the beam is zero, $\vec{A} = 0$. According to the Lorentz transformation, the longitudinal component of the vector potential in the laboratory system is $A_z = \beta_s\ U_b / c$. Therefore, to find solution of the problem it suffice to solve only equation for the scalar potential (5.50). Substitution of the value A_z into the wave equation (5.51) gives the equation for the scalar potential:

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2} = -\frac{1}{\varepsilon_o}\rho\ (x, y, \zeta). \qquad (5.53)$$

Equation (5.53) has to be solved together with the Vlasov equation for the beam distribution function:

$$\frac{df}{dt} = \frac{1}{m\gamma}\left(\frac{\partial f}{\partial x}p_x + \frac{\partial f}{\partial y}p_y + \frac{\partial f}{\partial \zeta}p_z\right) - q\left(\frac{\partial f}{\partial p_x}\frac{\partial U}{\partial x} + \frac{\partial f}{\partial p_y}\frac{\partial U}{\partial y} + \frac{\partial f}{\partial p_z}\frac{\partial U}{\partial \zeta}\right) = 0 \qquad (5.54)$$

where $U = U_{ext} + \gamma^{-2} U_b$ is a total potential of the structure. Eqs (5.53), (5.54) define the self-consistent distribution of a stationary beam which acts on itself in such a way, that this distribution is conserved.

The general approach to find a stationary, self-consistent beam distribution function is to represent it as a function of Hamiltonian $f = f(H)$ and then to solve Poisson's equation. Because the Hamiltonian is a constant of motion for a stationary process, any function of Hamiltonian is also a constant of motion which automatically obeys Vlasov's equation. A convenient way is to use an exponential function $f = f_o \, exp\,(- H / H_o)$:

$$f = f_o \, exp\left(- \frac{p_x^2 + p_y^2}{2\, m\, \gamma H_o} - \frac{p_z^2}{2\, m\, \gamma^3 H_o} - q\,\frac{U_{ext} + U_b \gamma^{-2}}{H_o}\right). \quad (5.55)$$

4.9. Beam equipartitioning in RF field

Consider an important consequence which follows immediately from Eq. (5.55). Let us rewrite the distribution function, Eq. (5.55), as

$$f = f_o \, exp\,(- 2\,\frac{p_x^2 + p_y^2}{p_t^2} - 2\,\frac{p_z^2}{p_l^2} - q\,\frac{U_{ext} + U_b \gamma^{-2}}{H_o}), \qquad (5.56)$$

where $p_t = 2\sqrt{<p_x^2>} = 2\sqrt{<p_y^2>}$ and $p_l = 2\sqrt{<p_z^2>}$ are double root-mean-square (rms) beam sizes in phase space. Transverse, ε_t, and longitudinal, ε_l, rms beam emittances are:

$$\varepsilon_t = 2\,\frac{p_t}{mc}\sqrt{<x^2>} = 2\,\frac{p_t}{mc}\sqrt{<y^2>}, \qquad (5.57)$$

$$\varepsilon_l = 2\,\frac{p_l}{mc}\sqrt{<\zeta^2>}. \qquad (5.58)$$

Taking together Eqs. (5.55) - (5.58), the value of H_o can be expressed as a function of the beam parameters:

$$16 \cdot H_o = \frac{m c^2}{\gamma}\,\frac{\varepsilon_t^2}{<x^2>} = \frac{m c^2}{\gamma}\,\frac{\varepsilon_t^2}{<y^2>} = \frac{m c^2}{\gamma^3}\,\frac{\varepsilon_l^2}{<\zeta^2>}. \qquad (5.59)$$

Equation (5.59) can be rewritten as

$$\frac{\varepsilon_t}{R} = \frac{\varepsilon_l}{\gamma\, l}, \tag{5.60}$$

where $R = 2\sqrt{<x^2>}$ is a beam radius and $l = 2\sqrt{<\zeta^2>}$ is a half-length of the bunch. Equation (5.60) expresses the equipartitioning condition for the beam in a RF field (R.Jameson, IEEE Trans Nucl Sci, NS-28, 2408 (1981)). From the above derivations it is clear, that the equipartitioning is a consequence of stationary nature of the collisionless beam distribution function, Eq. (5.55). If the distribution function is stationary (time independent), equipartitioning is fulfilled. The opposite statement is not valid in the general case: there are an infinitely many distribution functions which obey condition (5.60), but which are not necessarily stationary. To find the stationary distribution function, it is necessary to solve the nonlinear Poisson equation for the unknown space-charge potential of the beam.

Space charge field of the bunch

The space charge density of the beam is obtained as an integral of the beam distribution function over the particle momentum:

$$\rho(x,y,\zeta) = q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dp_x\, dp_y\, dp_z = \rho_o exp\left(-q\,\frac{U_{ext} + U_b\gamma^{-2}}{H_o}\right), \qquad (5.61)$$

where ρ_o is the space charge density in the center of the bunch. The value of ρ_o is unknown at this point due to the unknown space charge potential of the beam, U_b. For further analysis let us introduce an average value of the space charge density, $\bar{\rho}$, which is equal to the density of an equivalent uniformly-charged cylindrical bunch with the same beam radius, R, and the same half-bunch length, l, as that of unknown stationary bunch. The space charge density of the cylindrical bunch, $\bar{\rho} = Q/V$, is

$$\bar{\rho} = \frac{I\lambda}{2\pi R^2 l\, c} \qquad (5.62)$$

where $Q = I\lambda/c$ is the charge of the bunch, $V = \pi R^2 2l$ is the volume of the bunch and I is the beam current. Let us compare the value of $\bar{\rho}$, Eq. (5.62), with that for another distributions. The space charge density of a uniformly populated spheroid with semi-axises R and l is

$$\rho_s = \frac{3\, I\, \lambda}{4\pi\, R^2\, l\, c} = \frac{3}{2}\bar{\rho}. \qquad (5.63)$$

A bunch with the Gaussian distribution

$$\rho = \frac{I\lambda}{(2\pi)^{3/2} c\sqrt{<x^2>}\sqrt{<y^2>}\sqrt{<\zeta^2>}} exp\,(-\frac{x^2}{2<x^2>} - \frac{y^2}{2<y^2>} - \frac{\zeta^2}{2<\zeta^2>}), \qquad (5.64)$$

has a space charge density in its center

$$\rho_G = \frac{8}{(2\pi)^{3/2}} \frac{I\lambda}{c\,R^2 l} = \frac{8}{\sqrt{2\pi}}\bar{\rho}. \qquad (5.65)$$

Since different distributions give similar expressions for the space charge density in the bunch center within the factor of $k \approx 1....3$, one can assume that unknown value of the space charge density ρ_o in the bunch center, Eq. (5.61), also differs from the average value of the space charge density ρ within the same factor:

$$\rho_o = k\,\bar{\rho}. \qquad (5.66)$$

For further derivations introduce dimensionless variables:

$$V_{ext} = \frac{q\,U_{ext}}{H_o}, \qquad V_b = \frac{q\,U_b}{H_o}, \qquad \xi = \frac{r}{a}, \qquad \eta = \frac{\zeta}{a}, \qquad (5.67)$$

where a is a channel radius. Poisson's equation (5.53) in cylindrical polar coordinates becomes

$$\frac{1}{\xi}\frac{\partial V_b}{\partial \xi} + \frac{\partial^2 V_b}{\partial \xi^2} + \frac{\partial^2 V_b}{\partial \eta^2 \gamma^2} = -\,q\,\frac{\rho_o\,a^2}{\varepsilon_o\,H_o}\,exp\,-(V_{ext} + \frac{V_b}{\gamma^2}). \qquad (5.68)$$

Let us introduce a bunching factor, $B \equiv 2l/(\beta\lambda)$. Substitution of ρ_o, Eq. (5.66), and H_o, Eq. (5.59), with the introduced quantity of B into Eq. (5.68) gives:

$$\frac{1}{\xi}\frac{\partial V_b}{\partial \xi} + \frac{\partial^2 V_b}{\partial \xi^2} + \frac{\partial^2 V_b}{\partial \eta^2 \gamma^2} = - \frac{8\ k\ b}{B}(\gamma \frac{a}{R})^2 \ exp - (V_{ext} + \frac{V_b}{\gamma^2}). \tag{5.69}$$

where b is the dimenionless beam brightnes $b \equiv 2IR^2/(\beta\gamma I_c \varepsilon_t^2)$. Equation (5.69) is a nonlinear differential equation for the unknown beam space charge potential, V_b, which appears in the left and right hand side of the equation. The unknown space charge potential of the beam can be represented as a Fourier-Bessel series:

$$V_b = V_o + \sum_{n=o}^{\infty}\sum_{m=1}^{\infty} J_o(\upsilon_{om}\xi)[A_{nm}cos(k_z n\eta a) + B_{nm}sin(k_z n\eta a)], \tag{5.70}$$

where $J_o(\zeta)$ is the Bessel function, υ_{om} is the m-th root of the equation $J_o(\zeta) = 0$. Expansion (5.70) obeys the Dirichlet boundary condition $V_b(1, \eta) = V_o$ at the perfectly conductive surface of the channel and takes into account the periodic nature of potential due to the train of the bunches. To find the first approximation to the solution of Poisson's equation, let us take only the first term in the expansion of the exponential function

$$exp(-V_{ext} - V_b\gamma^{-2}) \approx 1 - V_{ext} - V_b\gamma^{-2}. \tag{5.71}$$

Poisson's equation (5.69) then becomes:

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}[1+ \frac{\upsilon_{om}^2 + (k_z na)^2\gamma^{-2}}{8kb} B(\frac{R}{a})^2]$$

4.

$$J_o(\upsilon_{om}\xi)[A_{nm}cos(k_z n\eta a) + B_{nm}sin(k_z n\eta a)] = (1 - V_{ext})\ \gamma^2 - V_o. \tag{5.72}$$

The space charge potential, Eq. (5.70), is mostly represented by several low-order terms. For example, for the train of uniformly populated cylindrical bunches, the values of Fourier-Bessel coefficients drop quickly with numbers m, n:

$$A_{nm} \sim \frac{1}{n\,\upsilon_{om}\,[\upsilon_{om}^2 + (k_z na)^2 \gamma^{-2}]}. \qquad (5.73)$$

For a space charge dominated beam, $b >> 1$, Eq. (5.72) can be simplified. The expression in square brackets in Eq. (5.72) is

$$1 + \frac{\upsilon_{om}^2 + (k_z na)^2\,\gamma^{-2}}{8\,k\,b}\,B\,(\frac{R}{a})^2 = 1 + \delta, \qquad (5.74)$$

where introduced parameter δ is:

$$\delta = \frac{\upsilon_{om}^2 + (k_z na)^2\,\gamma^{-2}}{8\,k\,b}\,B\,(\frac{R}{a})^2. \qquad (5.75)$$

Low-order roots of the Bessel function are $\upsilon_{01}=2.408$, $\upsilon_{11}=3.832$, $\upsilon_{02}=5.52$. The product of $k_z a$ is usually close to unity:

$$k_z\,a = 2\pi\,(\frac{a}{\beta\lambda}) \approx 1. \qquad (5.76)$$

Taking into account that $B \leq 1$ and $R / a \approx 0.5$, it is easy to see that the value of δ, Eq. (5.75), is much smaller than unity for a high brightness beam. It can be written as

$$\delta \approx \frac{1}{b_\varphi k} << 1, \tag{5.77}$$

where b_φ is a dimensionless beam brightness of the bunched beam:

$$b_\varphi = \frac{b}{B}\left(\frac{a}{R}\right)^2 = \frac{2}{\beta\gamma}\frac{I}{I_c B}\left(\frac{a}{\varepsilon_t}\right)^2 = \frac{2}{\beta\gamma}\frac{I_{peak}}{I_c}\left(\frac{a}{\varepsilon_t}\right)^2, \tag{5.78}$$

and $I_{peak} = I / B$ is the peak value of the bunched beam current. Therefore, the expression, Eq. (5.74), can be taken out of the sum in Eq. (5.72). With this approximation, Eq. (5.72) becomes:

$$(1+ \delta)(V_b - V_o) = (1 - V_{ext})\gamma^2 - V_o. \tag{5.79}$$

Let us define the constant V_o in such a way that the total potential of the structure vanishes at the bunch center:

$$V_{ext}(0, 0) + \frac{V_b(0, 0)}{\gamma^2} = 0. \tag{5.80}$$

The external potential is equal to zero at the beam center $V_{ext}(0,0) = 0$ (see Eq. (5.49)) therefore the condition (5.80) gives $V_b(0,0) = 0$. Substitution of $V_{ext}(0,0)$, $V_b(0,0)$ into Eq. (5.79) defines constant $V_o = - \gamma^2/\delta$. Then, from Eq. (5.79) the self-consistent space charge dominated beam potential is:

$$V_b = - \frac{\gamma^2}{1 + \delta} V_{ext}. \tag{5.81}$$

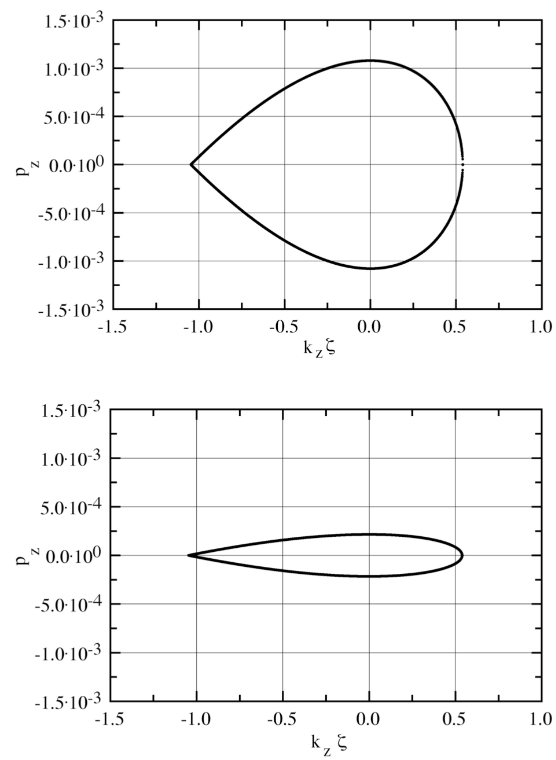
Taking the first approximation to the space charge potential of the beam, Eq. (5.81), the Hamiltonian corresponding to the self-consistent bunch distribution is as follows:

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\, (\frac{\delta}{1+\delta})\, U_{ext}. \tag{5.88}$$

Equation (5.88) indicates that in the presence of an intense, bright bunched beam ($\delta << 1$) the stationary longitudinal phase space of the beam becomes narrow in momentum spread, while the phase width of the distribution remains the same in the first approximation.

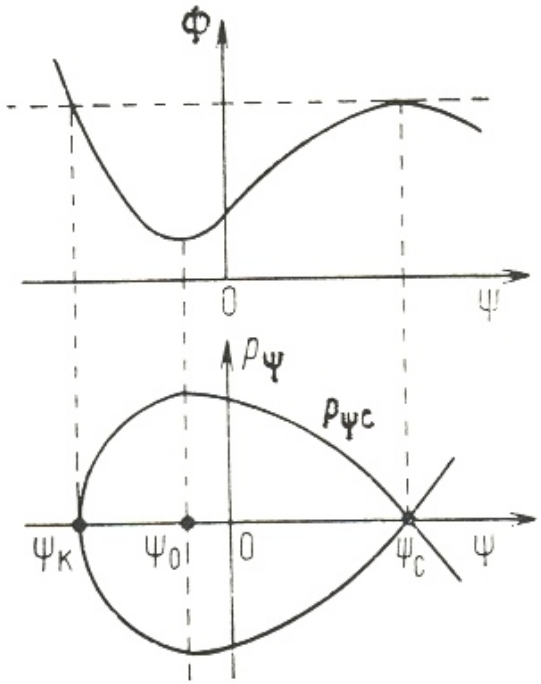
(a)

(b)

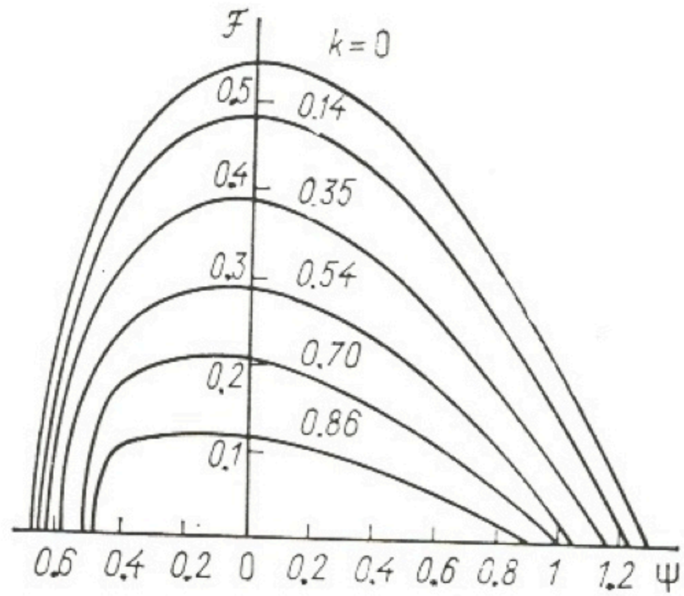


Separatrix of longitudinal motion: (a) low brightness beam, $b<<1$, and (b) high brightness beam, $b>>1$.

More precise analysis based on numerical solution of equation for beam potential indicates that synchronous phase is shifted in space charge dominated beam and phase width of the bunch also decreases but much slower than vertical size of the separatrix.



The potential function and separatrix of the beam with high space-charge density (from Kapchinsky, 1985).



The separatrix shape for different values of space charge parameter (from Kapchinsky, 1985).

Stationary bunch profile

The self consistent space charge density distribution of a matched beam can be found from Poisson's equation:

$$\rho(r,\zeta) = - \varepsilon_o \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_b}{\partial r}\right) + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2}\right]. \qquad (5.89)$$

Substitution of Eq. (5.84) into Eq. (5.89) gives the stationary particle density distribution inside the bunch:

$$\rho(r,\zeta) = 2\gamma^2 G_t\, \varepsilon_o \left\{ 1 - \frac{\delta}{\sqrt{(1+\delta)^2 - 2\delta V_{ext}}} - \frac{\delta^2}{32\gamma}\frac{\varepsilon_t^2}{<x^2>}\left(\frac{mc^2}{G_t q a^2}\right)\frac{\left(\frac{\partial V_{ext}}{\partial \xi}\right)^2 + \left(\frac{\partial V_{ext}}{\gamma \partial \eta}\right)^2}{[(1+\delta)^2 - 2\delta V_{ext}]^{3/2}} \right\}. \qquad (5.90)$$

For a high brightness beam parameter $\delta << 1$, therefore the space charge density is close to constant within the bunch:

$$\rho(r,\zeta) \approx 2\frac{\gamma^2}{1+\delta} G_t\, \varepsilon_o. \qquad (5.91)$$

From Eq. (5.81) it follows, that, in the first approximation, the space charge potential of the beam is the same function of coordinates, as the external potential, Eq. (5.49), with the opposite sign. Therefore, equation $U_{ext}(r, \zeta) = const$ gives the family of equipotential lines of the space charge field of the beam:

$$I_o\left(\frac{k_z r}{\gamma}\right) sin(\varphi_s - k_z\zeta) - sin\varphi_s + k_z\zeta cos\varphi_s + \frac{G_t k_z}{2E} r^2 = const. \qquad (5.92)$$

In the general case, the bunch boundary is not an equipotential surface; therefore Eq. (5.92) does not coincide with bunch profile. To find the self-consistent bunch profile, consider a uniformly populated bunch with boundary $R(\zeta)$, defined by the following nonlinear equation

$$I_o\left(\frac{k_z R}{\gamma}\right)\sin(\varphi_s - k_z\zeta) - \sin\varphi_s + k_z\zeta\cos\varphi_s + C(k_z R)^2 = const. \tag{5.93}$$

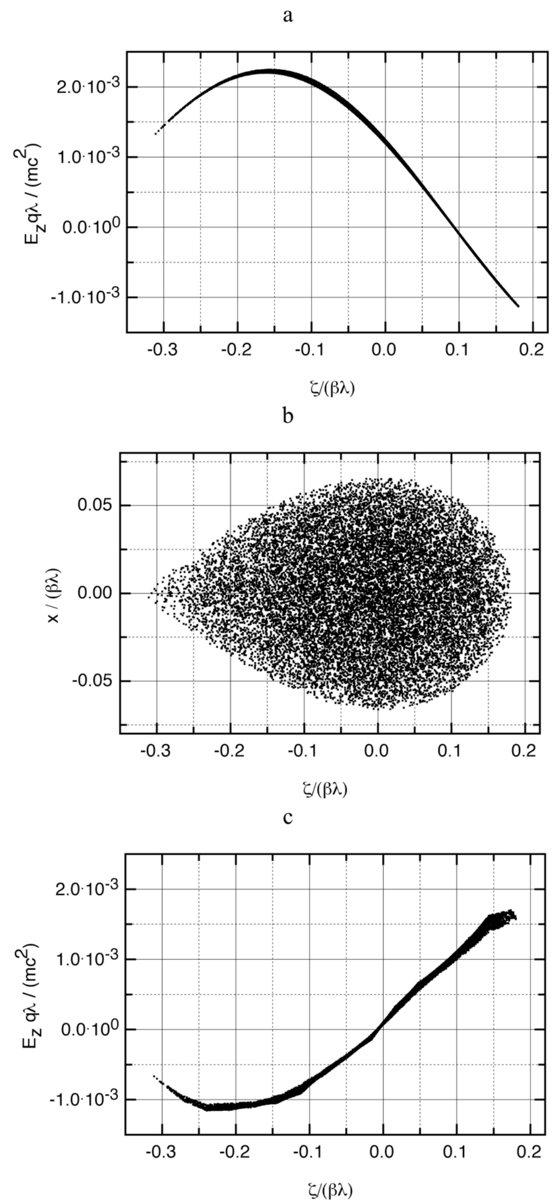
Equation (5.93) differs from Eq. (5.92) by the inserted parameter C, which will be used to adjust the bunch shape in such a way, that the self field of the bunch will be approximately opposite to the external field. The value of the constant in right side of Eq. (5.93) can be determined from the condition, that the longitudinal bunch size is, in the first approximation, the same as for the zero - current mode. Therefore, at $R(\zeta) = 0$ one boundary of the bunch is $k_z\zeta = 2\varphi_s$ and the value of the constant is

$$const = 2\varphi_s \cos\varphi_s - 2 \sin\varphi_s. \tag{5.94}$$

Substitution of Eq. (5.94) into Eq. (5.93) gives expression for the expected bunch profile:

$$I_o\left(\frac{k_z R}{\gamma}\right)\sin(\varphi_s - k_z\zeta) + \sin\varphi_s - (2\varphi_s - k_z\zeta)\cos\varphi_s + C(k_z R)^2 = 0. \tag{5.95}$$

Figure below illustrates a uniformly populated bunch with boundary, Eq. (5.95). The bunch profile in real space resembles a separatrix shape in longitudinal phase space. The space charge field of the bunch in longitudinal direction is a nonlinear function of the coordinate ζ, and approximately repeats the RF field inside the bunch with negative sign. In the transverse direction the space charge forces are close to a linear function of the coordinate and compensate for external focusing forces. It is clearly seen that the high brightness beam shields itself from the external field (Debye shielding). The shape and space charge forces of the bunch depend on the parameter C. Consider those dependencies in more detail.



4.

Stationary self-consistent particle distribution in RF field, φ_s = -60^o, C=3.8: (a) RF field, (b) particle distribution, (c) space charge field of the beam.

For a long bunch, $\beta\lambda >> R_{max}$, the Bessel function can be approximated as $I_o(\chi) \approx 1 + \chi^2/4$, and equation (5.95) for bunch boundary becomes:

$$R(\zeta) = \frac{\beta\lambda}{2\pi}\sqrt{\frac{(2\varphi_s - k_z\zeta)\cos\varphi_s - \sin\varphi_s - \sin(\varphi_s - k_z\zeta)}{C + \frac{1}{4\gamma^2}\sin(\varphi_s - k_z\zeta)}} . \qquad (5.96)$$

Transverse bunch size, R_{max}, is determined from the equation $\partial R(\zeta)/\partial\zeta = 0$, which has an approximate solution $\zeta(R_{max}) \approx 0$. Substitution of this value into Eq. (5.96) gives for maximum beam size:

$$R_{max} = \frac{\beta\lambda}{2\pi}\sqrt{\frac{2(\varphi_s\cos\varphi_s - \sin\varphi_s)}{C + \frac{1}{4\gamma^2}\sin\varphi_s}} . \qquad (5.97)$$

The exact value of $\zeta(R_{max})$ is slightly positive and the maximum value of the bunch profile is shifted to the head of the bunch. The phase length of a separatrix is approximately $3\varphi_s$ and the full bunch length is $l_b = \beta\lambda 3\varphi_s/(2\pi)$. The ratio of transverse to longitudinal bunch sizes for a given value of synchronous phase, φ_s, is:

$$\frac{R_{max}}{l_b} = \frac{1}{3|\varphi_s|}\sqrt{\frac{2(\varphi_s\cos\varphi_s - \sin\varphi_s)}{C + \frac{1}{4\gamma^2}\sin\varphi_s}} . \qquad (5.98)$$

Let us compare the space charge potential of the bunch with that of external RF filed. Consider for simplicity a non-relativistic case. The potential of an arbitrary charge distribution at the point ζ_o at the axis is:

$$U_b(\zeta_o, 0) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho\, dV}{|\vec{r}|} =$$

$$= \frac{1}{4\pi\varepsilon_o} \int_0^{R(\zeta)} \int_0^{2\pi} \int_{\zeta_{min}}^{\zeta_{max}} \frac{\rho(r, \zeta')\; r\, dr\; d\zeta'\, d\varphi}{\sqrt{r^2 + (\zeta' - \zeta_o)^2}}. \qquad (5.99)$$

Let us use the RF phase $\psi = -\, k_z\zeta$ instead of longitudinal coordinate ζ. After integration in Eq. (5.99) over radius and azimuth angle, the beam potential is:

$$U_b(\psi_o, 0) = U_o \int_{\psi_{min}}^{\psi_{max}} [\sqrt{k_z^2 R^2(\psi) + (\psi - \psi_o)^2} - \sqrt{(\psi - \psi_o)^2}\,]\, d\psi. \qquad (5.100)$$

For typical values of the parameter $C = 1....5$, the bunch profile, Eq. (5.96), can be approximated as follow:

$$(k_z R)^2 \approx \frac{1}{C} [(\psi + 2\varphi_s)\, cos\varphi_s - sin\varphi_s - sin\,(\psi + \varphi_s)]. \qquad (5.101)$$

Substitution of Eq. (5.101) into Eq. (5.100) gives the space charge potential of the bunch at the arbitrary axis point ψ_o:

$$U_b(\psi_o,0)=U_o\int_{\varphi_s}^{-2\varphi_s}[\sqrt{\psi^2+\psi(\frac{\cos\varphi_s}{C}-2\psi_o)+\frac{2\varphi_s\cos\varphi_s-\sin\varphi_s-\sin(\varphi_s+\psi)}{C}+\psi_o^2}$$

$$-\sqrt{(\psi-\psi_o)^2}]d\psi. \qquad (5.102)$$

Because the value of the synchronous phase in an RF field is negative $\varphi_s<0$, integration in (5.102) has to be performed in the limits of (φ_s, $-2\varphi_s$). In Fig. 5.4 results of space charge potential of the bunch, Eq. (5.102) are presented. Also, an inverse autophasing potential is given:

$$V(\psi) = -U_{ext}(\psi,0) = -\frac{E}{k_z}[\sin(\psi+\varphi_s) - \psi\cos\varphi_s]. \qquad (5.103)$$

It is clear that the values of both potentials are close to each other. Therefore, a uniformly populated bunch with boundary, Eq. (5.95), compensates for the "restoring" autophasing force inside the bunch, which indicates a good approximation of the bunch boundary by Eq. (5.95).

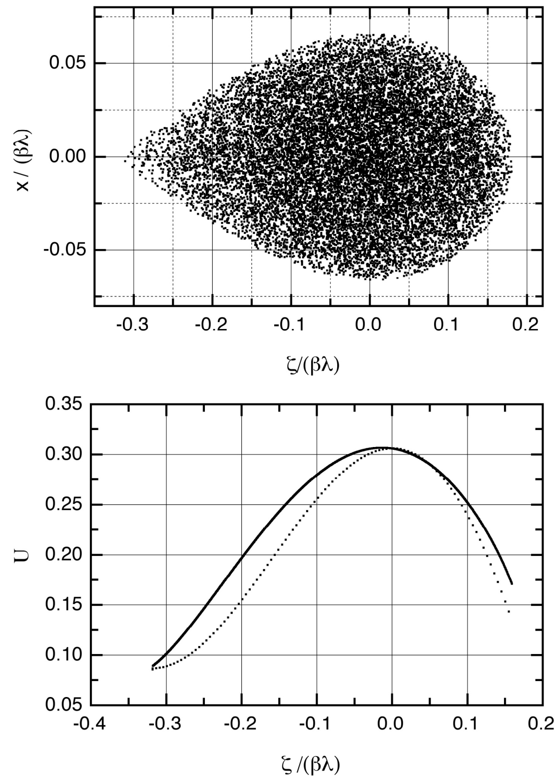


Fig. 5.4. Comparison of potential functions of the bean and RF field: (dotted line) space charge potential of bunched beam distribution at the axis, $\varphi_s = -60^o$, $C = 3.8$; (solid line) inverse external potential at the axis, $-U_{ext}(\psi,0)$.

Parameter C can be expressed as a function of ratio of transverse, G_t^b, and longitudinal, G_z^b, gradients of space charge forces inside the bunch

$$C = C\left(\frac{G_t^b}{G_z^b}\right) \tag{5.104}$$

Figs. 5.5 and 5.6 illustrate dependencies, Eq. (5.104), for different values of synchronous phase and beam energy. Components of electric field of a relativistic bunch in a moving frame, $E_x^{'}$, $E_y^{'}$, $E_z^{'}$ were calculated via numerical solution of the Poisson's equation and then the Lorentz transform was applied to get components of electric field, E_x, E_y, E_z, in the laboratory system:

$$E_x = \gamma E_x^{'}, \qquad E_y = \gamma E_y^{'}, \qquad E_z = E_z^{'} . \tag{5.105}$$

In the laboratory system, the transverse field was reduced by the factor of γ^{-2} due to self magnetic field of the beam:

$$F_x = E_x - v_z B_y = \frac{E_x}{\gamma^2}, \qquad F_y = E_y + v_z B_x = \frac{E_y}{\gamma^2} . \tag{5.106}$$

Gradients of the space charge field were calculated as derivatives of space charge forces in the vicinity of the synchronous phase:

$$G_t^b = \frac{\partial F_x}{\partial x} = \frac{1}{\gamma^2}\frac{\partial E_x}{\partial x}, \qquad G_z^b = \frac{\partial F_z}{\partial \zeta} = \frac{\partial E_z}{\partial \zeta} . \tag{5.107}$$

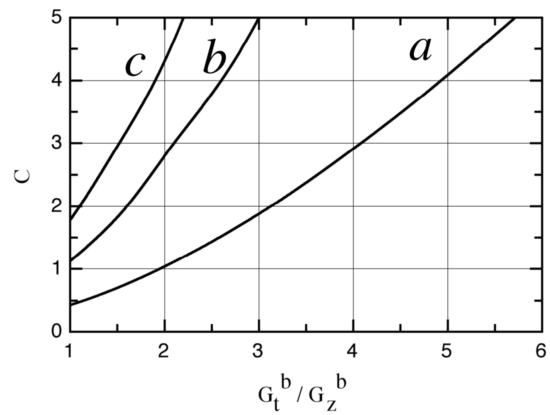


Fig. 5.5 Coefficient C in bunch shape for φ_s = -30° as a function of ratio of transverse and longitudinal gradients of space charge field of the beam: a) γ= 1, b) γ= 3, c) γ= 6.

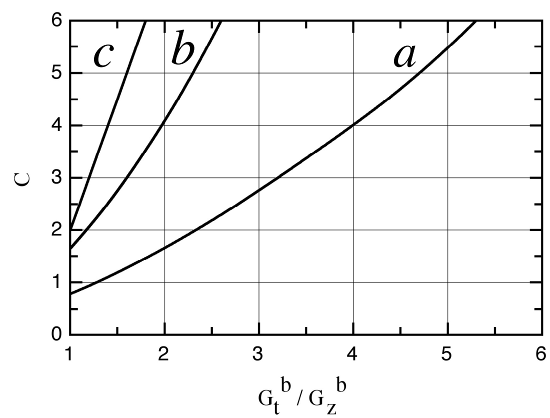


Fig. 5.6 Coefficient C in bunch shape for φ_s = -60° as a function of ratio of transverse and longitudinal gradients of space charge field of the beam: a) γ= 1, b) γ= 3, c) γ= 6.

According to Eq. (5.81), the space charge field of a stationary bunch compensates for the external accelerating and focusing field within the bunch. Therefore, if space charge forces are known, the opposite field defines the required external filed. Gradients of external field are calculated from Eq. (5.49) in the vicinity of synchronous phase utilizing expansions

$$sin(\varphi_s - k_z\zeta) \approx sin\varphi_s - (k_z\zeta)cos\varphi_s - \frac{1}{2}(k_z\zeta)^2 sin\varphi_s, \quad k_z\zeta << 1, \tag{5.108}$$

$$I_o(\frac{k_z r}{\gamma}) \approx 1 + \frac{1}{4}(\frac{k_z r}{\gamma})^2. \tag{5.109}$$

Substitution of Eqs. (5.108), (5.109) into Eq. (5.49) gives for the external potential:

$$U_{ext} = G_z\frac{\zeta^2}{2} + G_t\frac{r^2}{2}[1 - \frac{G_z}{2\gamma^2 G_t}\frac{sin(\varphi_s - k_z\zeta)}{sin\varphi_s}] \approx G_z\frac{\zeta^2}{2} + G_{t,\,eff}\frac{r^2}{2}, \tag{5.110}$$

where G_z is the longitudinal gradient of external field

$$G_z = 2\pi\frac{E\,|sin\ \varphi_s|}{\beta\lambda}. \tag{5.111}$$

and $G_{t,\,eff}$ is the effective transverse gradient of external field, depressed due to RF defocusing:

$$G_{t,\,eff} = G_t(1 - \frac{G_z}{2\,\gamma^2 G_t}). \tag{5.112}$$

Taking into account Eq. (5.81), one finds the relationships between gradients of space charge field and that of external field are

$$G_z^b = - \left(\frac{1}{1+\delta}\right) \frac{2\pi E \left|\sin \varphi_s\right|}{\beta\lambda}, \tag{5.113}$$

$$G_t^b = - \left(\frac{1}{1+\delta}\right) \left[G_t - \frac{\pi E \left|\sin \varphi_s\right|}{\gamma^2 \beta\lambda}\right]. \tag{5.114}$$

Eqs. (5.113), (5.114) together with dependencies, presented in Figs. 6, 7, uniquely define the shape of the stationary bunch for given values of the accelerating field, E, focusing gradient, G_t, synchronous phase, φ_s, wavelength, λ, and beam energy, γ.

4.10. Maximum beam current

The volume of the bunch is defined by

$$V = \pi \int_{z_{min}}^{z_{max}} R^2(\zeta)\, d\zeta = \frac{\beta\lambda}{2} \int_{\varphi_s}^{-2\varphi_s} R^2(\psi)\, d\psi. \qquad (5.115)$$

For a long bunch, $\beta\lambda >> R_{max}$, an approximate bunch boundary, $R(\psi)$, is determined by Eq. (5.101). Integration in Eq. (5.115) gives for the bunch volume:

$$V = \frac{(\beta\lambda)^3}{8\pi^2 C}(3\varphi_s \sin\varphi_s - \frac{9}{2}\varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s). \qquad (5.116)$$

The total charge of the bunch is $Q = \rho \cdot V$ and the beam current, $I = \frac{Q}{2\pi}\omega$, is

$$I = \frac{I_{max}}{1+\delta}, \qquad (5.117)$$

$$I_{max} = I_c\left(\frac{\beta^3\gamma^2}{16\pi^3 C}\right)\left(\frac{G_t q\lambda^2}{mc^2}\right)[3\varphi_s \sin\varphi_s - \frac{9}{2}\varphi_s^2\cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s], \qquad (5.118)$$

where I_{max} is the maximum beam current for an infinitely bright beam. The expression in square brackets in Eq. (5.118) is close to the cubic function of the synchronous phase, φ_s^3, (see Fig. 8), which exhibits that the maximum beam current is proportional to the cube of the synchronous phase. It is in qualitative agreement with analysis, based on the well-known ellipsoidal approximation to a bunched beam.

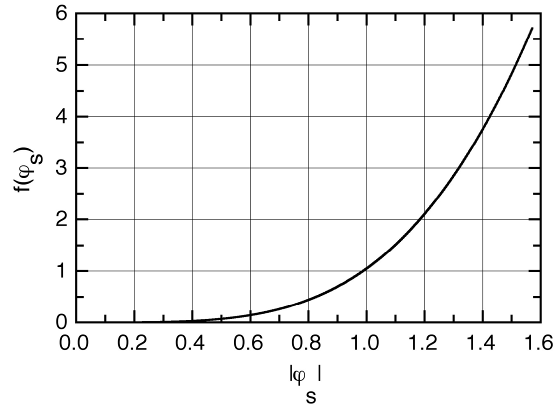


Fig. 5.7. Function $f(\varphi_s) = 3\varphi_s \sin\varphi_s - \frac{9}{2}\varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s$ in maximum beam current, Eq. (5.118).

Substitution of Eq. (5.78), into Eq. (5.117) gives an explicit expression for the beam current:

$$I = I_{max}\,(1 - \frac{\varepsilon_t^2}{\alpha^2}), \tag{5.119}$$

where α defines the normalized acceptance of the channel in presence of transverse focusing and RF fields:

$$\alpha = a\sqrt{\frac{\beta^2\gamma}{8\pi^3 BC}\left(\frac{G_t q\lambda^2}{mc^2}\right)\left[3\varphi_s \sin\varphi_s - \frac{9}{2}\varphi_s^2\cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s\right]}\,. \tag{5.120}$$

Eq. (5.118) gives a unique expression for the beam current limit (without separate transverse and longitudinal limits) for every combination of the values of E, G_t, φ_s and λ.

4.

Comparison with ellipsoidal model

Let us discuss the applicability of the well-known approximation of the bunch by uniformly populated ellipsoid. In the derivations of the self-consistent solution of the beam distribution resulting in Eq. (5.81) no assumptions were made regarding the external potential. Therefore, Eq. (5.81) is valid for an arbitrary external field. In the vicinity of the synchronous particle, where external forces are approximately linear functions of coordinates, the external potential is given by Eq. (5.110). Substitution of Eq. (5.110) into Eq. (5.81) gives for potential of a stationary bunch:

$$U_b = -\frac{\rho}{2\varepsilon_o G_t}\left(G_z \frac{\zeta^2}{2} + \frac{G_{t,\,eff}}{2} r^2\right), \qquad (5.121)$$

where ρ is given by Eq. (5.91). Potential, Eq. (5.121), corresponds to a uniformly populated ellipsoid. In a moving system of coordinates, the potential of the ellipsoid, U_b', with space charge density $\rho' = \rho/\gamma$ is given by

$$U_b' = -\frac{\rho'}{2\varepsilon_o}\left(M\zeta'^2 + \frac{1-M}{2} r^2\right), \qquad (5.122)$$

where $\zeta' = \zeta\gamma$ is the longitudinal deviation from the center of ellipsoid and M is the function of semi-axes of an ellipsoid:

$$M(R, \gamma l) = \frac{R^2 \gamma l}{2}\int_o^\infty \frac{ds}{(R^2 + s)\,(\gamma^2 l^2 + s)^{3/2}}. \qquad (5.123)$$

After transformation to laboratory system, the beam potential, $U_b = \gamma U_b'$, is

$$U_b = -\frac{\rho}{2\varepsilon_o}\left[M\gamma^2\zeta^2 + \frac{1-M}{2} r^2\right]. \qquad (5.124)$$

Comparison of Eq. (5.121) and Eq. (5.124) gives for the coefficient M

$$M(R, \gamma l) = \frac{G_z}{2\,\gamma^2 G_t}. \tag{5.125}$$

The coefficient M, Eq. (5.125), and the space charge density ρ, Eq. (5.91), define a family of ellipsoidal bunches with the same ratio of semi-axes R/l, which are in equilibrium with the external field. Taking into account that the volume of an ellipsoid is $V = (4/3)\pi\, R^2 l$, the maximum bunched beam current, $I_{max} = \rho V \omega/(2\pi)$, which can be carried by an ellipsoid is

$$I_{max} = I_c \frac{2}{3}\gamma^2 \left(\frac{R^2 l}{\lambda^3}\right)\left(\frac{G_t\, q\, \lambda^2}{m\, c^2}\right). \tag{5.126}$$

Since a bunch with current, Eq. (5.126), completely cancels the external field, expression (5.126) gives both a transverse and longitudinal current limit. Let us substitute the gradient of the focusing field, G_t, by the value of the zero-current phase advance, σ_o, of betatron oscillations per period $S = N\beta\lambda$ of a pure focusing structure (without RF field):

$$\sigma_o = \sqrt{\frac{q\, G_t}{m\gamma}}\,\frac{S}{\beta c}. \tag{5.127}$$

In presence of a RF field, the effective focusing gradient is $G_{t,\,eff} = G_t(1 - M)$, see Eqs. (5.112) (5.125). Therefore, the zero-current phase advance per period, $\sigma_{o,t}$, including both the focusing and RF defocusing term is defined by:

$$\sigma_{o,t}^2 = \sigma_o^2\,(1 - M). \tag{5.128}$$

The phase width of the bunch, which contains most of the particles, can be approximately taken as $2\varphi_s$ and, therefore, half of the bunch length is

$$l = \beta\lambda\varphi_s/(2\pi). \qquad (5.129)$$

Substitution of expressions (5.127) - (5.129) into Eq. (5.126) gives for the current limit

$$I_{max} = \frac{4}{3}\frac{mc^2}{Z_o q}\beta\gamma^3 \frac{\varphi_s \sigma_{ot}^2}{(1 - M) N^2}\left(\frac{R}{\lambda}\right)^2, \qquad (5.130)$$

where $Z_o = (c\varepsilon_o)^{-1} = 376.73\ \Omega$ is the impedance of free space. Expression (5.130) is the well known transverse current limit. Let us show that Eq. (5.126) gives also longitudinal current limit. Substitution of the parameter M, Eq. (5.125), and the amplitude of the accelerating field E from Eq. (5.111) into Eq. (5.126) gives for current limit:

$$I_{max} = \frac{8\pi^2}{3Z_o}\frac{E\sin\varphi_s}{\beta M}\frac{R^2 l}{\lambda^2}, \qquad (5.131)$$

which is the well-known expression for longitudinal current limit in a RF field. Usually the parameter M can be approximated as $M \approx R/(3\gamma l)$. With that approximation the current limit, Eq. (5.131), is:

$$I_{max} = \frac{2\beta\gamma}{Z_o} E\varphi_s^2 \left|\sin\varphi_s\right| R. \qquad (5.132)$$

For small absolute values of synchronous phase one can assume $|\sin\varphi_s| \approx |\varphi_s|$, and the current limit, Eq. (5.132), is proportional to the cube of synchronous phase [1, 2, 6], which is consistent with derivations of Section 6. Analysis shows that approximation of the bunched beam by an uniformly populated ellipsoid is valid for small bunches, $R << \beta_s\lambda$, $l << \beta_s\lambda$, while more general analysis results in a bunch shape, described by Eq. (5.95).

4.