Measuring Bunch Length Using Fluctuations in Synchrotron Radiation

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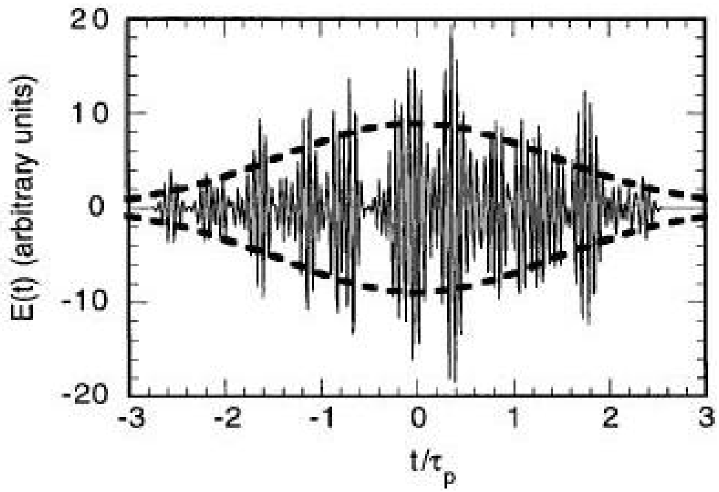
Introduction

- Bunch lengths are getting shorter:
  - 30 ps in typical rings
  - $<$ 10 ps with special low-momentum-compaction lattices
  - 10 to 100 fs in new linac-based light sources (LCLS at SLAC)
- Fastest streak camera has a resolution of 200 fs/pixel.
  - Also expensive and complex for a routine monitor.
- Various new techniques have been devised.
- A technically simple, but subtle, scheme (Zolotorev and Stupakov, 1996) studies the statistics of single-bunch emission, either examining:
  - Turn-to-turn variations in the energy in a narrow band, or
  - Single-shot variations in the spectrum

Electric Field of a Bunch

- The electrons in the bunch are randomly distributed:
  - Normalized distribution $f(t)$: $\int_{-\infty}^{\infty} f(t)dt = 1, \quad f(t)$ is real
  - Characteristic time duration σ_t
  - Later we will use a Gaussian: $$f(t) = \frac{1}{\sqrt{2\pi}\sigma_t} \exp\left(-\frac{t^2}{2\sigma_t^2}\right)$$
- Electric field is the sum of the fields of the $N >> 1$ electrons:
$$E(t) = \sum_{k=1}^{N} \mathrm{e}(t - t_k)$$
- Fourier transform of the field:
$$\hat{E}(\omega) = \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt = \hat{\mathrm{e}}(\omega) \sum_{k=1}^{N} e^{i\omega t_k}$$
- The total field is noisy.
  - ê is smooth; the noise in Ê comes from the random spacing of the t_k.



Energy Radiated by the Bunch

- The energy radiated by the bunch is:

$$
\begin{aligned}
W &= \int_{-\infty}^{\infty} \left|E(t)\right|^2 dt \\
&= \int_{-\infty}^{\infty} dt \frac{1}{2\pi} \sum_{k=1}^{N} \int_{-\infty}^{\infty} d\omega\, \hat{e}(\omega) e^{-i\omega(t-t_k)} \frac{1}{2\pi} \sum_{l=1}^{N} \int_{-\infty}^{\infty} d\omega' \hat{e}^*(\omega') e^{i\omega'(t-t_l)} \\
&= \frac{1}{(2\pi)^2} \sum_{k,l=1}^{N} \iiint d\omega d\omega' dt\, \hat{e}(\omega) \hat{e}^*(\omega') e^{i(\omega t_k - \omega' t_l)} e^{-i(\omega-\omega')t} \\
&= \frac{1}{2\pi} \sum_{k,l} \int d\omega \left|\hat{e}(\omega)\right|^2 e^{i\omega(t_k - t_l)}
\end{aligned}
$$

  - We used $|E|^2$ for power to simplify notation.
  - We made use of the identity: $\int_{-\infty}^{\infty} e^{i\omega t} dt = 2\pi\delta(\omega)$

Mean, Variance, and Standard Deviation

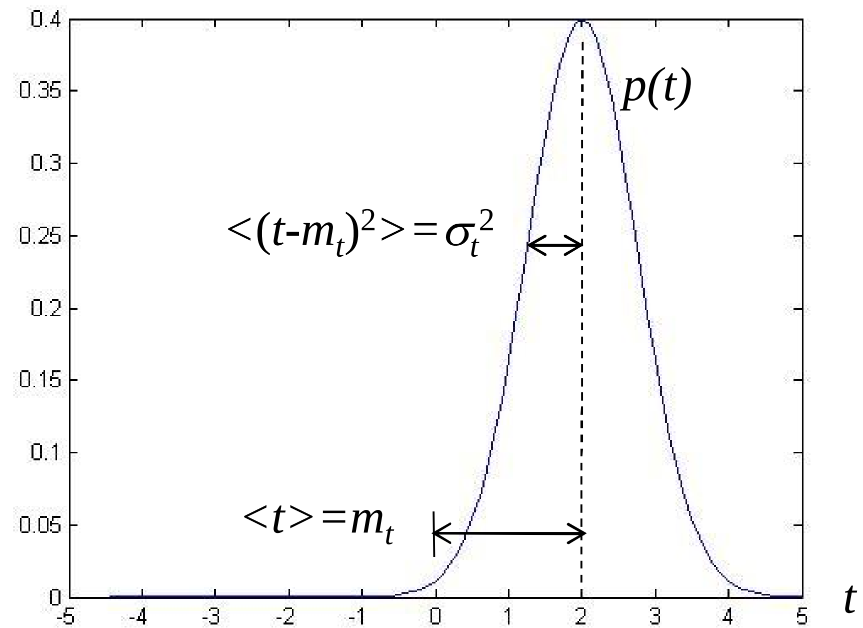
- To get the bunch length, we find the mean (1st moment) and variance (2nd moment) of the energy per pulse *W*.
- For any distribution $p(t)$ and function $q(t)$, define:
  - Mean

$$m_q = \langle q(t) \rangle = \int q(t)\,p(t)\,dt$$

  - Variance

$$\sigma_q^2 = \left\langle \left[q(t) - m_q \right]^2 \right\rangle = \int \left[q(t) - m_q \right]^2 p(t)\,dt = \left\langle q^2(t) \right\rangle - m_q^2$$

  - Standard deviation = σ_q



Mean of the Energy

- The ensemble-averaged (also time-averaged) energy is then:

$$\langle W \rangle = m_W = \frac{1}{2\pi}\sum_{k,l}\iint dt_k dt_l\, f(t_k)\, f(t_l)\int d\omega \left|\hat{e}(\omega)\right|^2 e^{i\omega(t_k - t_l)}$$

$$= \frac{1}{2\pi}\int d\omega \left|\hat{e}(\omega)\right|^2 \left[\sum_{k=l}\iint dt_k dt_l\, f(t_k)\, f(t_l) + \sum_{k\neq l}\iint dt_k dt_l\, f(t_k)\, f(t_l) e^{i\omega(t_k - t_l)}\right]$$

$$= \frac{1}{2\pi}\int d\omega \left|\hat{e}(\omega)\right|^2 \left[N + N^2 \left|\hat{f}(\omega)\right|^2\right] \quad (\text{for } N >> 1)$$

- The first term is incoherent radiation from the *N* electrons.
- The second term is coherent radiation:
  - The characteristic width of $\hat{f}(\omega)$ is $\sigma_\omega = 1/\sigma_t$
  - Coherent term is insignificant when $\omega >> \sigma_\omega$

Bandwidth and Coherence Time

- The light is filtered to a narrow bandwidth σ_{filt} centered at ω_{filt}
  - The characteristic *coherence time* for oscillations of the filtered electric field is: $\tau_{\text{coh}} = 1/\sigma_{\text{filt}}$
  - We are interested in the statistics of the incoherent part of the emission.
  - The filter is chosen so that $M = \sigma_t/\tau_{\text{coh}} >> 1$, or $1/\sigma_t = \sigma_\omega << \sigma_{\text{filt}}$
    - We can neglect the coherent-radiation term.
    - The filter band is also narrow compared to ω_{filt}, and so $\sigma_\omega << \sigma_{\text{filt}} << \omega_{\text{filt}}$
    - Since the bunch duration is many coherence times, it can be pictured as M independently radiating modes, each with random amplitude.
    - The filtered power $|\hat{e}(\omega)|^2$ from each electron, which is not random (but has random timing), has a characteristic width of σ_{filt}.

Variance of the Energy

$$\sigma_W^2 = \left\langle |W|^2 \right\rangle - \left|\langle W \rangle\right|^2$$

$$= \frac{1}{(2\pi)^2} \sum_{k,l,m,n} \iiint d\omega dt_k dt_l \iiint d\omega' dt_m dt_n \left|\hat{e}(\omega)\right|^2 \left|\hat{e}(\omega')\right|^2 f(t_k) f(t_l) f(t_m) f(t_n) e^{i\omega(t_k - t_l) - i\omega'(t_m - t_n)}$$

$$- \left[\frac{N}{2\pi} \int d\omega \left|\hat{e}(\omega)\right|^2 \right]^2$$

$$= \frac{1}{(2\pi)^2} \left[\sum_{k=l,m=n} (\ldots) + \sum_{k=m,l=n} (\ldots) \right] - \left[\frac{N}{2\pi} \int d\omega \left|\hat{e}(\omega)\right|^2 \right]^2$$

$$= \frac{1}{(2\pi)^2} \sum_{k,l} \iint d\omega d\omega' \left|\hat{e}(\omega)\right|^2 \left|\hat{e}(\omega')\right|^2 \iint dt_k dt_l f(t_k) f(t_l) e^{i(\omega-\omega')(t_k - t_l)}$$

$$= \left(\frac{N}{2\pi} \right)^2 \iint d\omega d\omega' \left|\hat{e}(\omega)\right|^2 \left|\hat{e}(\omega')\right|^2 \left| \hat{f}(\omega - \omega') \right|^2$$

$$= \left(\frac{N}{2\pi} \right)^2 \int d\omega \left|\hat{e}(\omega)\right|^4 \int d\omega' \left| \hat{f}(\omega') \right|^2$$

The next slide explains some steps used here.

Wait…How Was That Done?

- As before, we kept only the significant combinations:
  - $k = l$, $m = n$: Canceled by the last term (the mean-squared term).
  - $k = n$, $l = m$: Gives $\hat{f}(\omega + \omega')$ terms, which are small.
  - Neglect coherent-radiation terms.
- $\hat{f}(\omega - \omega')$ has width σ_ω, much narrower than width of $\hat{e}(\omega')$.
  - We can set $\hat{e}(\omega') \approx \hat{e}(\omega)$ when integrating over ω'.

Ratio to Mean

- Ratio of the variance to the mean squared:

$$\frac{\sigma_W^2}{m_W^2} = \frac{\int d\omega \left|\hat{\mathrm{e}}(\omega)\right|^4}{\left[\int d\omega \left|\hat{\mathrm{e}}(\omega)\right|^2\right]^2} \int d\omega \left|\hat{f}(\omega)\right|^2$$

- A beam in a storage ring is Gaussian in time (slide 3). In the frequency domain, the distribution becomes:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}\sigma_t} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2\sigma_t^2} + i\omega t\right) dt = \exp\left(-\frac{\omega^2 \sigma_t^2}{2}\right)$$

- Assume that the filter is also Gaussian.
  - A filter's RMS width σ_{filt} is generally expressed in terms of intensity (E^2), not field. So, after the filter, the single-electron spectrum is:

$$\left|\hat{\mathrm{e}}(\omega)\right|^2 = \frac{p_1}{\sqrt{2\pi}\sigma_{\text{filt}}} \exp\left[-\frac{(\omega - \omega_{\text{filt}})^2}{2\sigma_{\text{filt}}^2}\right]$$

Finding the Length of a Gaussian Bunch

$$\int d\omega \left|\hat{e}(\omega)\right|^2 = p_1$$

$$\int d\omega \left|\hat{e}(\omega)\right|^4 = \frac{p_1^2}{2\sqrt{\pi}\sigma_{\text{filt}}}$$

$$\int d\omega \left|\hat{f}(\omega)\right|^2 = \frac{\sqrt{2\pi}}{\sigma_t}$$

$$\frac{\sigma_W^2}{m_W^2} = \frac{1}{\sqrt{2}\sigma_t\sigma_{\text{filt}}} = \frac{\tau_{\text{coh}}}{\sqrt{2}\sigma_t} = \frac{1}{\sqrt{2}M}$$

Conclusion: The bunch length σ_t can be determined by finding the mean and variance of many measurements of the radiated energy W through a narrow filter of known bandwidth σ_{filt}.

Example

- View 550-nm light through a filter with a 1-nm bandwidth (in intensity):
  - $\omega_{\text{filt}} = 2\pi c/\lambda_{\text{filt}} = 3.425\times10^{15}\ \text{s}^{-1}$
  - $\sigma_{\text{filt}} = \omega_{\text{filt}}\ \sigma_{\lambda}\ /\lambda_{\text{filt}} = 6.227\times10^{12}\ \text{s}^{-1}$
  - $\tau_{\text{coh}} = 1/\sigma_{\text{filt}} = 0.16$ ps
- Measure the statistics:
  - $\sigma_W/m_W = 0.08$
- The bunch length $\sigma_t = 18$ ps.

What if the Pulse isn't Gaussian?

- Interferometric method
  - Split the pulse, delay one part by a time τ, and recombine at the detector, for a total field:

$$E_{\text{total}}(t,\tau) = E(t) + \alpha E(t-\tau)$$

    - A Michelson interferometer can be used.
    - In the frequency domain:

$$\hat{E}_{\text{total}}(\omega,\tau) = \hat{\mathrm{e}}(\omega)\sum_k e^{i\omega t_k}\left(1+\alpha e^{i\omega\tau}\right)$$

  - When $\tau = 0$, this is the same as the previous approach.
  - We will see that the result is the *autocorrelation* of the distribution $f(t)$ as a function of the delay τ:

$$\int f(t)\, f(t-\tau)\, dt$$

    - When $f(t)$ is real and symmetric, the autocorrelation can normally be inverted to find f.

Energy in the Pulse

$$W(\tau)=\frac{1}{(2\pi)^2}\sum_{k,l}\iiint dt\,d\omega\,d\omega'\,\hat{e}(\omega)\hat{e}^*(\omega')\left(1+\alpha e^{i\omega\tau}\right)\left(1+\alpha^* e^{-i\omega'\tau}\right)e^{-i\omega(t-t_k)+i\omega'(t-t_l)}$$

$$=\frac{1}{2\pi}\sum_{k,l}\int d\omega\left|\hat{e}(\omega)\right|^2\left|1+\alpha e^{i\omega\tau}\right|^2 e^{i\omega(t_k-t_l)}$$

$$\left\langle W(\tau)\right\rangle = m_W(\tau)=\frac{1}{2\pi}\sum_{k,l}\int d\omega\left|\hat{e}(\omega)\right|^2\left|1+\alpha e^{i\omega\tau}\right|^2\iint dt_k\,dt_l\,f(t_k)\,f(t_l)e^{i\omega(t_k-t_l)}$$

$$=\frac{N}{2\pi}\int d\omega\left|\hat{e}(\omega)\right|^2\left|1+\alpha e^{i\omega\tau}\right|^2+\frac{N^2}{2\pi}\int d\omega\left|\hat{e}(\omega)\right|^2\left|1+\alpha e^{i\omega\tau}\right|^2\left|\hat{f}(\omega)\right|^2$$

- As before, we neglect the second term, for coherent radiation, because the filter passes light only at a high frequency ω_{filt} .

Variance

$$\sigma_W^2(\tau) = \frac{1}{(2\pi)^2}\sum_{k,l,m,n}\iint d\omega d\omega' \left|\hat{e}(\omega)\right|^2 \left|\hat{e}(\omega')\right|^2 \left|1+\alpha e^{i\omega\tau}\right|^2 \left|1+\alpha e^{i\omega'\tau}\right|^2$$
$$\times \int dt_k\, f(t_k) \int dt_l\, f(t_l) \int dt_m\, f(t_m) \int dt_n\, f(t_n) e^{i\omega(t_k-t_l)-i\omega'(t_m-t_n)}$$
$$-\left[\frac{N}{2\pi}\int d\omega \left|\hat{e}(\omega)\right|^2 \left|1+\alpha e^{i\omega\tau}\right|^2\right]^2$$
$$= \frac{1}{(2\pi)^2}\sum_{k=m,l=n}\iint d\omega d\omega' \left|\hat{e}(\omega)\right|^2 \left|\hat{e}(\omega')\right|^2 \left|1+\alpha e^{i\omega\tau}\right|^2 \left|1+\alpha e^{i\omega'\tau}\right|^2$$
$$\times \int dt_k\, f(t_k) \int dt_l\, f(t_l) e^{i(\omega-\omega')(t_k-t_l)}$$
$$= \left(\frac{N}{2\pi}\right)^2 \iint d\omega d\omega' \left|\hat{e}(\omega)\right|^2 \left|\hat{e}(\omega')\right|^2 \left|1+\alpha e^{i\omega\tau}\right|^2 \left|1+\alpha e^{i\omega'\tau}\right|^2 \left|\hat{f}(\omega-\omega')\right|^2$$

The next slide explains some steps used here.

The Usual Tricks

- Only certain combinations of the sum are significant:
  - $k = l$, $m = n$: Canceled by the last term (mean squared).
  - $k = m$, $l = n$: Gives the ω-ω' terms that provide our result.
- $\hat{f}(\omega - \omega')$ has width σ_ω, much narrower than width of $\hat{e}(\omega')$.
  - We can set $\hat{e}(\omega') \approx \hat{e}(\omega)$ when integrating over ω'.
- Recall that:
  - $\hat{e}(\omega)$ is centered at a high frequency ω_{filt}
  - The delay τ is comparable to the pulse width σ_t
  - $\omega\tau \sim \omega_{\text{filt}}\sigma_t >> 1$
- As a result:
  - In expanding the $|1+\alpha e^{i\omega\tau}|$ factors, all but the constant terms and those involving ω-ω' oscillate rapidly, vanishing in the ω' integral.
  - But for $\tau = 0$ this argument does not apply, and we simply pull the $|1+\alpha|$ factors out of the integral.

Variance and Autocorrelation

$$\sigma_W^2(\tau) = \left(\frac{N}{2\pi}\right)^2 \left(1+|\alpha|^2\right)^2 \iint d\omega d\omega' |\hat{e}(\omega)|^4 \left|\hat{f}(\omega-\omega')\right|^2$$

$$+\left(\frac{N}{2\pi}\right)^2 |\alpha|^2 \iint d\omega d\omega' |\hat{e}(\omega)|^4 \left|\hat{f}(\omega-\omega')\right|^2 \left(e^{i(\omega-\omega')\tau} + e^{-i(\omega-\omega')\tau}\right)$$

$$= \left(\frac{N}{2\pi}\right)^2 \left(1+|\alpha|^2\right)^2 \int d\omega |\hat{e}(\omega)|^4 \int d\omega' \left|\hat{f}(\omega')\right|^2$$

$$+ 2\left(\frac{N}{2\pi}\right)^2 |\alpha|^2 \operatorname{Re}\left[\int d\omega |\hat{e}(\omega)|^4 \int d\omega' \left|\hat{f}(\omega')\right|^2 e^{-i\omega'\tau}\right]$$

$$= \left(\frac{N}{2\pi}\right)^2 \int |\hat{e}(\omega)|^4 d\omega \left[\left(1+|\alpha|^2\right)^2 \int f(t)^2 dt + 2|\alpha|^2 \int f(t) f(\tau-t) dt\right]$$

Again, see the next slide
for some steps used here.

And More Tricks

- We used a theorem of Fourier transforms: The product of two transforms is an autocorrelation in the time domain.

$$\int g(t)h(\tau - t)dt = \frac{1}{(2\pi)^2}\int d\omega \int d\omega' \hat{g}(\omega)\hat{h}(\omega')\int dt\, e^{-i\omega t - i\omega'(\tau - t)}$$
$$= \frac{1}{2\pi}\int \hat{g}(\omega)\hat{h}(\omega)e^{-i\omega\tau}d\omega$$

  - We also used a special case of this, Parseval's theorem:

$$\int |g(t)|^2 dt = \frac{1}{2\pi}\int |g(\omega)|^2 d\omega$$

  - And we used the fact that $f(t)$ is real and assumed to be symmetric.
- When we look at the change in the variance as τ is scanned, we can ignore the first, τ-independent term.
- For $\alpha = 0$ (no interference), the result reverts to the prior case.

Ratio to Mean

$$\frac{\sigma_W^2(\tau)}{m_w^2(0)} = \frac{\int \left|\hat{e}(\omega)\right|^4 d\omega \left[\left(1+\left|\alpha\right|^2\right)^2 \int f(t)^2 dt + 2\left|\alpha\right|^2 \int f(t) f(\tau - t) dt \right]}{\left|1+\alpha\right|^4 \left[\int \left|\hat{e}(\omega)\right|^2 d\omega \right]^2}$$

- As τ is scanned, the ratio of the variance to the central (peak) value of the mean gives a constant and a varying term.
- When $f(t)$ is real and symmetric, the autocorrelation from the varying term can be inverted to determine f.

Additional Complications

- Transverse beam size
  - If the beam is too wide for transversely coherent emission, or if there is diffraction at a limiting aperture, then the measured variance is reduced.
- Detector noise
  - Detector noise adds to the measured fluctuations, and must be accounted for to find the correct bunch length.
- Photon count
  - If the number of photons on the detector is too low, shot noise will increase the measured fluctuations.

Another Variation

- Spectrographic method
  - Use a spectrometer to make *many* narrow filters.
  - The fluctuations from one wavelength bin to the next then give the bunch length in a single measurement of the pulse.

Conclusion

- We can find the length σ_t of a short bunch using a simple statistics of many measurements of the radiated energy W through a narrow filter.
- A more elaborate setup can provide more information about the temporal profile.