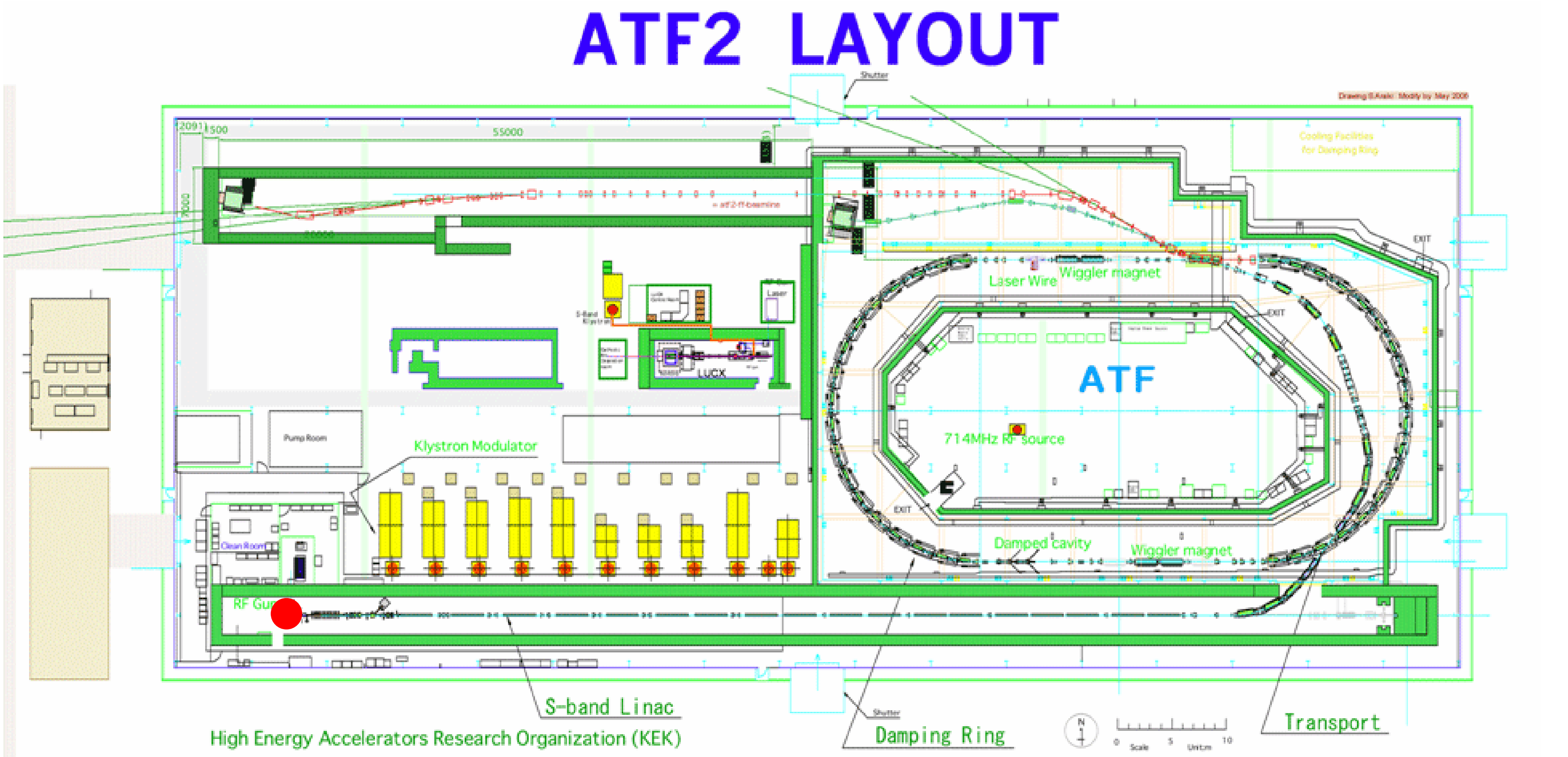
Introduction to Accelerator Optics

(several slides from previous USPAS classes)

A Quick Review for the USPAS'11 Particle Interaction Region Course

Introduction



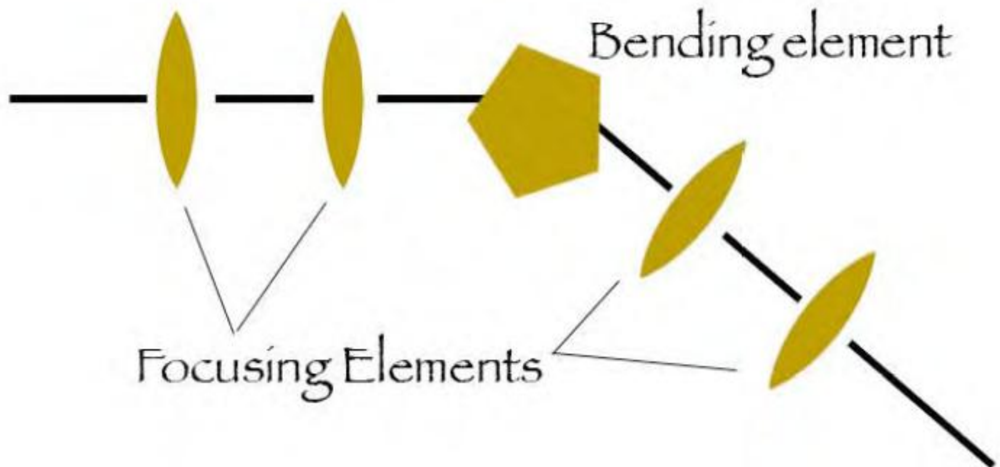
Introduction

- The design orbit is the ideal orbit on which the particles should move.
- We need to i) bend the particles and ii) continuously focus the beam into the orbit.
- Both bending and focusing is accomplished with electromagnetic forces.
- The Lorentz Force is

$$\boxed{\vec{F} = ma = e(\vec{E} + \vec{v} \times \vec{B})}$$

Optics are essential to guide the beam through the accelerator

- Optics (lattice): distribution of magnets that direct & focus beam



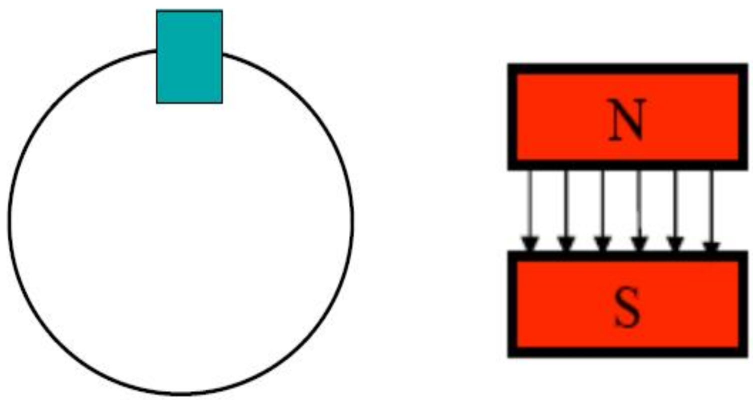
- Lattice design depends upon the goal & type of accelerator
  - Linac or synchrotron
  - High brightness: small spot size & small divergence
  - Physical constraints (building or tunnel)

The lattice must transport a real beam not just an ideal beam

Types of magnets & their fields:dipoles

Dipoles:
Used for steering

$$B_x = 0$$
$$B_y = B_o$$

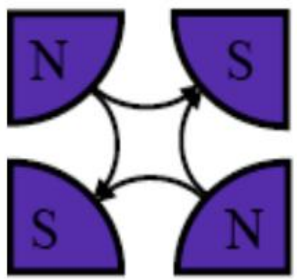


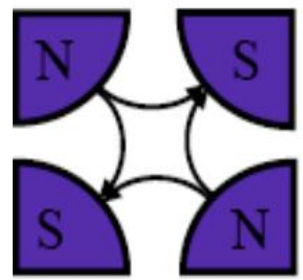
Types of magnets & their fields: quadrupoles

Quadrupoles:
Used for focusing

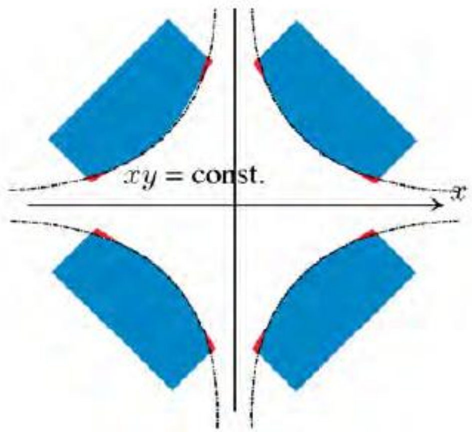
$B_x = Ky$

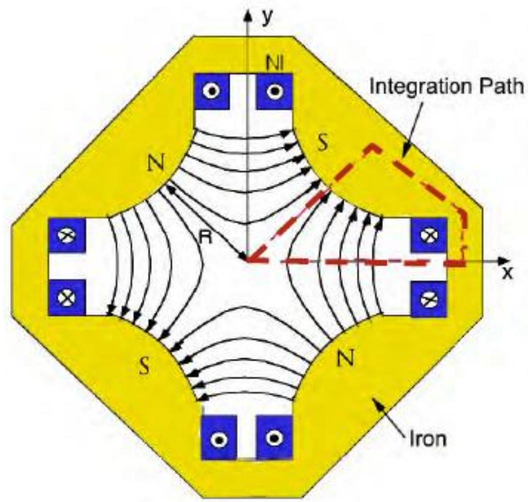
$B_y = Kx$

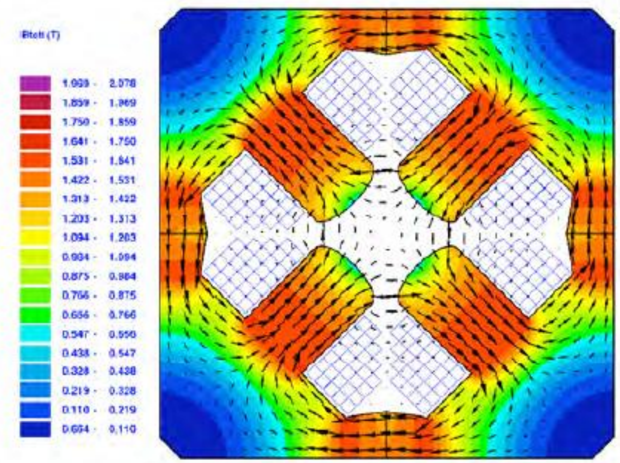




The quadrupole magnet & its field

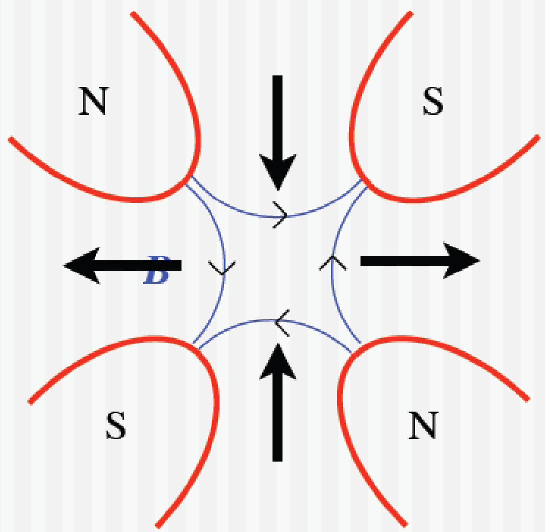






Quadrupole Field

- Note: A quadrupole magnet will focus in one plane, and defocus in the other



Because of the v x B term in $\vec{F} = ma = e(\vec{E} + \vec{v} \times \vec{B})$

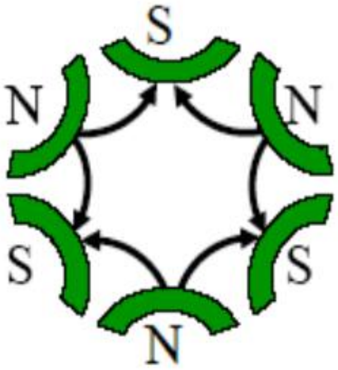
Types of magnets & their fields: sextupoles

Sextupoles:

Used for chromatic correction

$B_x = 2Sxy$

$B_y = S(x^2 - y^2)$

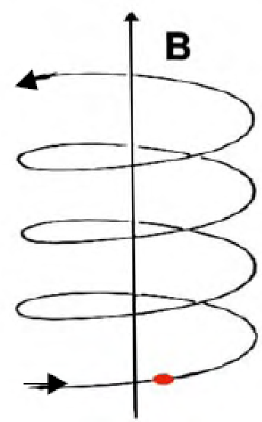


Charged particle motion in a uniform (dipole) magnetic field

- Let $\mathbf{B} = B_o\hat{\mathbf{y}}$
- Write the Lorentz force equation in two components, z and ⊥

$$\frac{dp_y}{dt} = 0 \quad \text{and} \quad \frac{d\mathbf{p}_\perp}{dt} = q(\mathbf{v}_\perp \times \mathbf{B}) = \frac{qB_o}{\gamma m_o}(\mathbf{p}_\perp \times \hat{\mathbf{y}})$$

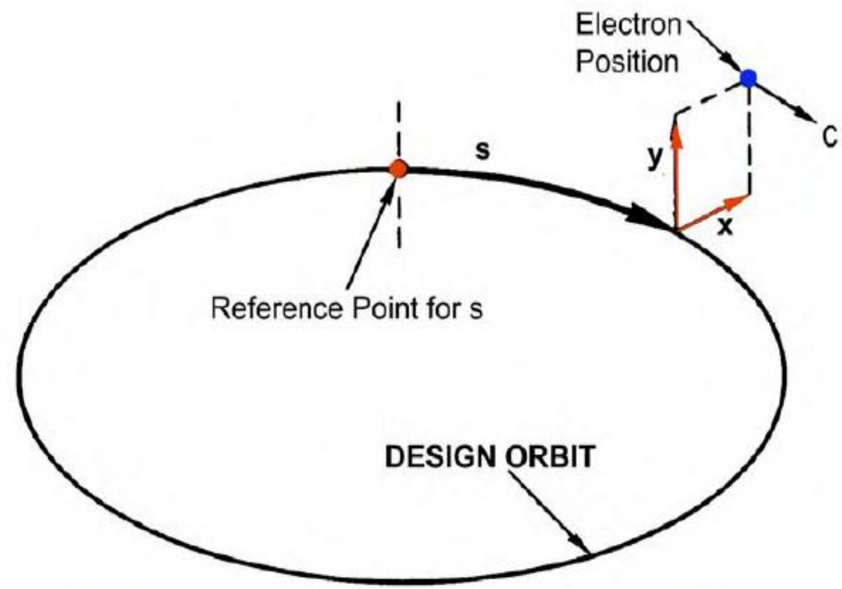
- ==> p_y is a constant of the motion
- Since B does no work on the particle, $|p_\perp|$ is also constant
  - → The total momentum & total energy are constant
  - → For $p_{y,o} \neq 0$, the orbit is a helix



To analyze particle motion we will use local Cartesian coordinates

Change dependent variable from time, t, to longitudinal position, s

The origin of the local coordinates is a point on the *design trajectory* in the *bend plane*

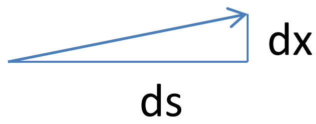


$$x,\quad x' = \frac{dx}{ds},\quad y,\quad y' = \frac{dy}{ds},\quad \delta = \frac{\Delta p}{p_0}, \tau = \frac{\Delta L}{L}$$

The bend plane is generally called the horizontal plane
The vertical is y in American literature & often z in European literature

Equation of motion in Circular Accelerator

- As coordinate, it is more convenient to use the slope or angle:



$$x' = \frac{dx}{ds} \qquad \text{or equivalently} \qquad x' = \frac{p_x}{p_0}$$

- In circular accelerator the particles equation of motion to first order are written:

$$\boxed{\begin{gathered} x'' - \left(k(s) - \frac{1}{\rho(s)^2} \right) x = \frac{1}{\rho(s)} \frac{\Delta p}{p} \\ y'' + k(s) y = 0 \end{gathered}}$$

off-momentum particles

where $1/\rho^2$ is the dipole weak focusing term and the $\Delta p/p$ term is present for off-momentum particles

Betatron oscillation and beta function

- In the case of on-momentum particle $p=p_0$ or Δp=0

$$\boxed{x'' + K(s)x = 0}$$

Hill's equation

It can be shown that the solution of the Hill equation is given by:

$$\boxed{x(s) = a\sqrt{\beta(s)}\, e^{\pm i\Phi(s)}}$$

$$with \quad \Phi'(s) = \frac{1}{\beta(s)} \quad and \quad a = const$$

Betatron oscillations and beta function

Thus, the most general solution to the Hill equation is a pseudo-harmonic oscillation. Amplitude and wavelenght depend on the coordinate *s* and are both given in term of the **beta function:**

$$amplitude \quad \propto \sqrt{\beta(s)}; \qquad \lambda(s) = 2\pi\beta(s)$$

Another key parameter is the “alpha” function:

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

which represents the slope of the beta function.

Tune and resonances

- The “phase” function is also computed in term of the beta function

$$\Phi(s) = \int_{s_0}^{s} \frac{1}{\sqrt{\beta(t)}}\, dt$$

Phase computed between two locations of the beam line

- The “tune” or Q value (often denoted also with ν) is defined as the number of betatron oscillations per revolution in a circular accelerator:

$$Q = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

Integral over the circumference

Matrix formalism

- Typically to track particles, instead of solving the equations of motion we use matrices to represent the action of the magnetic elements in the beam line. Simpler and more manageable.
- Each beam line element is represented by a matrix.
- The 6 particle coordinates are represented by a vector.
- Transport is obtained by a series of matrix multiplications. Total transport “map”.

Matrix formalism

Start from Hill equation $\boxed{x'' + K(s)x = 0}$

Build matrices from the solutions. Example:

- DRIFT for K=0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Can you show this?

- Quadrupole:
  - K > 0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

  - K < 0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sinh\left(\sqrt{K}L\right) \\ -\sqrt{K}\sinh\left(\sqrt{K}L\right) & \cosh\left(\sqrt{K}L\right) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Matrix formalism

- "Thin Lens" Quadrupole
  - Consider a short enough quadrupole so that the particle offset doesn't change while the slope x' does.
  - Assume length L→0 while KL remains finite, thus

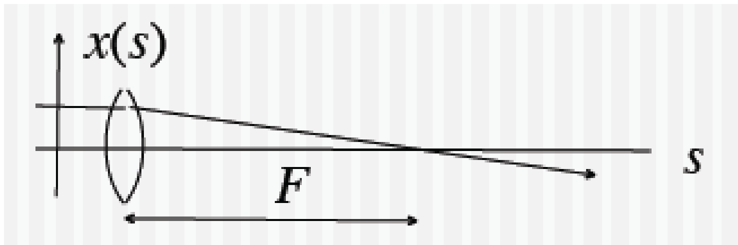
K > 0 Focusing Quad:

$$Q_F = \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

(change sign for Defocusing Quad)

Valid if length of focus: F>>L.

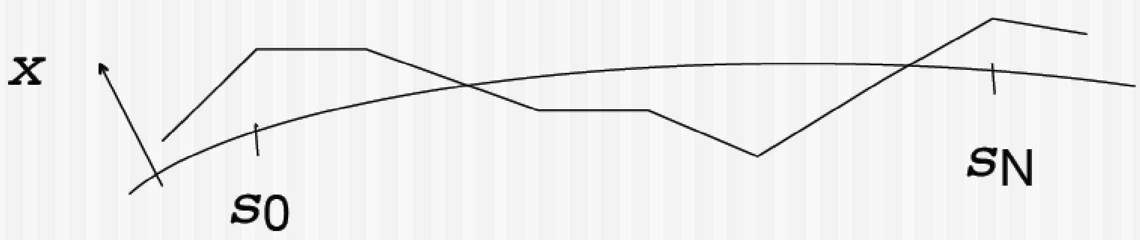
$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$



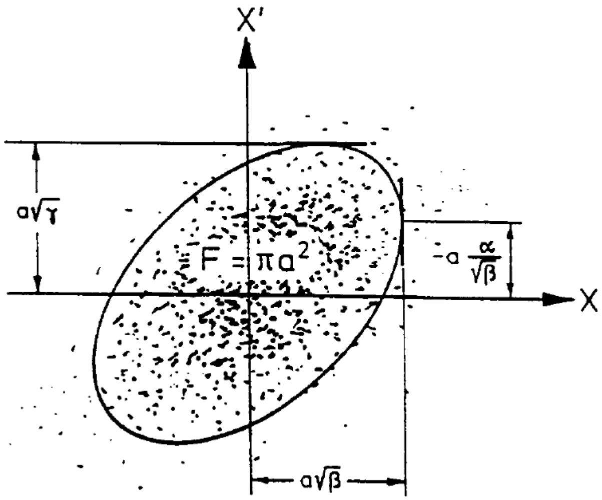
Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed *via* matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

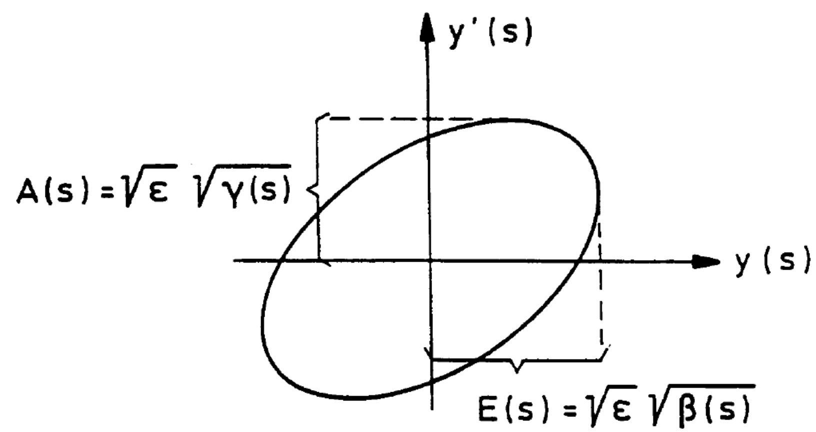


Emittance



Emittance is the area in the phase space (x,x') or (y,y') containing a certain fraction (90%) of beam particles.

The emittance and the beta function are used to compute the beam size (Envelope) and beam divergence at position s along the beam line.



Beam envelope E(s) and divergence A(s). Note also $\gamma(s) = \left(1+\alpha^2(s)\right)/\,\beta(s)$

Emittance

We talk about "un-normalized" (previous definition) and "normalized" emittance

$$\varepsilon_N = (\frac{p_0}{m_0 c})\varepsilon \qquad or \qquad \varepsilon_N = \gamma\varepsilon$$

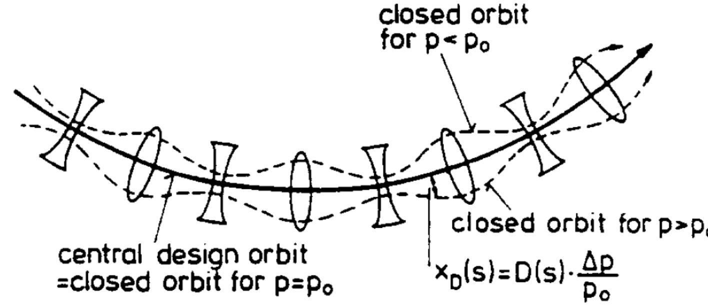
For protons and electrons in linacs, the normalized emittance is a quantity that stays constant during acceleration.

Dispersion function

- The central design orbit it the ideal closed curve that goes through the center of all quadrupoles. An ideal particle with nominal $p=p_0$, zero displacement and zero slope will move on the design orbit for an arbitrary number of turns.
- A particle with nominal $p=p_0$ and with non-vanishing initial conditions will conduct betatron oscillations around the closed orbit.

Dispersion function

- Particles with larger momentum will need a circumference with larger radius on which they can move indefinitely.



- Particles will perform betatron oscillations about this new larger circles.
- A particle with $\Delta p=p-p_0 \neq 0$ satisfies the inhomogeneous Hill equation in the horizontal plane

$$x'' + K(s)x = \frac{1}{\rho}\frac{\Delta p}{p_0}$$

- The total deviation of the particle is:

$$\boxed{x(s) = x_D(s) + x_\beta(s)}$$

where $x_D(s) = D(s)\cdot\frac{\Delta p}{p_0}$ is the deviation of the closed orbit for a particle with Δp.

- D(s) is the dispersion **function** that satisfies the Hill eq. $D'' + K(s)D = \frac{1}{\rho(s)}$ along the circumference. $D' = dx'/d\delta$ is the slope of the dispersion.

Chromaticity

1st:Transport definition

Storage Rings: chromaticity defined as a change of the betatron tunes versus energy.

In single path beamlines, it is more convenient to use other definitions.

$$x_i = \begin{pmatrix} x \\ x' \\ y \\ y' \\ \Delta l \\ \delta \end{pmatrix} \qquad\qquad x_i^{out} = R_{ij}\; x_j^{in}$$

The second, third, and so on terms are included in a similar manner:

$$x_i^{out} = R_{ij}\; x_j^{in} + T_{ijk}\; x_j^{in}\; x_k^{in} + U_{ijkn}\; x_j^{in}\; x_k^{in}\; x_n^{in} + \ldots$$

In FF design, we usually call 'chromaticity' the second order elements T_{126} and T_{346}. All other high order terms are just 'aberrations', purely chromatic (as T_{166}, which is second order dispersion), or chromo-geometric (as U_{3246}).

A. Seryi, USPAS 2007

By the way …

Relativistic speaking:

$$\boxed{\beta = v / c}$$

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \beta^2}}}$$

Particle energy and momentum

$$\boxed{E = \sqrt{p^2 c^2 + (mc^2)^2}}$$

$$\boxed{E = \gamma m c^2}$$

$$\boxed{p = \gamma \beta m c}$$

References

For example:

1) M. Syphers, A. Warner (Fermilab) R. Miyamoto (U. Texas) USPAS 2008 lectures: http://home.fnal.gov/~syphers/Education/uspas/USPAS08/w1-2Tue.pdf

2) W. Barletta USPAS 2010 lectures: http://uspas.fnal.gov/materials/09UNM/Unit_7_Lecture_15_Linear_optics.pdf

Questionnaire

- Circle what you already know about:

  luminosity, relativistic γ, GeV, quadrupole, sextupole, beta function, alpha, closed orbit, BPMs, MAD, emittance, energy spread, dispersion, momentum compaction, tune, tune shift, tune footprint, chromaticity, transverse feedback, beam based alignment, beam line matching.