Lecture 4

RF Acceleration in Linacs

Part 2

Outline

- Traveling-wave linear accelerators
- Longitudinal beam dynamics

- Material from *Wangler, Chapters 3, 6*

Guided Electromagnetic Waves in a Cylindrical Waveguide

- Can we accelerate particles by transporting EM waves in a waveguide?
- Consider a cylindrical geometry. The wave equation in cylindrical coordinates for the z field component is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \phi^2}+\frac{\partial^2 E_z}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}=0$$

- Assume the EM wave propagates in the Z direction. Let's look for a solution that has a finite electric field in that same direction:

$$E_z = E_z(r,\phi,z,t) = E_0(r,\phi)\cos(k_z z-\omega t)$$

- The azimuthal dependence must be repetitive in ϕ:

$$E_z = R(r)\cos(n\phi)\cos(k_z z-\omega t)$$

- The wave equation yields:

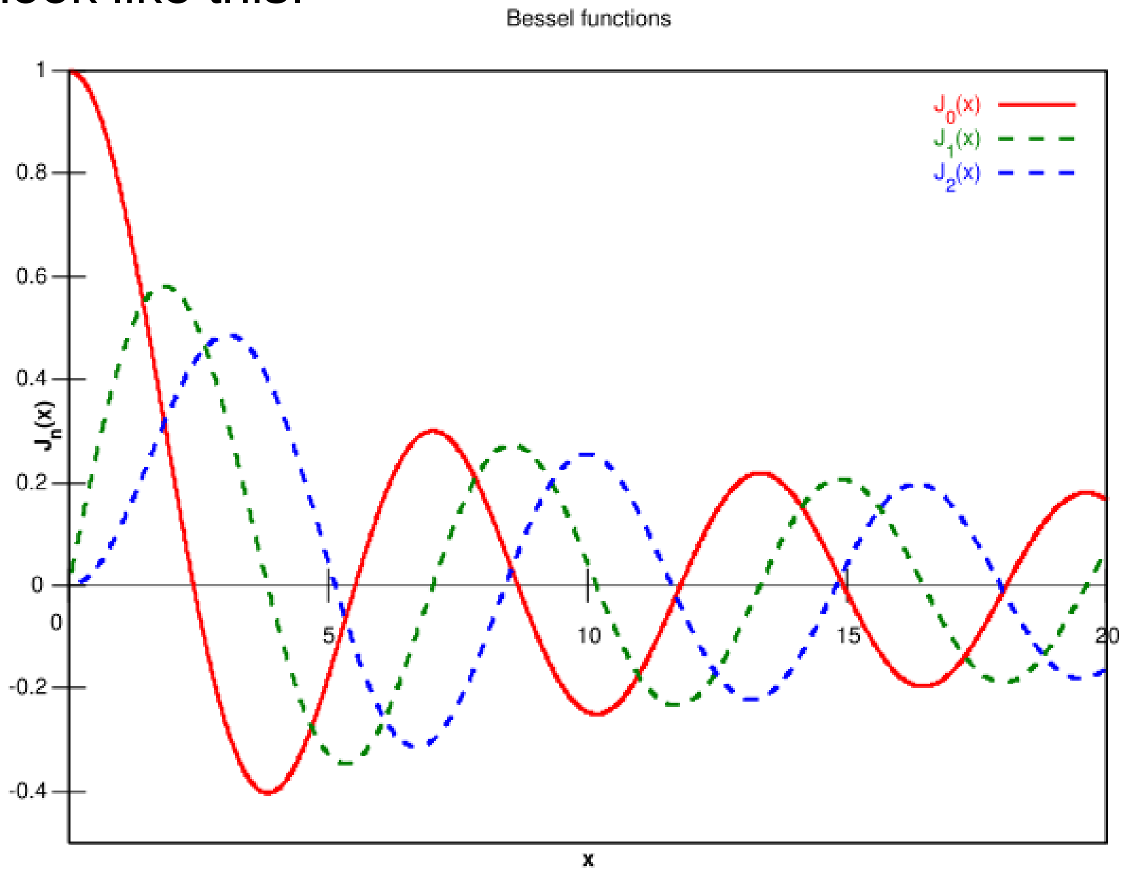
$$\frac{\partial^2 R(r)}{\partial r^2}+\frac{1}{r}\frac{\partial R(r)}{\partial r}+\left(\underbrace{\frac{\omega^2}{c^2}-k_z^2}_{k_c^2}-\frac{n^2}{r^2}\right)R(r)=0$$

Cylindrical Waveguides

- Which results in the following differential equation for *R(r) (with $x=k_c r$)*

$$\frac{d^2R}{dx^2}+\frac{1}{x}\frac{dR}{dx}+(1-n^2/x^2)R=0$$

- The solutions to this equation are *Bessel functions of order n*, $J_n(k_c r)$, which look like this:



Cylindrical Waveguides

- The solution is:

$$E_z = J_n(k_c r)\cos(n\phi)\cos(k_z z - \omega t)$$

- The boundary conditions require that

$$E_z(r = a) = 0$$

- Which requires that

$$J_n(k_c a) = 0 \text{ for all n}$$

- Label the *n*-th zero of J_m: $J_m(x_{mn}) = 0$
- For m=0, $x_{01} = 2.405$

$$\frac{\omega^2}{c^2} = k_c^2 + k_z^2 = \left(\frac{2.405}{a}\right)^2 + k_z^2$$

Cutoff Frequency and Dispersion Curve

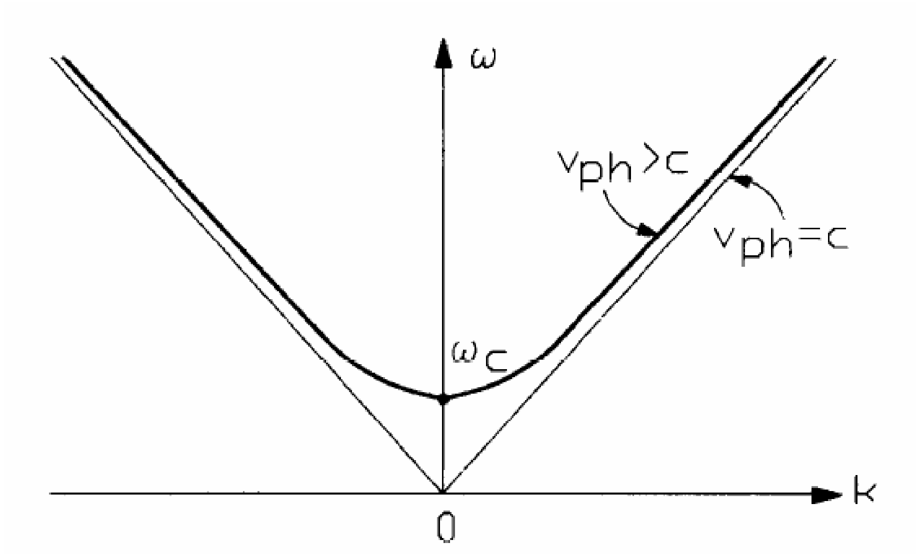
- The cylindrically symmetric waveguide has

$$k_0^2 = k_c^2 + k_z^2 \qquad \omega^2 = \omega_c^2 + (k_z c)^2$$

- A plot of ω vs. k is a hyperbola, called the Dispersion Curve.

Two cases:

- $\omega > \omega_c$: k_z is a real number and the wave propagates.
- $\omega < \omega_c$: k_z is an imaginary number and the wave decays exponentially with distance.
- Only EM waves with frequency above cutoff are transported!



Phase Velocity and Group Velocity

- The propagating wave solution has

$$E_z = E_0(r,z)\cos(\phi) \qquad \phi = k_z z - \omega t$$

- A point of constant ϕ propagates with a velocity, called the phase velocity,

$$v_p = \frac{\omega}{k_z}$$

- The electromagnetic wave in cylindrical waveguide has phase velocity that is faster than the speed of light:

$$v_p = \frac{c}{\sqrt{1-\omega_c^2/\omega^2}} > c$$

- This won't work to accelerate particles. We need to modify the phase velocity to something smaller than the speed of light to accelerate particles.
- The *group velocity* is the velocity of energy flow:

$$P_{RF} = v_g U$$

- And is given by:

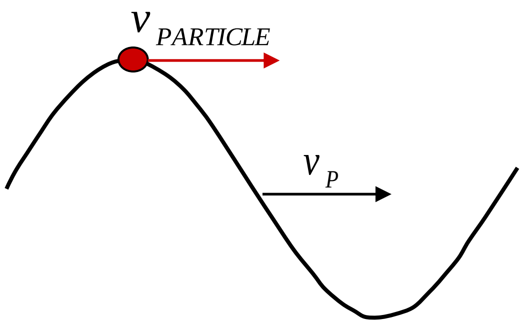
$$v_g = \frac{d\omega}{dk}$$

Traveling Wave Structures

- Recall that in the cylindrical waveguide, the electromagnetic wave has phase velocity that is faster than the speed of light:

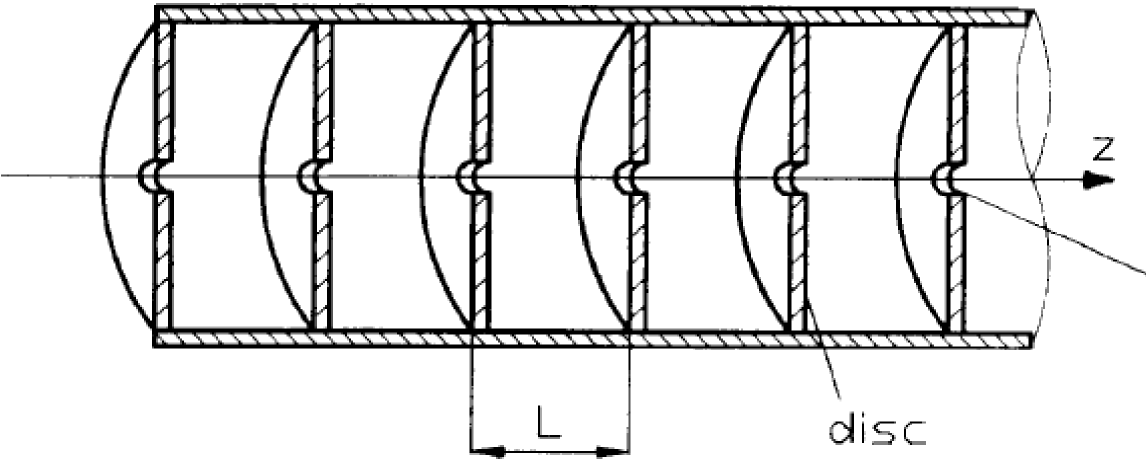
$$v_p = \frac{c}{\sqrt{1-\omega_c^2/\omega^2}} > c$$

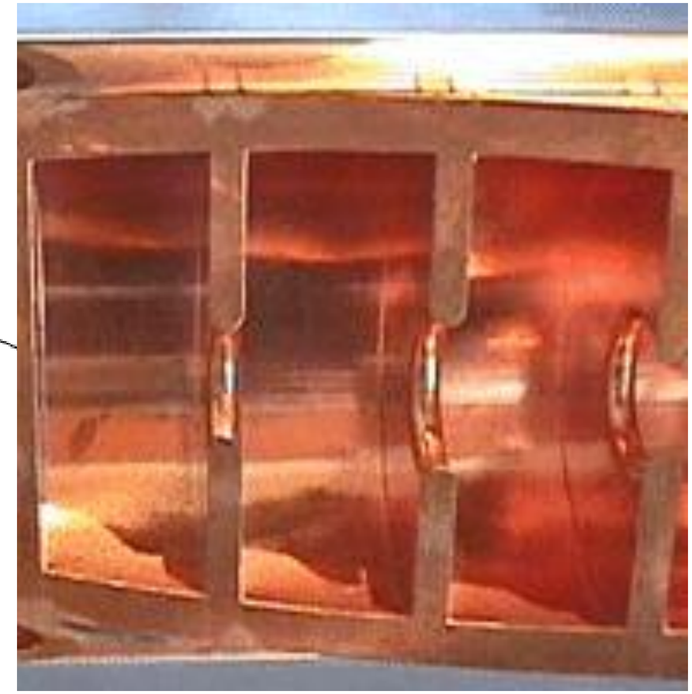
- This won't work to accelerate particles. We need to modify the phase velocity to the speed of light (or slower) to accelerate particles in a traveling wave.
- Imagine a situation where the EM wave phase velocity equals the particle velocity.
- Then the particle "rides the wave".



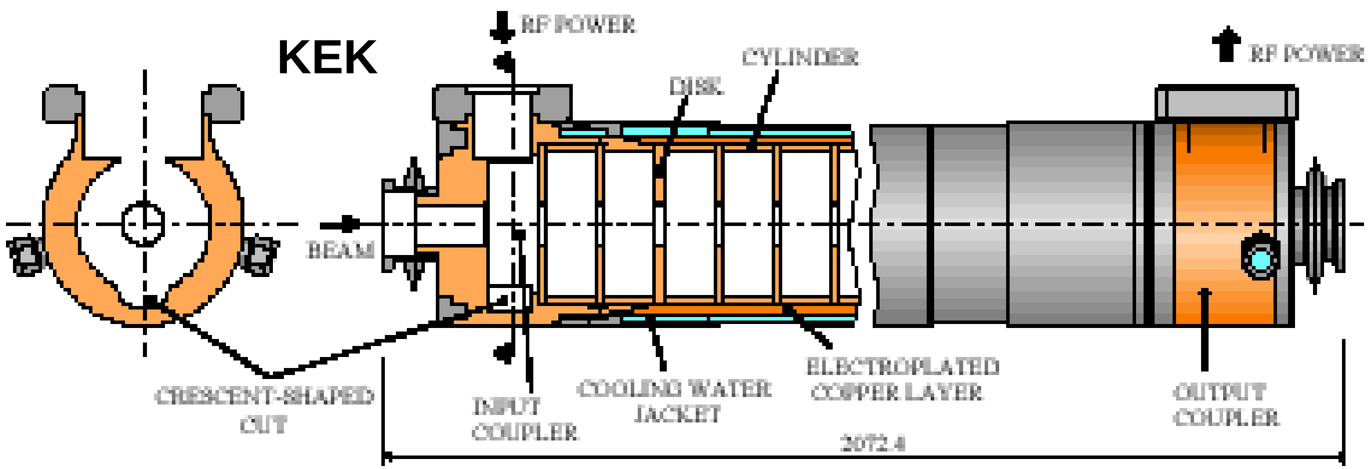
- A "disk-loaded waveguide" can be made to have a phase velocity equal to the speed of light. These structures are often used to accelerate electrons.
- The best and largest example of such an accelerator is the SLAC two-mile long linac.

Disk-loaded waveguide structure





KEK



Energy Gain in a Disk-Loaded Waveguide

Define

- E_a: longitudinal accelerating field amplitude
- U: stored energy per unit length
- P_w: traveling wave power
- dP_w/dz: power dissipation per unit length
- Shunt impedance per unit length $r_L = E_a^2/(-dP_w/dz)$
- We have

$$Q = \omega U/(-dP_w/dz)$$

$$P_w = v_g U$$

$$E_a^2 = \omega r_L P_w / Q v_g$$

$$\frac{dP_w}{dz} = -\frac{\omega}{Q v_g} P_w = -2\alpha_0 P_w$$

We have two choices for the accelerating structure, considered now in turn:

Constant Impedance Traveling Wave Structure

- Consider a disk-loaded waveguide with uniform cell geometry along the length, then Q, v_g, r_L, α_0 are independent of z:

$$P_w(z) = e^{-2\alpha_0 z}$$

- Power decays exponentially along the length of the structure.
- The Electric field amplitude is

$$dE_a / dz = -\alpha_0 E_a$$

$$E_a(z) = E_0 e^{-\alpha_0 z}$$

- At the end of a waveguide of length L

$$P_w(L) = P_0 e^{-2\tau_0} \qquad E_a(L) = E_0 e^{-\tau_0}$$

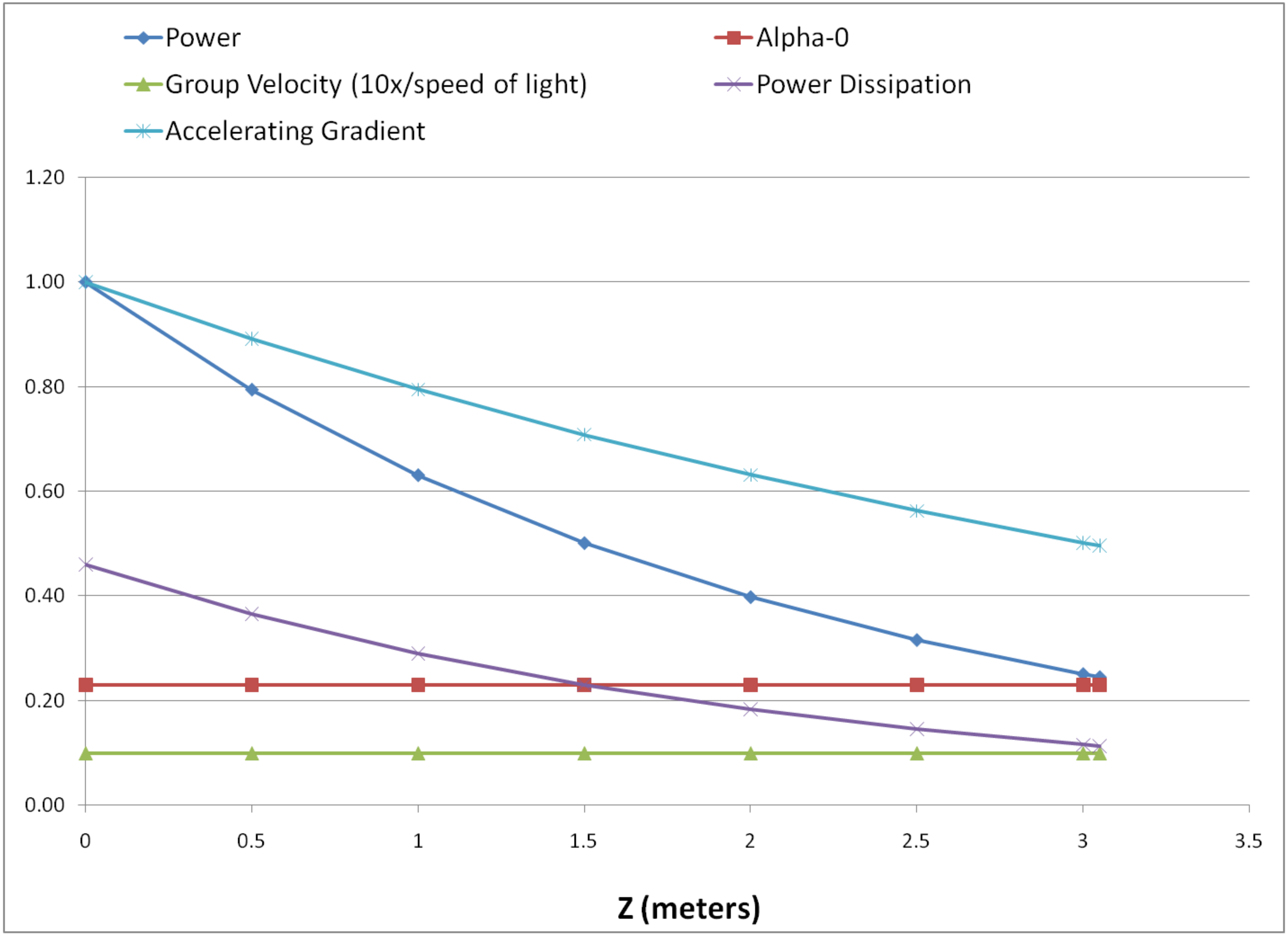
$$\tau_0 = \alpha_0 L = \frac{\omega L}{2Qv_g}$$

- The energy gain is

$$\Delta W = q \cos\phi \int_0^L E_a(z) dz = qE_0 L \frac{1-e^{-\tau_0}}{\tau_0} \cos\phi$$

$$\Delta W = q\sqrt{2 r_L P_0 L}\, \frac{1-e^{-\tau_0}}{\sqrt{\tau_0}} \cos\phi$$

Constant Impedance Structure Parameters



Constant Gradient Traveling Wave Structure

- A more common design keeps the gradient constant over the length, which requires that the attenuation α_0 depend on z

$$\frac{dP_w}{dz} = -2\alpha_0(z)P_w$$

- Which can be integrated to yield

$$P_w(z) = P_0\left[1 - \frac{z}{L}(1 - e^{-2\tau_0})\right]$$

- The attenuation factor is

$$\alpha_0(z) = \frac{1}{2L}\frac{1 - e^{-2\tau_0}}{1 - (z/L)(1 - e^{-2\tau_0})}$$

- The energy gain is

$$\Delta W = q\cos\phi\int_0^L E_a(z)dz = qE_0 L\cos\phi$$

$$\Delta W = q\sqrt{r_L P_0 L(1 - e^{-2\tau_0})}\cos\phi$$

- To achieve a constant gradient, the SLAC linac structure tapers from a radius of 4.2 to 4.1 cm, and the iris radii taper from 1.3 to 1.0 cm over 3 meters.

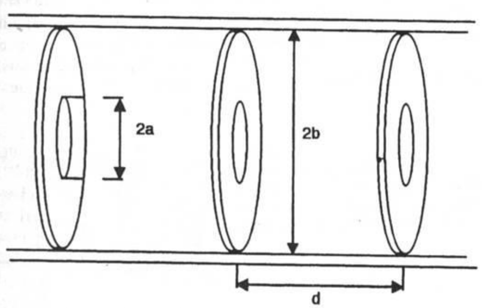


Figure 3.14 Iris (disk)-loaded traveling-wave structure.

[4] For a comprehensive article on this subject, see G. A. Loew and R. B. Neal, in *Linear Accelerators*, P. M. Lapostolle and A. L. Septier, Wiley, New York, 1970, pp. 39–113.

Constant Gradient Traveling Wave Structure

- The group velocity is

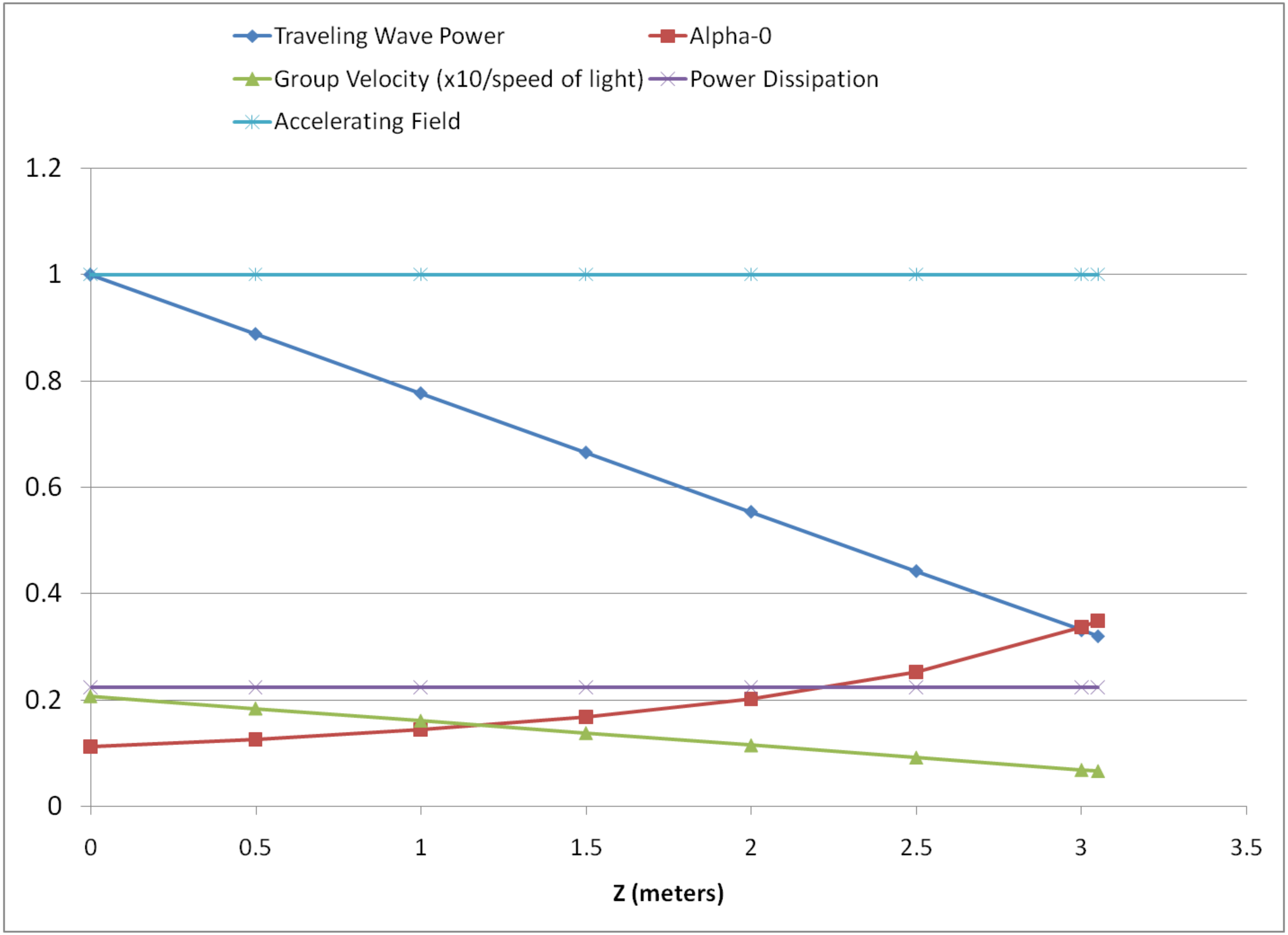
$$v_g(z) = \frac{\omega}{2Q\alpha_0(z)} = \frac{\omega L}{Q} \frac{1-(z/L)(1-e^{-2\tau_0})}{1-e^{-2\tau_0}}$$

- The filling time is

$$t_F = \int_0^L \frac{dz}{v_g(z)} = \frac{Q}{\omega L}(1-e^{-2\tau_0})\int_0^L \frac{dz}{1-(z/L)(1-e^{-2\tau_0})} = \tau_0 \frac{2Q}{\omega}$$

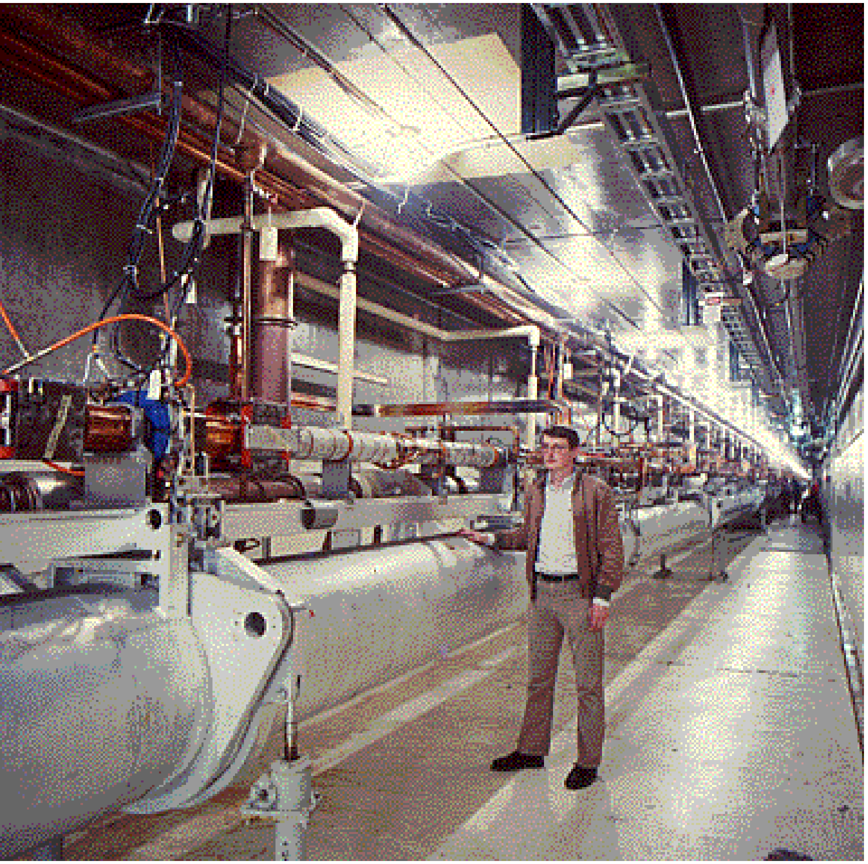
- For typical parameters, the filling time is ~1 μsec, and the beam pulse is 1-2 μsec.

Constant Gradient Structure Parameters

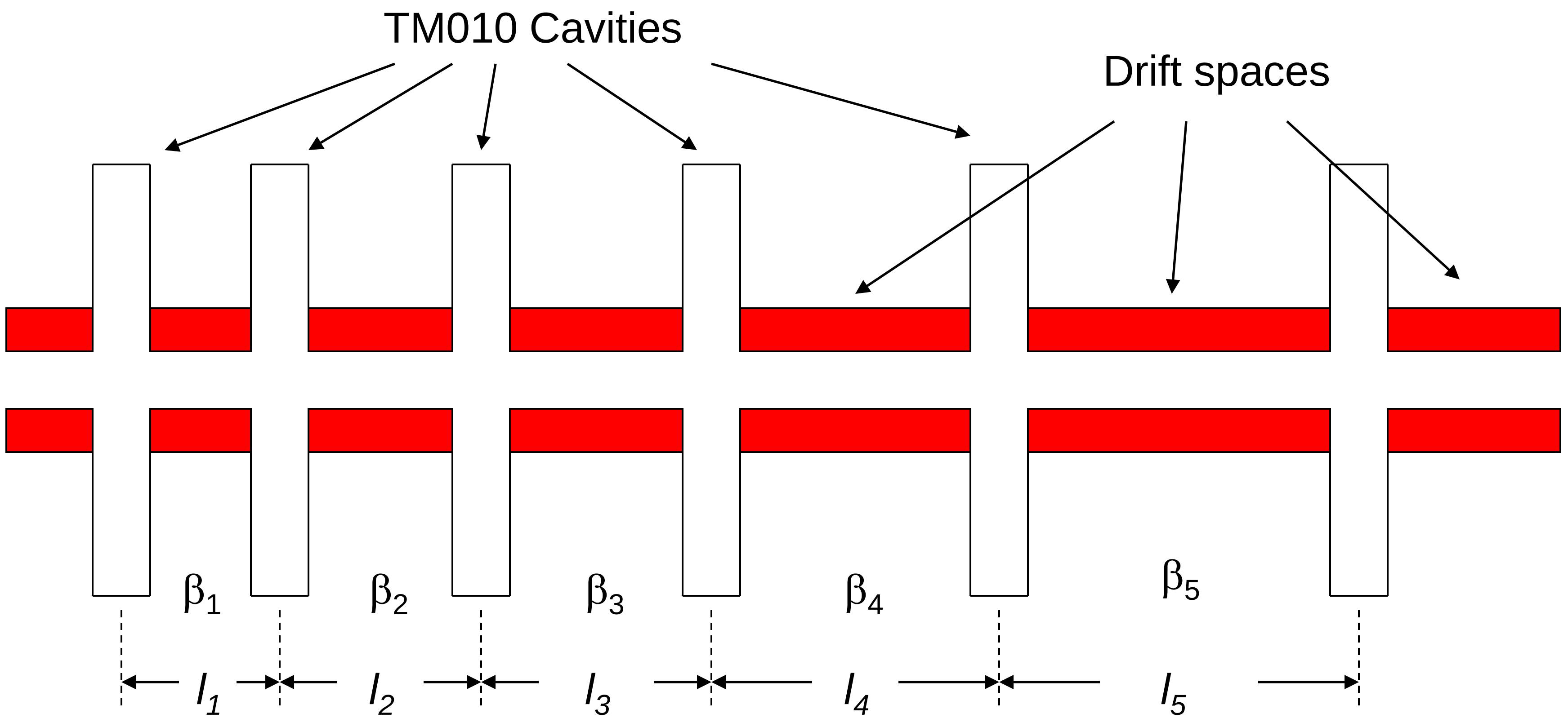


SLAC Linac

- Largest in the world. Reached energies of 50 GeV.



Synchronicity condition in multicell RF structures



- Suppose we want a particle to arrive at the center of each gap at $\phi=0$. Then we would have to space the cavities so that the RF phase advanced by
  - 2π if the coupled cavity array was driven in zero-mode,
  - Or by π if the coupled cavity array was driven in pi-mode.

Synchronicity Condition

Zero-mode:

$$\phi = \omega t = \frac{2\pi c}{\lambda} t = \frac{2\pi c}{\lambda} \frac{l_n}{c\beta_n} = 2\pi$$

$$l_n = \beta_n \lambda$$

- RF gaps (cells) are spaced by βλ, which increases as the particle velocity increases.

Pi-mode:

$$\phi = \omega t = \frac{2\pi c}{\lambda} t = \frac{2\pi c}{\lambda} \frac{l_n}{c\beta_n} = \pi$$

$$l_n = \beta_n \lambda / 2$$

- RF gaps (cells) are spaced by βλ/2, which increases as the particle velocity increases.

Longitudinal Dynamics

- The drift space length between gaps is calculated for a particular particle with a very specific energy. This is the *reference* particle, or the *synchronous* particle.
- What happens to particles slightly faster or slower than the *synchronous* particle that the linac was designed to accelerate?
- Linacs are operated to provide *longitudinal focusing* to properly accelerate particles over a range in energies or arrival time.
- Slower particles arrive at the next gap later than the synchronous particle.
  - They experience a larger accelerating field.
- Faster particles arrive at the next gap earlier than the synchronous particle.
  - They experience a smaller accelerating field.

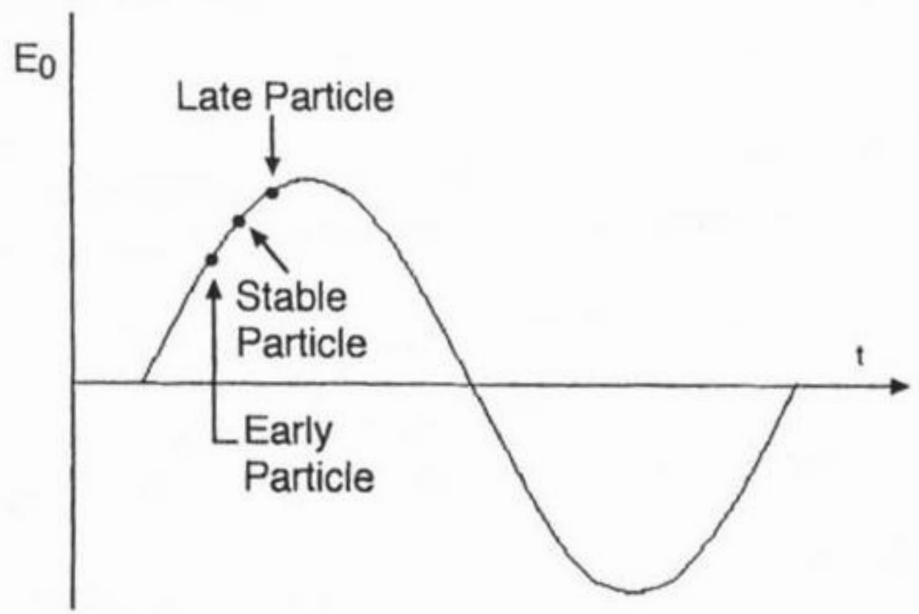


Figure 6.1. Stable phase.

Equations of Motion I

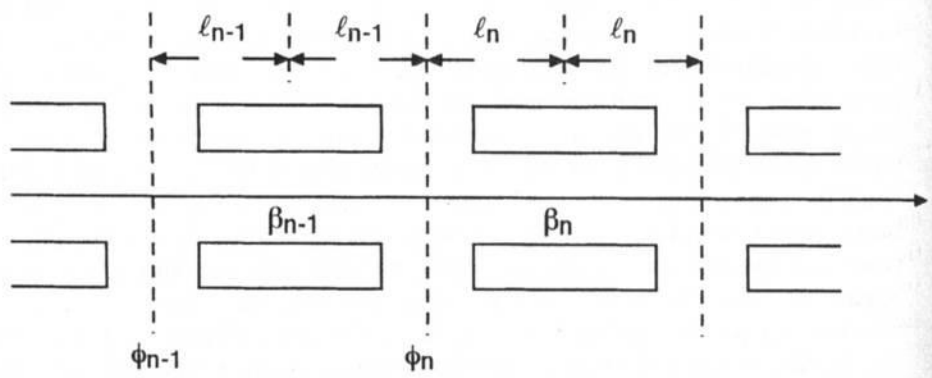


Figure 6.2. Accelerating cells for describing the longitudinal motion.

- Consider an array of accelerating cells with drift tubes and accelerating gaps.
- The array is designed at the n-th cell for a particle with synchronous phase, kinetic energy, and velocity ϕ_{sn}, W_{sn}, β_{sn}. Note that the synchronous phase is not zero!
- We express the phase, energy and velocity for an arbitrary particle in the n-th cell as ϕ_n, W_n, β_n.
- Assume that the particles receive a longitudinal kick at the geometric center of the cell, and drift freely to the center of the next cell.
- The half-cell length is

$$l_{n-1} = \frac{N\beta_{s,n-1}\lambda}{2}$$

where N=1/2 for Pi-mode and 1 for zero-mode.

- The cell length (center of one drift tube to center of next) is therefore

$$L_n = N(\beta_{s,n-1} + \beta_{s,n})\lambda / 2$$

Equations of Motion II

- The RF phase changes as the particle advances from one gap to the next according to

$$\phi_n = \phi_{n-1} + \omega \frac{2l_{n-1}}{\beta_{n-1}c} + \begin{cases} \pi & \textbf{π mode} \\ 0 & \textbf{0 mode} \end{cases}$$

- The phase change during the time an arbitrary particle travels from gap n-1 to gap n, relative to the synchronous particle is

$$\Delta(\phi - \phi_s)_n = \Delta\phi_n - \Delta\phi_{s,n} = 2\pi N \beta_{s,n-1}\left[\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}}\right] \cong -2\pi N \beta_{s,n-1} \frac{\delta\beta_{n-1}}{\beta_{s,n-1}^2}$$

where we have used

$$\frac{1}{\beta} - \frac{1}{\beta_s} = \frac{1}{\beta_s + \delta\beta} - \frac{1}{\beta_s} \cong -\frac{\delta\beta}{\beta_s^2}, \text{ for } \delta\beta \ll 1$$

- Using

$$\delta\beta = \frac{\delta W}{mc^2\gamma_s^3\beta_s}$$

- We get

$$\boxed{\Delta(\phi - \phi_s)_n = -2\pi N \frac{\left(W_{n-1} - W_{s,n-1}\right)}{mc^2\gamma^3_{s,n-1}\beta^2_{s,n-1}}}$$

Equations of Motion III

- Next, derive the difference in kinetic energies of the arbitrary particle and the synchronous particle:

$$\Delta\left(W - W_s\right)_n = qE_0 TL_n (\cos\phi_n - \cos\phi_{s,n})$$

- To figure out the dynamics, we could track particles through gaps on a computer using these difference equations.
- To get a feeling for the dynamics "on paper", we can convert these difference equations to differential equations by replacing the discrete action of the fields with a continuous field.
- So we replace

$$\Delta(\phi - \phi_s) \rightarrow \frac{d(\phi - \phi_s)}{dn} \qquad \Delta(W - W_s) \rightarrow \frac{d(W - W_s)}{dn} \qquad n = \frac{s}{N\beta_s\lambda}$$

- giving

$$\gamma_s^3 \beta_s^3 \frac{d(\phi - \phi_s)}{ds} = -2\pi \frac{W - W_s}{mc^2\lambda} \qquad \frac{d(W - W_s)}{ds} = qE_0 T(\cos\phi - \cos\phi_s)$$

Equations of Motion IV

- Assume acceleration rate is small, and that E_0T, ϕ_s and β_s are constant.
- We arrive at the equations of motion:

$$w' = \frac{dw}{ds} = B\left(\cos\phi - \cos\phi_s\right) \qquad \text{and} \qquad \phi' = \frac{d\phi}{ds} = -Aw$$

$$\text{with} \qquad w = \frac{W - W_s}{mc^2} \qquad \text{and} \qquad \mathrm{A} = \frac{2\pi}{\beta_s^3\gamma_s^3\lambda} \qquad \mathrm{B} = \frac{\mathrm{qE}_0 T}{mc^2}$$

$$\frac{d^2\phi}{ds^2} = -AB\left(\cos\phi - \cos\phi_s\right)$$

- Finally

$$\frac{Aw^2}{2} + B(\sin\phi - \phi\cos\phi_s) = H_\phi$$

$$\frac{1}{2}Aw^2 + V_\phi = H_\phi$$

- Where V is the potential energy term, and H (the Hamiltonian) is total energy. Technically, $-\phi$ is the canonical conjugate of *w*.

Stable RF Bucket

- There is a potential well when $-\pi < \phi_s < 0$.
- There is acceleration for $-\pi/2 < \phi_s < \pi/2$.
- The stable region for phase motion is $\phi_2 < \phi < -\phi_s$.
- The "separatrix" defines the area within which the trajectories are stable.
- The stable area is called the "bucket".
- Stable motion means that particles follow a trajectory about the stable phase, with constant amplitude given by H_ϕ.

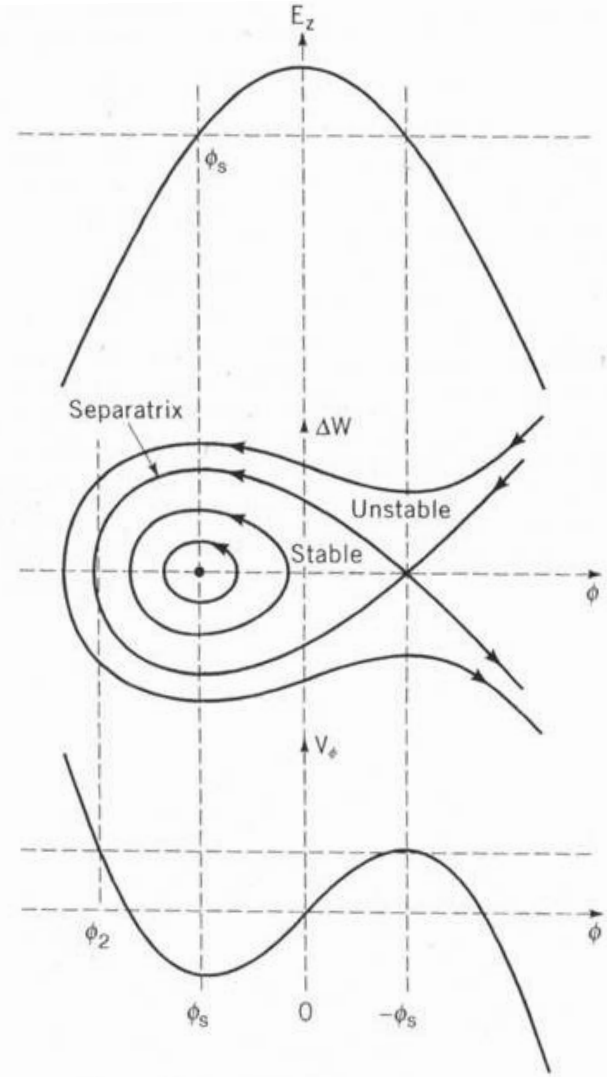


Figure 6.3. At the top, the accelerating field is shown as a cosine function of the phase; the synchronous phase ϕ_s is shown as a negative number, which lies earlier than the crest where the field is rising in time. The middle plot shows some longitudinal phase-space trajectories, including the separatrix, the limiting stable trajectory, which passes through the unstable fixed point at $\Delta W = 0$, and $\phi = -\phi_s$. The stable fixed point lies at $\Delta W = 0$ and $\phi = \phi_s$, where the longitudinal potential well has its minimum, as shown in the bottom plot.

Hamiltonian and Separatrix Parameters

- We can calculate the Hamiltonian to complete the discussion.
- At the potential maximum where, $\phi = -\phi_s$, $\phi'=0$ and w=0

$$H_\phi = B(\sin(-\phi_s) - (-\phi_s \cos\phi_s))$$

- The points on the separatrix must therefore satisfy

$$\frac{Aw^2}{2} + B(\sin\phi - \phi\cos\phi_s) = -B(\sin\phi_s - \phi_s\cos\phi_s)$$

- We can calculate the "size" of the separatrix. We will do the energy width. The maximum energy width corresponds to $\phi = \phi_s$

$$\frac{Aw_{\max}^2}{2} + B(\sin\phi_s - \phi_s\cos\phi_s) = -B(\sin\phi_s - \phi_s\cos\phi_s)$$

- Giving for the energy half-width of the separatrix. The ***energy acceptance*** is twice this value:

$$w_{\max} = \frac{\Delta W_{\max}}{mc^2} = \sqrt{\frac{2qE_0T\beta_s^3\gamma_s^3\lambda}{\pi mc^2}(\phi_s\cos\phi_s - \sin\phi_s)}$$

Phase Width

- The maximum ***phase width*** is determined from the two solutions for *w=0*. One solution is $\phi_1 = -\phi_s$. The other solution ϕ_2 is given by

$$\sin\phi_2 - \phi_2\cos\phi_s = \phi_s\cos\phi_s - \sin\phi_s$$

- The total phase width is $\Psi = -\phi_s - \phi_2$
- The phase width is zero at $\phi_s=0$ and maximum at $\phi_s=-\pi/2$, giving $\psi=2\pi$ (see Wangler figure 6.4).

Small Amplitude Oscillations

- Look at small amplitude oscillations. Letting ϕ-ϕ_s be small,

$$(\phi - \phi_s)'' + AB \sin(-\phi_s)(\phi - \phi_s) = 0$$

- This is an equation for simple harmonic motion with an angular frequency given by

$$\omega_l^2 = \frac{\omega^2 q E_0 T \lambda \sin(-\phi_s)}{2\pi mc^2 \gamma_s^3 \beta_s}$$

- Note that as the beam becomes relativistic, the frequency goes to zero.
- From the equation of motion we can calculate the trajectory of a particle:

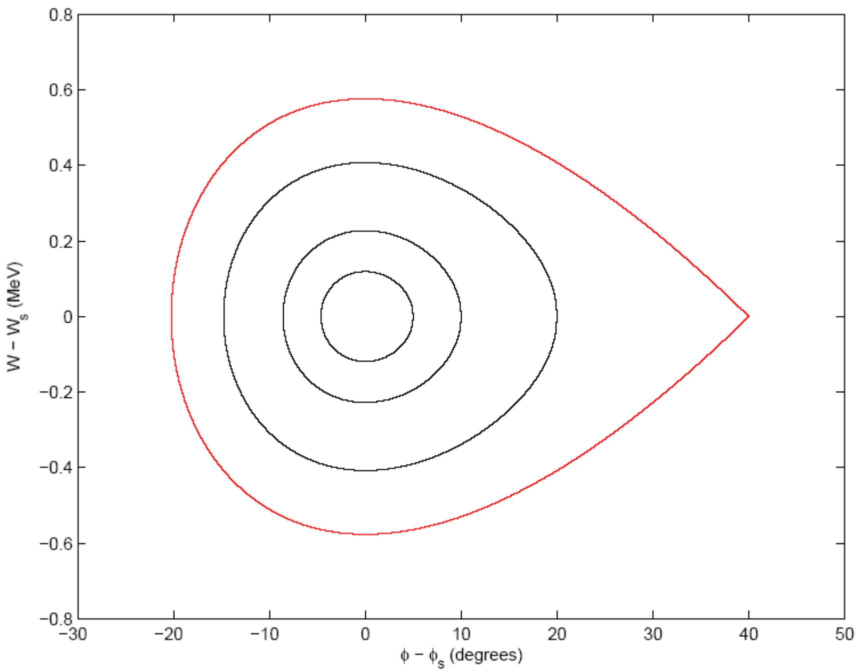
$$\frac{w^2}{w_0^2} + \frac{(\phi - \phi_s)^2}{(\Delta\phi_0)^2} = 1 \qquad w_0 = \frac{\Delta W}{mc^2} = \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda \sin(-\phi_s) \Delta\phi_0^2 / 2\pi mc^2}$$

- This is the equation of an ellipse in w, ϕ-ϕ_s *phase space.*
- Particles on a particular ellipse circulate indefinitely on that trajectory.

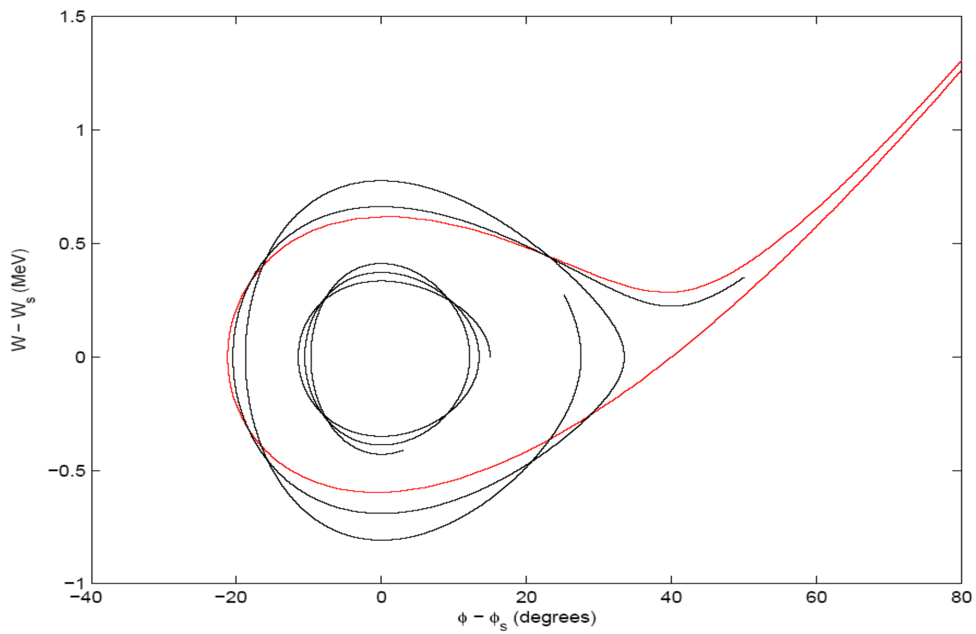
Longitudinal Phase Space Motion

- We studied the approximation of small acceleration rate, and constant velocity, synchronous phase, etc.
- In a real linac, the velocity increases, and the phase space motion and separatrix becomes more complicated.
- The "acceptance" takes a shape called the "golf-club".

$\beta\gamma = \text{const}$



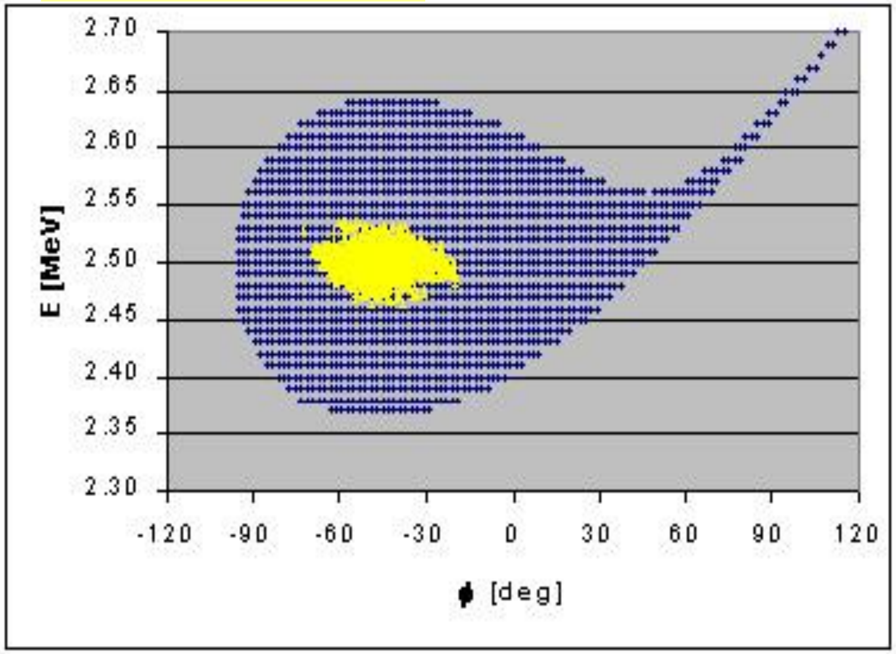
$\beta\gamma \neq \text{const}$



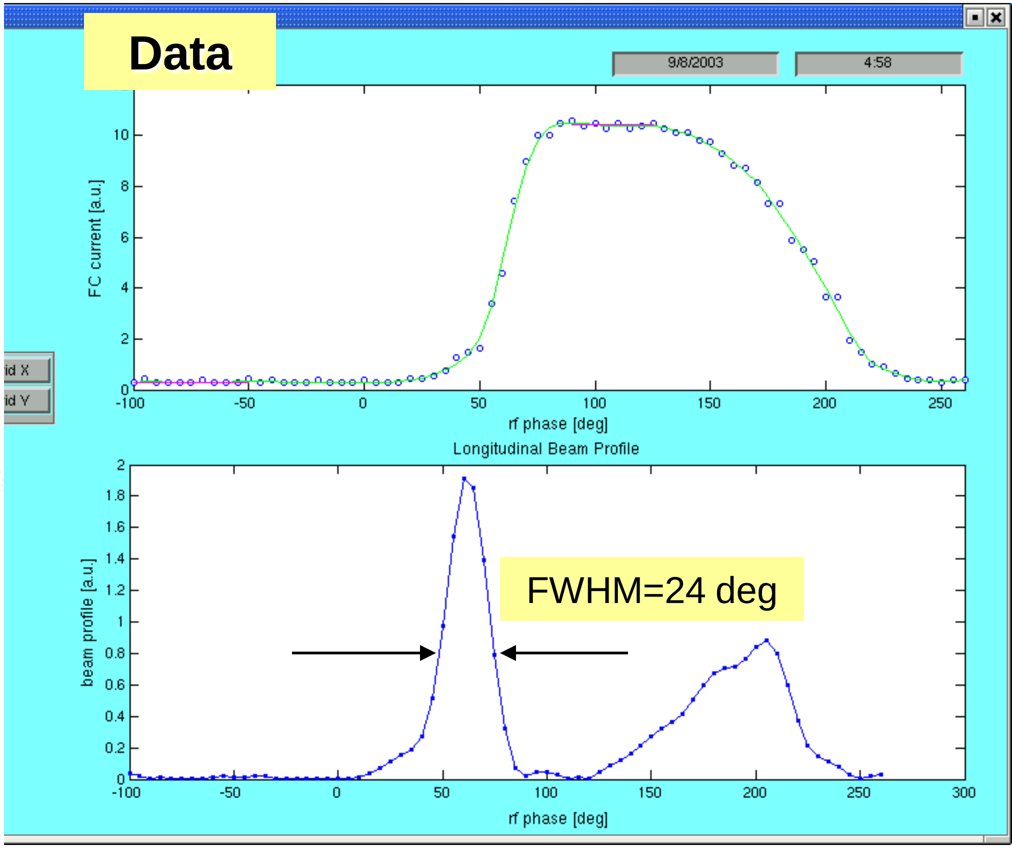
Longitudinal Dynamics: Real data from SNS Drift Tube Linac

- Longitudinal "Acceptance Scan"

Simulated DTL1 Acceptance

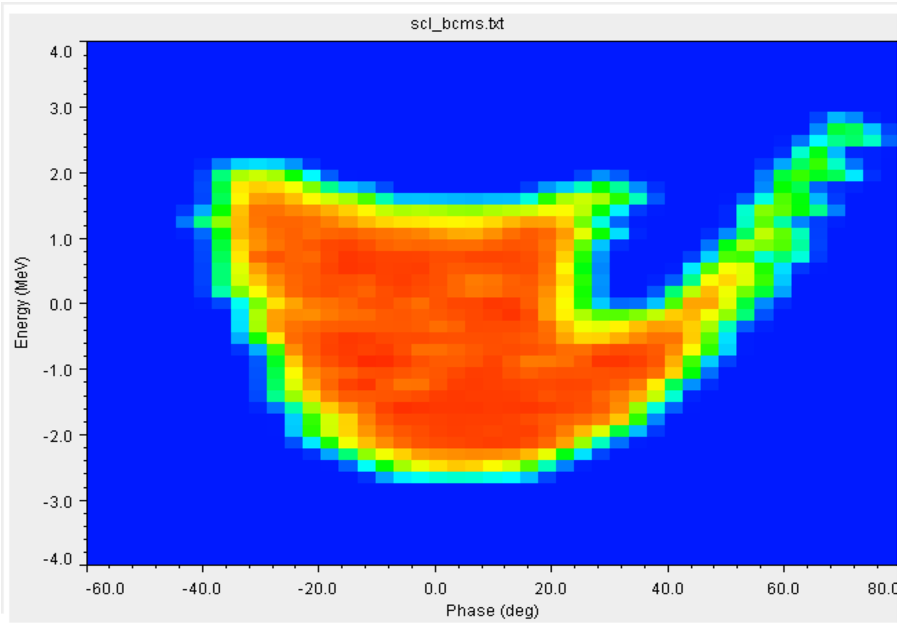


Data

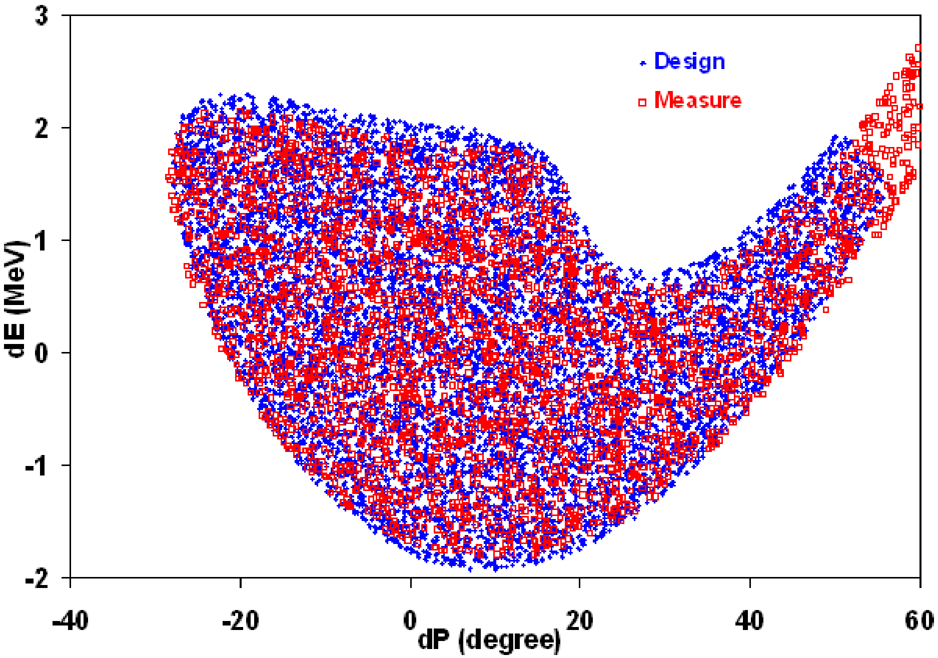


Measurement of SNS SC Linac Acceptance (Y. Zhang)

Measurement



Simulation



Adiabatic Phase Damping

- Louiville's theorem:

  The density in phase space of non-interacting particles in a conservative or Hamiltonian system measured along the trajectory of a particle is invariant.

- Or, if you prefer: *phase space area is conserved.*
- Area of ellipse:

$$Area = \pi \Delta \phi_0 \Delta W_0$$

- Which gives

$$\Delta \phi_0 = \frac{const}{(\beta_s \gamma_s)^{3/4}} \qquad \Delta W_0 = const \times (\beta_s \gamma_s)^{3/4}$$

- Since area is conserved an initial distribution with phase width $(\Delta\phi)_i$ acquired a smaller phase width after acceleration:

$$\frac{(\Delta \phi_0)_f}{(\Delta \phi_0)_i} = \frac{(\beta \gamma)_i^{3/4}}{(\beta \gamma)_f^{3/4}}$$

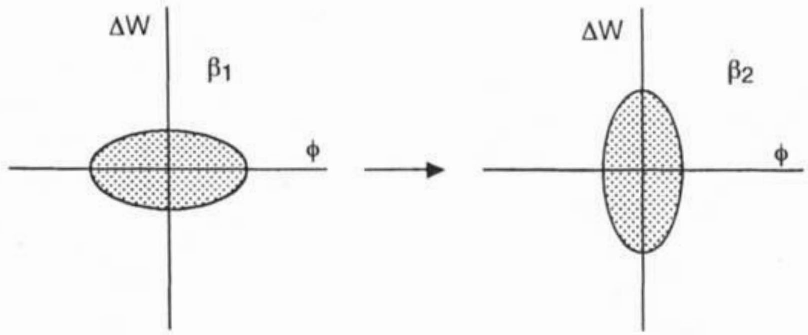


Figure 6.8. Phase damping of a longitudinal beam ellipse caused by acceleration. The phase width of the beam decreases and the energy width increases while the total area remains constant.

The End

- That concludes our whirlwind tour of Linear Accelerators
- Now, on to Rings….