Chapter 2 Beam Dynamics I Transverse

Transverse Beam Dynamics

Focusing Sequences

Strong Focusing

Computer Modeling

Interaction of Charged Particles with Fields

The force on a particle of charge *q* in fields ***E*** and ***B*** is

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

We can break this vector equation down to its components

$$\begin{aligned} F_x &= q(E_x + v_y B_z - v_z B_y) \\ F_y &= q(E_y + v_z B_x - v_x B_z) \\ F_z &= q(E_z + v_x B_y - v_y B_x) \end{aligned}$$

The force is parallel to the E-vector.
The force is perpendicular to the plane formed by the velocity and the B-vector.

The (non-relativistic) effect on a particle is given by Newton's law

$$\vec{F} = m\vec{a} = m\ddot{\vec{x}}$$

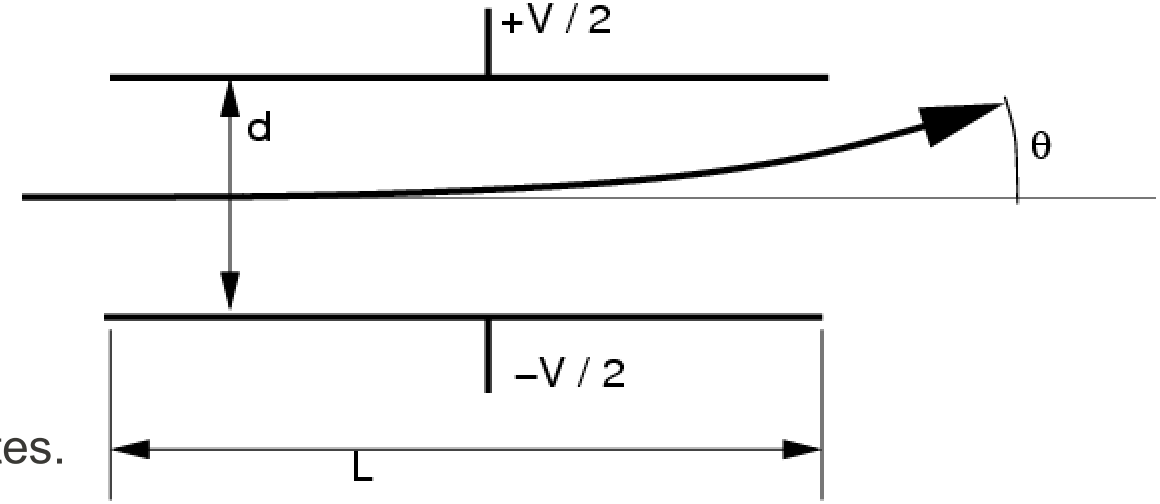
The change in momentum over a time interval is the *impulse*

$$\Delta\vec{p} = \vec{F}\cdot\Delta t$$

Electrostatic Deflector

A field E_y exists between the plates of length L and separation d

$$E_y = \frac{V}{d}$$



The proton spends a time

$$t = \frac{L}{\beta c}$$ between the plates.

A proton with momentum p_z comes in from the left. It acquires a change in vertical momentum $\Delta p_y = F_y \Delta t = e E_y L / \beta c$

The change in angle of the trajectory is (non-relativistically)

$$\theta = \frac{\Delta p_y}{p_z} = \frac{e E_y L}{\beta c} \cdot \frac{1}{\beta m_p c}$$

Remember that the beam rigidity of a proton is defined by

$$R = \frac{\beta \gamma}{c} \left(\frac{m_p c^2}{e} \right)$$

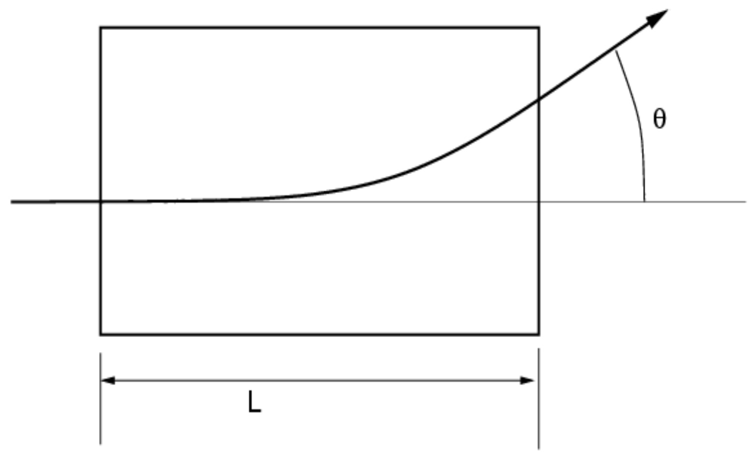
So the deflection angle is

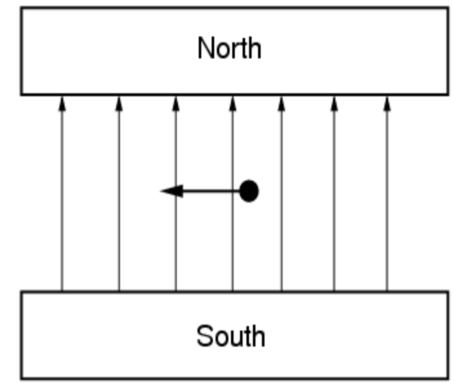
$$\theta = \left(\frac{E_y}{\beta c} \right) \frac{L}{R}$$

Magnetic Deflector (Dipole Bend Magnet)

A proton with momentum p_z enters a field $\boldsymbol{B_y}$ from the left.

The proton is deflected to the left as it enters the paper with the north pole of the magnet on top.





$$F_x = -q v_z B_y$$

$$\Delta p_x = -q v_z B_y \frac{L}{\beta c}$$

$$\theta = \frac{\Delta p_x}{p_z} = -\frac{c B_z L}{\beta}\left(\frac{e}{m_p c^2}\right)$$

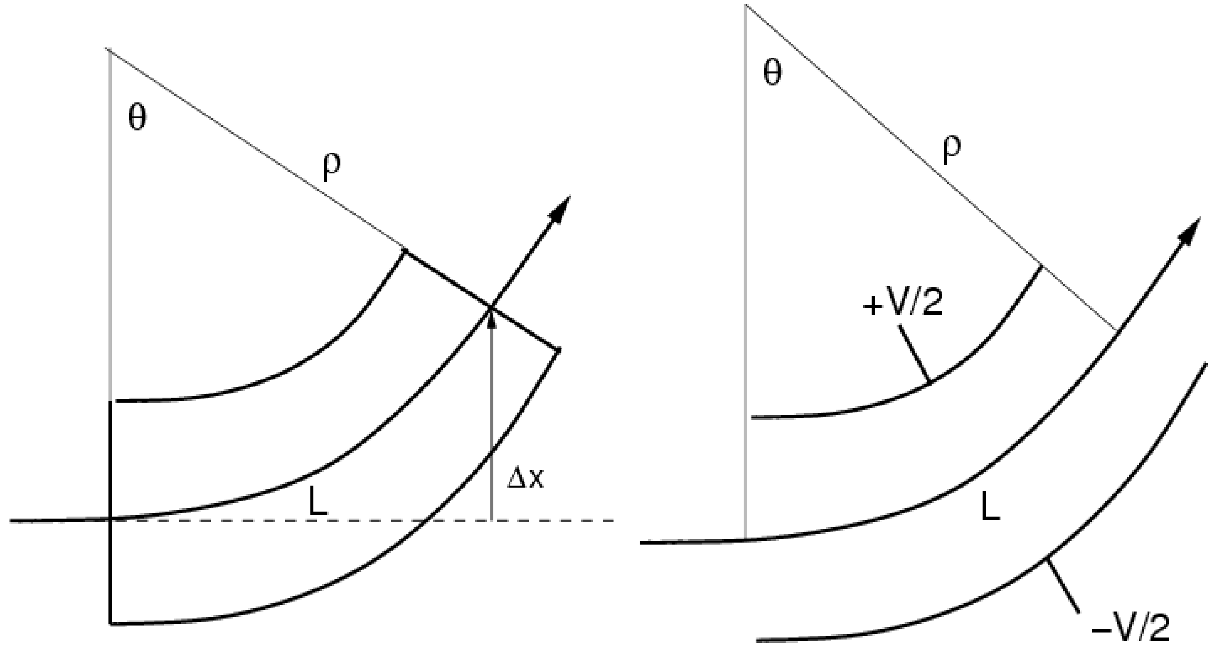
Or, substituting the definition of rigidity, proportional to momentum

$$\theta = \frac{B_y L}{R} \quad \frac{Tesla \cdot meters}{Tesla\, meters}$$

Radius of Curvature of the Orbit

Magnet $B L = R \theta$

Electrostatic $\frac{E_r}{\beta c} = R \theta$



$$\textit{radius of curvature} = \rho = \frac{L}{\theta}$$

The displacement of the orbit from the original trajectory at the exit is

$$\Delta x = \rho(1-\cos\theta) \approx \frac{\rho\theta^2}{2}$$ For small angle

Compare Relative Electrostatic and Magnetic Strengths

$$ES: \qquad \Delta\theta = \frac{E_y L}{\beta c}\frac{L}{R}$$

$$EM: \qquad \Delta\theta = B_z \frac{L}{R}$$

Equate the deflection angles

$$E_y = \beta c B_z$$

Typical strengths: E = 10 MV/m (100 kV/cm), quite high
B = 2 Tesla (iron pretty well saturated)

$$\beta = 0.0167$$

$$T = \frac{1}{2} m_p c^2 \beta^2 = 130\, keV$$

For a 130 keV proton beam, the deflecting strengths are identical.
Below that, electrostatic deflectors can be more powerful, above that, magnets.

The Wien Filter

The Wien filter is a velocity selector. A particle travels through cross B_y and E_x fields with the proper polarity that the forces cancel out for one particular velocity. Particles with other velocities will be deflected.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

$$F_x = q(E_x - \beta c B_y)$$

Using the previous numbers of crossed fields of 2 Tesla and 10 MV/m, a 130 keV proton beam will be undeflected. The deflection in a Wien filter of length L for a small angle deflection is

$$\theta_E - \theta_B \simeq \left(\frac{E}{\beta^2} - \frac{B c}{\beta} \right) L \left(\frac{e}{m_p c^2} \right)$$

Verify this equation.

B-Field in a Magnetic Dipole

$$\nabla \times H = J$$

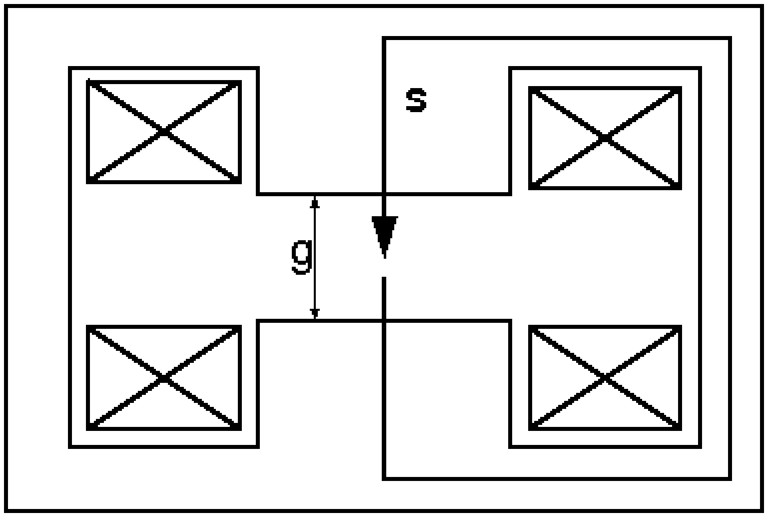
$$\oint H\,dl = N\,I$$

$$H\left(\frac{g}{\mu=1}+\frac{s}{\mu_{steel}}\right) = N\,I$$

If $\mu_{steel} \gg 1$

$$H = \frac{N\,I}{g} = \frac{B}{\mu_0}$$

$$[B]=\frac{volt\,sec}{m^2}, \quad [H] = \frac{amp}{m}$$



Field concentrated in gap, assume the permeability of steel to be very high.

N I is the current in the conductor times the number of turns in both the upper and lower coil.

This equation is good to usually better than 95%. A more exact expression includes the permeability of the steel, particularly if it is approaching saturation at 15 kGauss, and flux lines (leakage flux) that do not encircle the windings and the gap.

Homework Problems 2.1

Calculate the frequency and E and H field amplitudes for a laser

Calculate the parameters of a Wien filter.

Calculate the parameters of a magnet dipole.

Strong Focusing

Alternating gradient (AG) focusing is known as strong focusing. It was first discovered in the design of synchrotrons by Courant and Snyder at BNL and Christofolis in Europe. (BNL promptly hired Christofolis to avoid certain problems.)

Previous weak focusing synchrotrons were:

Cosmotron (BNL) 3 GeV
Bevatron (LBNL) 6.3 GeV (BeV) sized to discover the antiproton
Synchrophasotron (Dubna) 10 GeV - still running, with heavy ions
ZGS (ANL) 12.5 GeV – largest weak-focusing synchrotron ever built.

One could crawl around in the vacuum chambers of these machines.

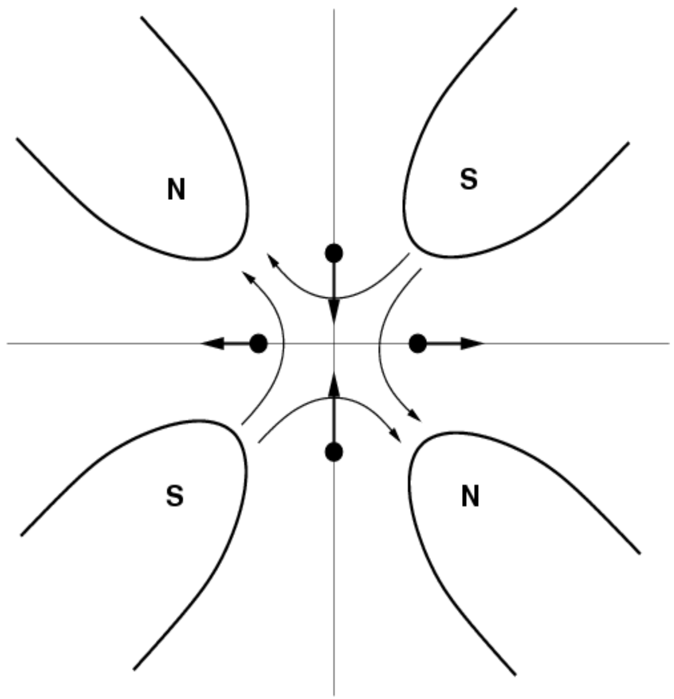
BNL then built the AGS, simultaneously with the PS at CERN. Both are still very active, after 50 years, supplying protons, antiprotons and heavy ions to still larger rings.

The first linacs used no focusing, then grid and solenoid focusing, so the transmission was small. The introduction of strong focusing to linacs allowed the transmission to approach 100%, and allowed the acceleration of very high intensities.

The Magnetic Quadrupole, A Focusing Device

The four poles are excited with a quadrupole magnetic field with the following components.

$$B_x = g\,y$$
$$B_y = g\,x$$



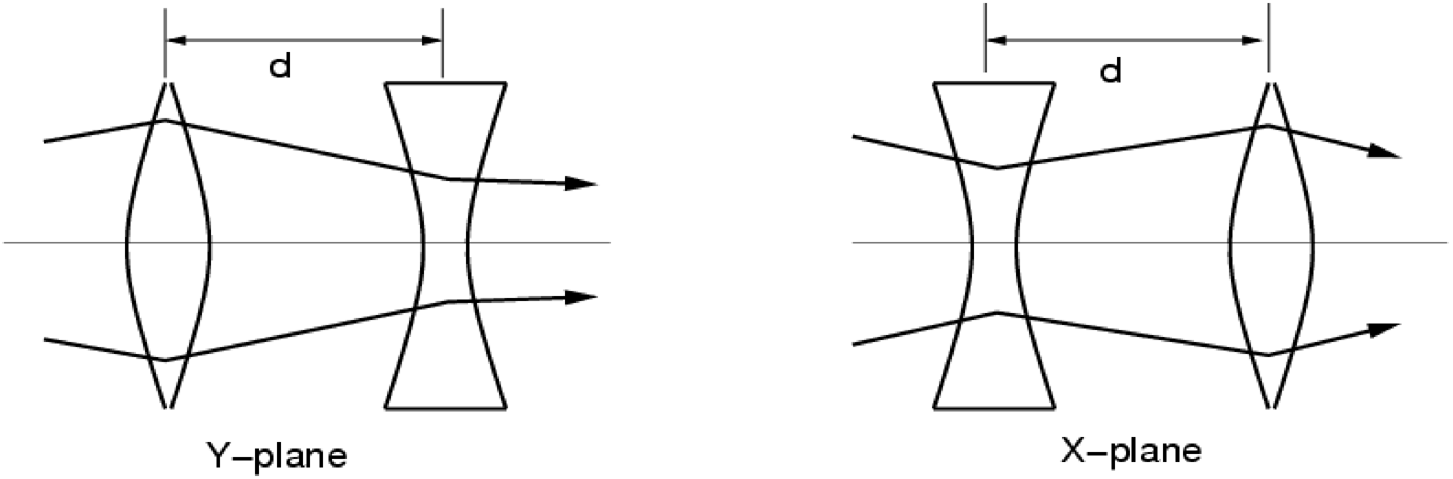
A proton going into the paper is deflected inward along the y-axis, and outward along the x-axis.

Notice that the quadrupole focuses in one plane, and defocuses in the other. Reversing the field polarity reverses the focusing/defocusing planes.

An approximate optical analogy is a cylindrical lens. Quadrupoles do not focus simultaneously in both planes.

FODO Lattice

FODO refers to Focus, Drift, Defocus, Drift, the basis of strong or alternate gradient focusing.



For lenses of strengths f_1 and f_2, separated by distance d, the strength is

$$\frac{1}{f_{tot}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

If $f_1 = -f_2$,

$$\frac{1}{f_{tot}} = \frac{1}{f_1} - \frac{1}{f_1} + \frac{d}{f_1^2} = \frac{d}{f_1^2} > 0$$

And the overall focusing strength is positive (focusing). for both FODO and DOFO sequences.

FODO / DOFO Transport Matrix

A sequence that is FODO in one plane is DOFO in the other plane.

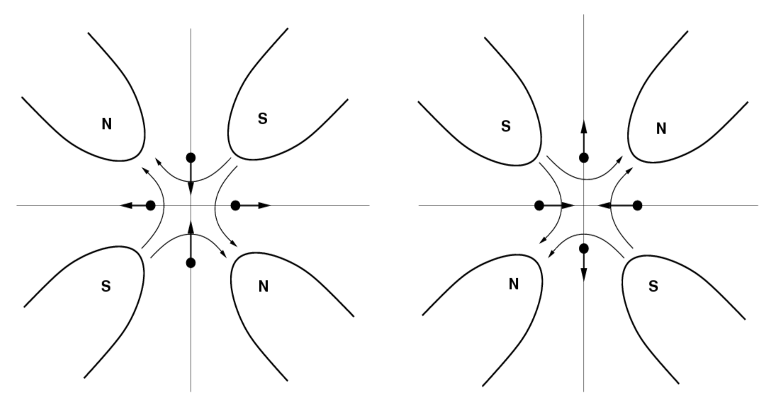
A ODOF lattice is represented by the series of 2x2 transport matrices

$$\underset{\text{F}}{\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}} \underset{\text{O}}{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}} \underset{\text{D}}{\begin{bmatrix} 1 & 0 \\ 1/f & 1 \end{bmatrix}} \underset{\text{O}}{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1+\dfrac{d}{f} & 2d+\dfrac{d^2}{f} \\ \dfrac{-d}{f^2} & 1-\dfrac{d}{f}-\dfrac{d^2}{f^2} \end{bmatrix}$$

The m_{21} element is less than zero, hence overall focusing.

Homework: show the same for the OFOD sequence.

Nomenclature:
F-quad: focusing in x-plane
D-quad: focusing in y-plane

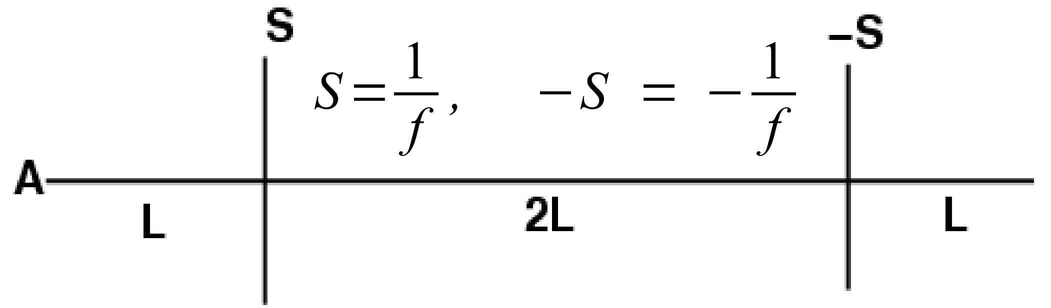


D-quad F-quad

Focusing Down a Periodic Channel

We cast it in a form that allows analysis of an infinitely long periodic OFOODO lattice. This treatment is due to Courant and Snyder. We will assume thin lenses here.

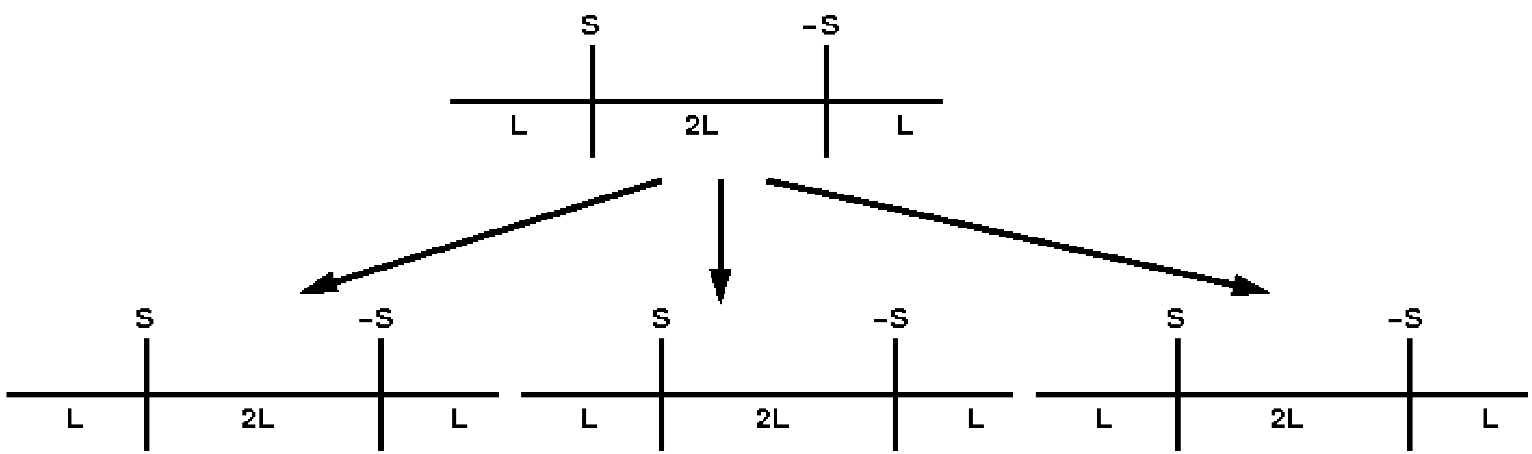
Lenses of strength *S* and *-S* occupy segment of length 4*L*.



$$S = \frac{1}{f}, \qquad -S = -\frac{1}{f}$$

$$M_{A\to B} = \begin{bmatrix} 1 - 2LS - 2L^2S^2 & 4L - 2L^3S^2 \\ -2LS^2 & 1 + 2LS - 2L^2S^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Take the unit focusing segment, and iterate it k times.



Focusing Down a Periodic Channel II

For one OFOODO unit in the lattice, the matrix is:

$$M_{A \to B} = \begin{bmatrix} 1 - 2LS - 2L^2S^2 & 4L - 2L^3S^2 \\ -2LS^2 & 1 + 2LS - 2L^2S^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let

$$\cos\mu = \frac{1}{2}(a+d) = 1 - 2L^2S^{2,} \quad \sin\mu = 2LS\sqrt{1-L^2S^2}$$

$$\begin{aligned} a-d &= 2\alpha\sin\mu \\ b &= \beta\sin\mu \\ c &= -\gamma\sin\mu \end{aligned}$$

$$\alpha = -\frac{1}{\sqrt{1-L^2S^2}}$$

$$\beta = \frac{\left(\frac{2}{S} - L^2S\right)}{\sqrt{1-L^2S^2}}$$

α, β and γ are the Twiss parameters

$$\gamma = \frac{S}{\sqrt{1-L^2S^2}}$$

Focusing Down a Periodic Channel III

Then the transform matrix for a unit OFOODO sequence

$$M_{A\to B} = \begin{bmatrix} 1-2LS-2L^2S^2 & 4L-2L^3S^2 \\ -2LS^2 & 1+2LS-2L^2S^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Can be recast as the sum of two matrices *I* and *J*

$$\begin{aligned} M_{A\to B} &= \vec{I}\cos\mu + \vec{J}\sin\mu \\ M^k_{A\to B} &= \vec{I}\cos k\mu + \vec{J}\sin k\mu \\ M^{-1}_{A\to B} &= \vec{I}\cos\mu - \vec{J}\sin\mu \end{aligned}$$

where $\vec{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \vec{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$

As $\cos\mu = \frac{1}{2}(a+d) = 1-2L^2S^2$

The motion is stable if $|a+b| < 2, \qquad -1 < \left(1-2L^2S^2\right) < 1$

Focusing Down a Periodic Channel IV

The repetitions can be expressed as trig functions. A single FODO unit is

$$M_{A \to B} = \vec{I} \cos\mu + \vec{J} \sin\mu$$

where

$$\vec{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \vec{J} = \begin{bmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{bmatrix}$$

For a lattice of k unit FODO cells, the overall transfer matrix is

$$M^{k}_{A \to B} = \vec{I} \cos k\,\mu + \vec{J} \sin k\,\mu$$

So matrix multiplication is replaced by multiplying the argument of a trig function and μ is the phase advance per unit period.

Matched Functions in a Lattice

In a periodic FODO lattice, there is a particular set of betatron functions that repeats every 2π of phase advance. (This is the solution to an eigenvalue equation with unity eigenvalue.)

$$S\begin{bmatrix}\beta\\ \alpha\\ \gamma\end{bmatrix}_i = \begin{bmatrix}\beta\\ \alpha\\ \gamma\end{bmatrix}_f$$

Recall that for a periodic lattice with phase advance *m* per period, that for k periods

$$M = \begin{bmatrix}a & b\\ c & d\end{bmatrix} = \begin{bmatrix}\cos k\mu + \alpha\sin k\mu & \beta\sin k\mu\\ -\gamma\sin k\mu & \cos k\mu - \alpha\sin k\mu\end{bmatrix}, \quad \cos\mu = 1-2L^2S^2$$

Solving the eigenvalue equation gives the matched functions at the beginning of a beam line specified by matrix *M*

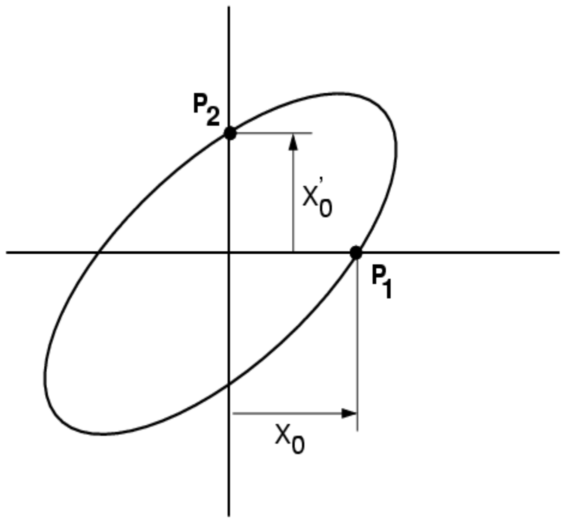
$$\beta_x = \frac{|b|}{\sqrt{1-(\frac{a+d}{2})^2}}, \quad \alpha_x = \frac{a-d}{2b}\beta_x$$

The phase advance per period is $\mu = \tan^{-1}\left(\frac{b}{\beta_i a - \alpha_i b}\right)$

Sine-like and Cosine-like Rays

The transport matrix down the regular FODO channel for a given phase advance μ is

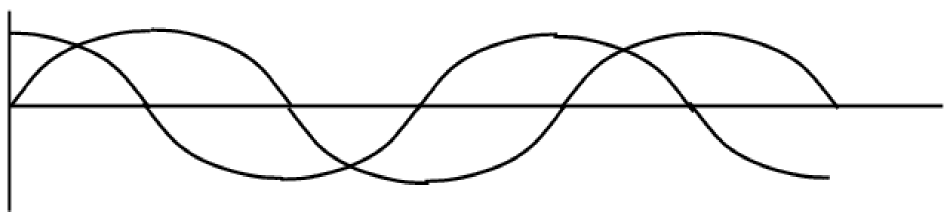
$$M^{\mu} = \begin{bmatrix} \cos k\mu + \alpha \sin k\mu & \beta \sin k\mu \\ -\gamma \sin k\mu & \cos k\mu - \alpha \sin k\mu \end{bmatrix}$$



We can choose two points on the ellipse which trace the ellipse

$$P_1 = \begin{bmatrix} \sqrt{\dfrac{\epsilon}{\gamma}} \\ 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0 \\ \sqrt{\dfrac{\epsilon}{\beta}} \end{bmatrix}$$

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon$$

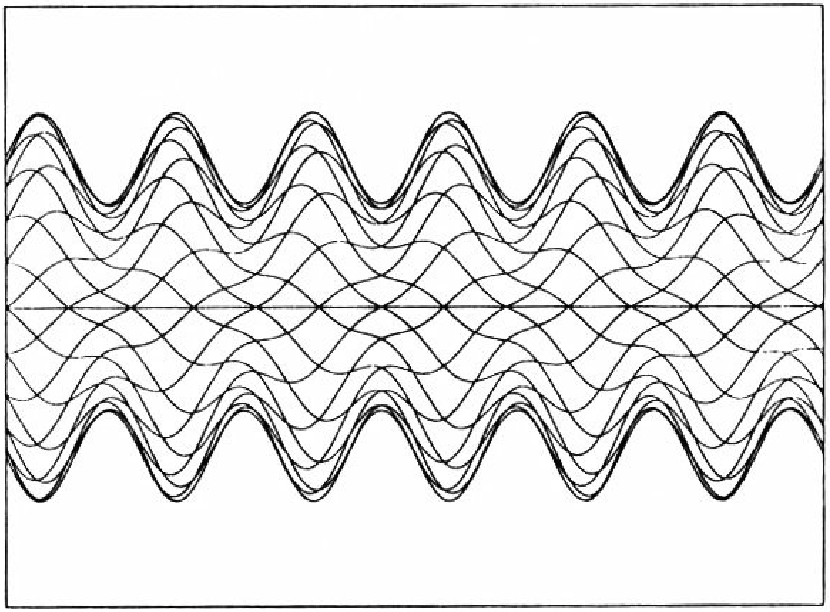


The beam envelope is given by the quadratic sum of the sin and cosine-like rays.

$$x = \sqrt{[M^{\mu} P_1]_1^2 + [M^{\mu} P_2]_1^2}$$

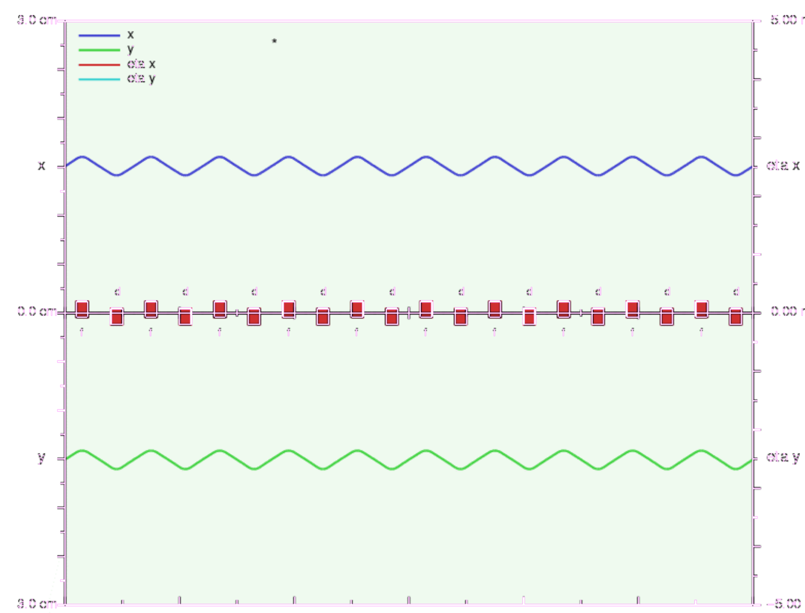
The periodicity of each ray is given by the phase advance μ per focusing period.

Beam Envelope of Many Rays



The beam envelope is comprised of all the rays in the beam.

The envelope, which encloses all the rays in the beam, will be modulated by the presence of the focusing element in the beam line.



Focusing Passband

Recall the matched Twiss parameters of a FODO lattice as a function of the focusing strength S and spacing between the focusing elements L and the phase advance μ for a unit FODO cell.

$$\alpha = -\frac{1}{\sqrt{1-L^2S^2}}$$

$$\beta = \frac{\left(\frac{2}{S} - L^2 S\right)}{\sqrt{1-L^2S^2}}$$

$$\gamma = \frac{S}{\sqrt{1-L^2S^2}}$$

As the parameter $2L^2S^2$ approaches unity, β becomes large, as does the beam size

$$x = \sqrt{\beta\epsilon}$$

If the focusing strength S becomes small, the factor $2/S$ in β becomes large, and the beam size also becomes large.

There is an optimum value for LS that minimizes the betatron functions for the lattice, and a corresponding phase advance μ for that optimum.

$$\cos\mu = 1 - 2L^2S^2$$

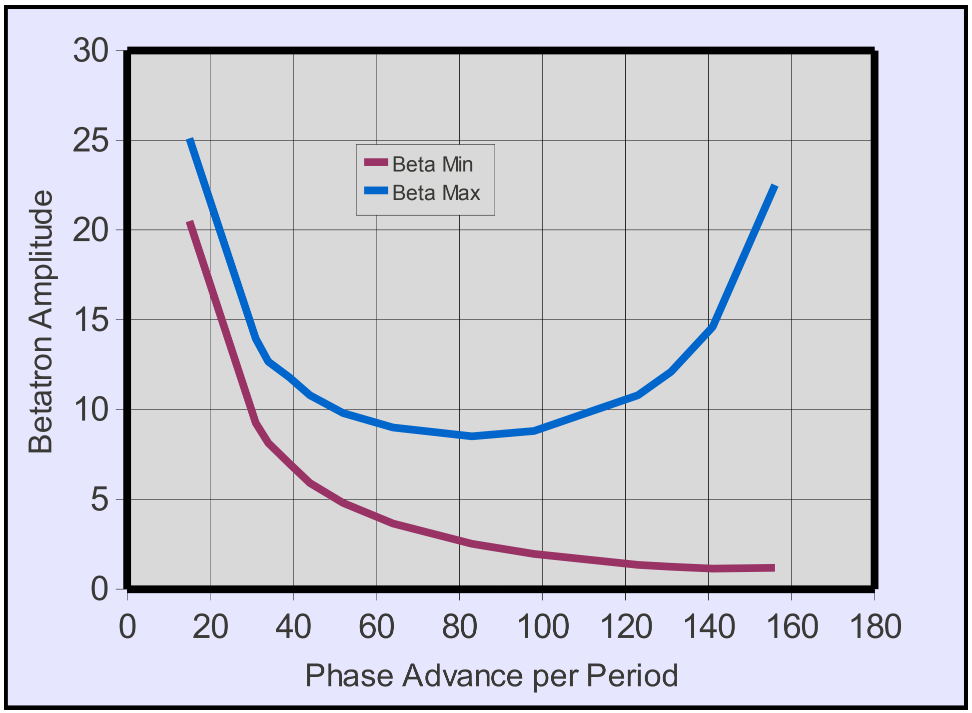
For L = 0.5 meters,

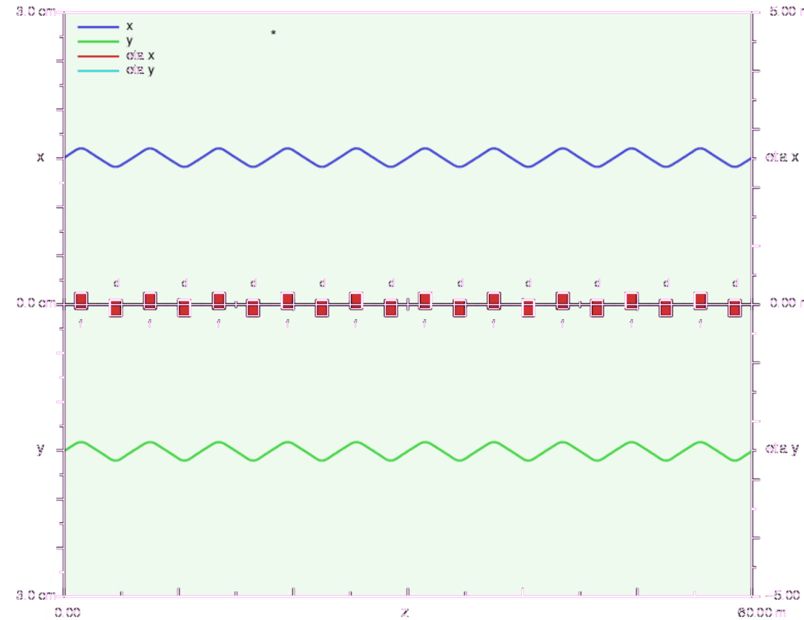
$$S = \sqrt{\frac{8}{3}}, \quad \cos\mu = -\frac{1}{3}, \quad \mu = 109.47 \text{ degrees}$$

Peak Betatron Amplitude vs. Tune

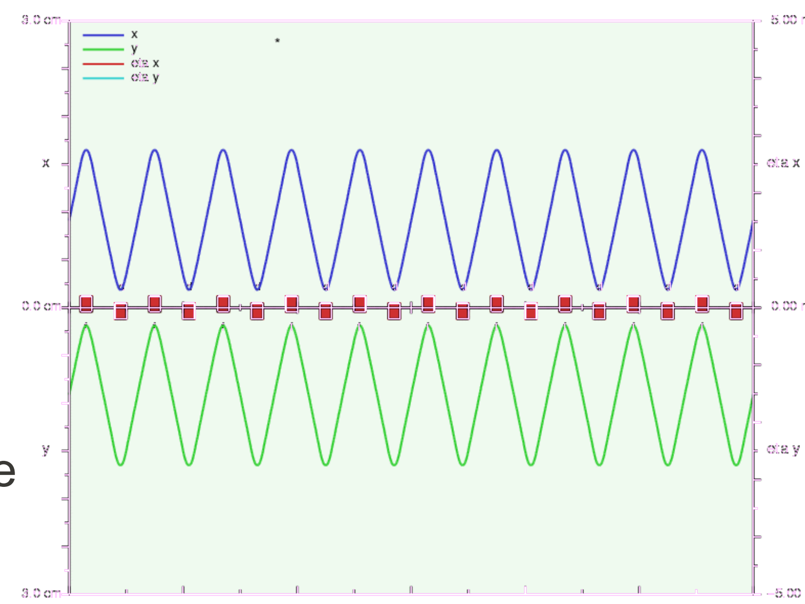
The ratio of maximum to minimum betatron amplitude depends on the tune. The beam size scales as

$$x = \sqrt{\beta \epsilon}$$





15 degrees per period

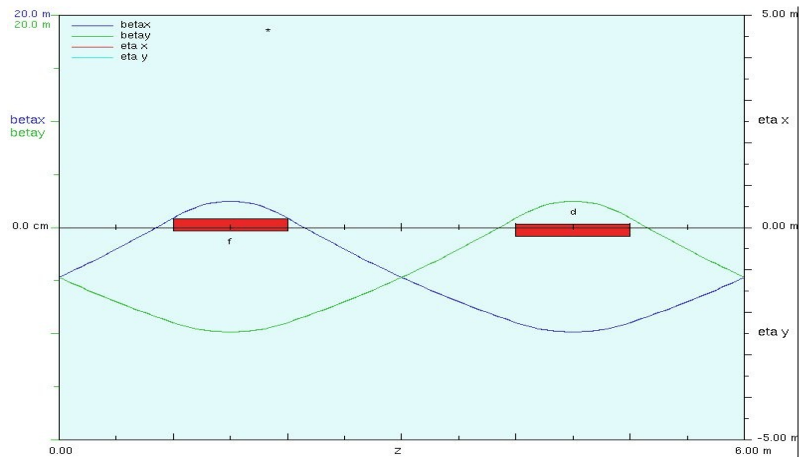


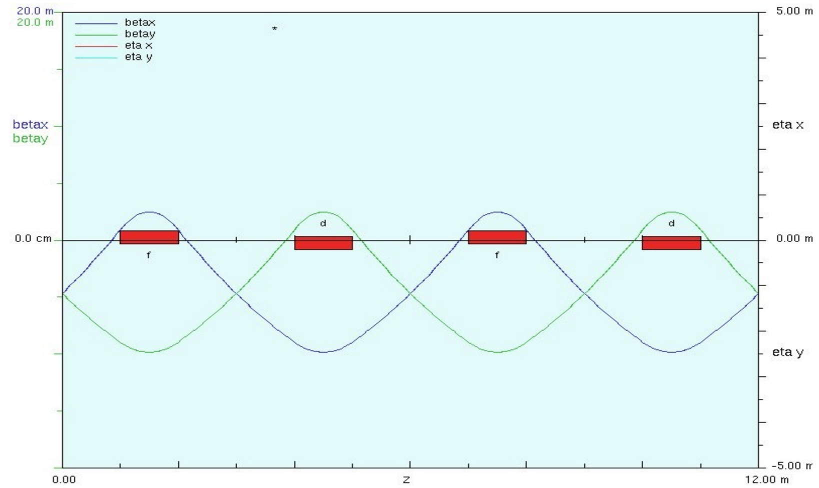
155 degrees per period

β_{max} is stable below 180 degrees/period, with a minimum around 80 degrees. The flutter increases with tune. Small beam waists increase influence of **space charge**.

What would be an optimal phase advance? Why?

Matched Beam in Periodic Lattice

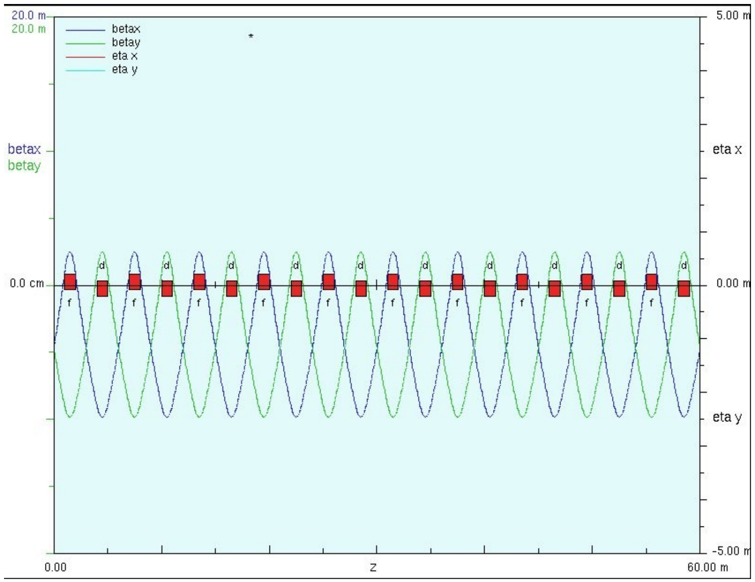




The unit OFOODO cell is 6 meters long:
1 meter drifts, 1 meter quadrupoles. The phase advance per cell is 46 degrees.

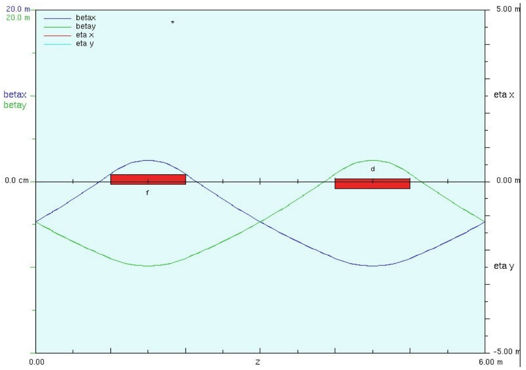
The plots show β function variation over the cell. The β function value is the same for both the x- and y-planes at each end of the cell.

This variation in the beam envelope is called the "flutter", and mirrors the distribution of the focusing elements.

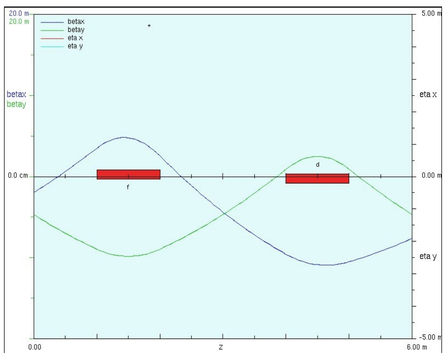


Mismatched Beam in the Lattice

β_x is changed from the matched value. The value of β_x is plotted over 10 lattice periods. Note that the peak values oscillate with a wavelength of about 3.9 lattice periods, or a phase advance of the envelope "breathing" of about 92 degrees per unit lattice period.



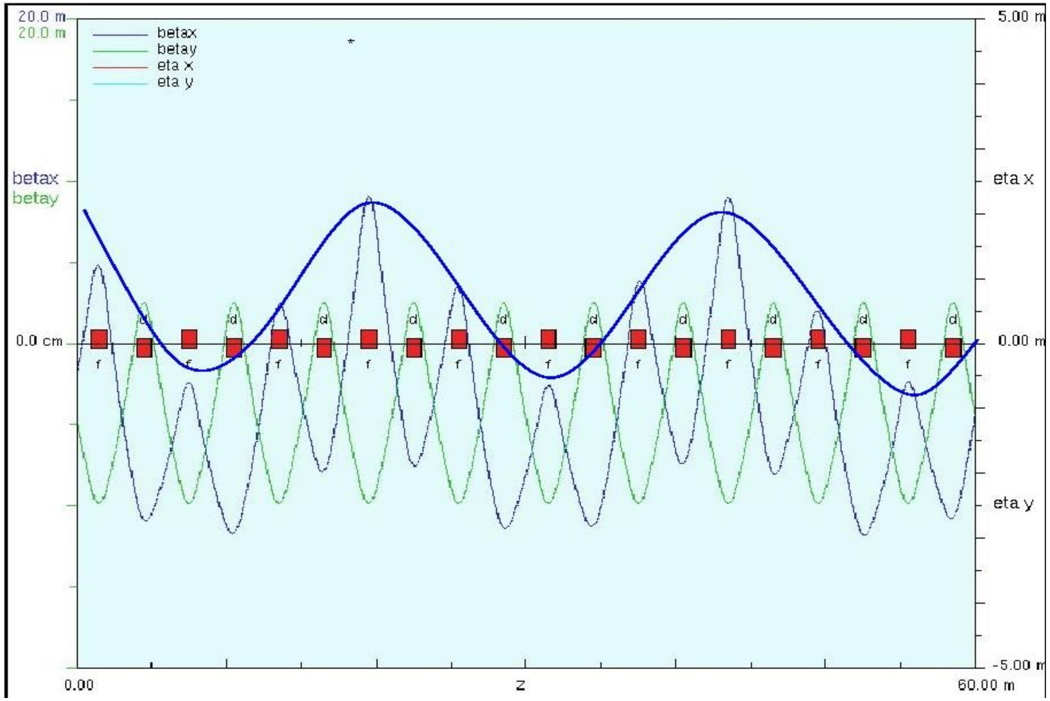
Matched Beam



Unmatched x-beam

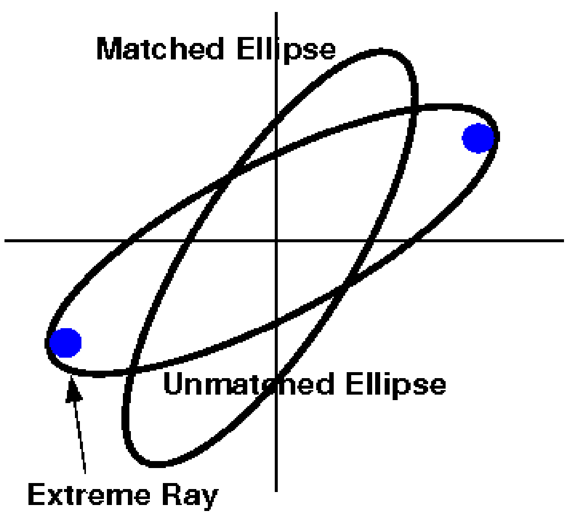
The phase advance of an orbit is 46 degrees per unit cell.

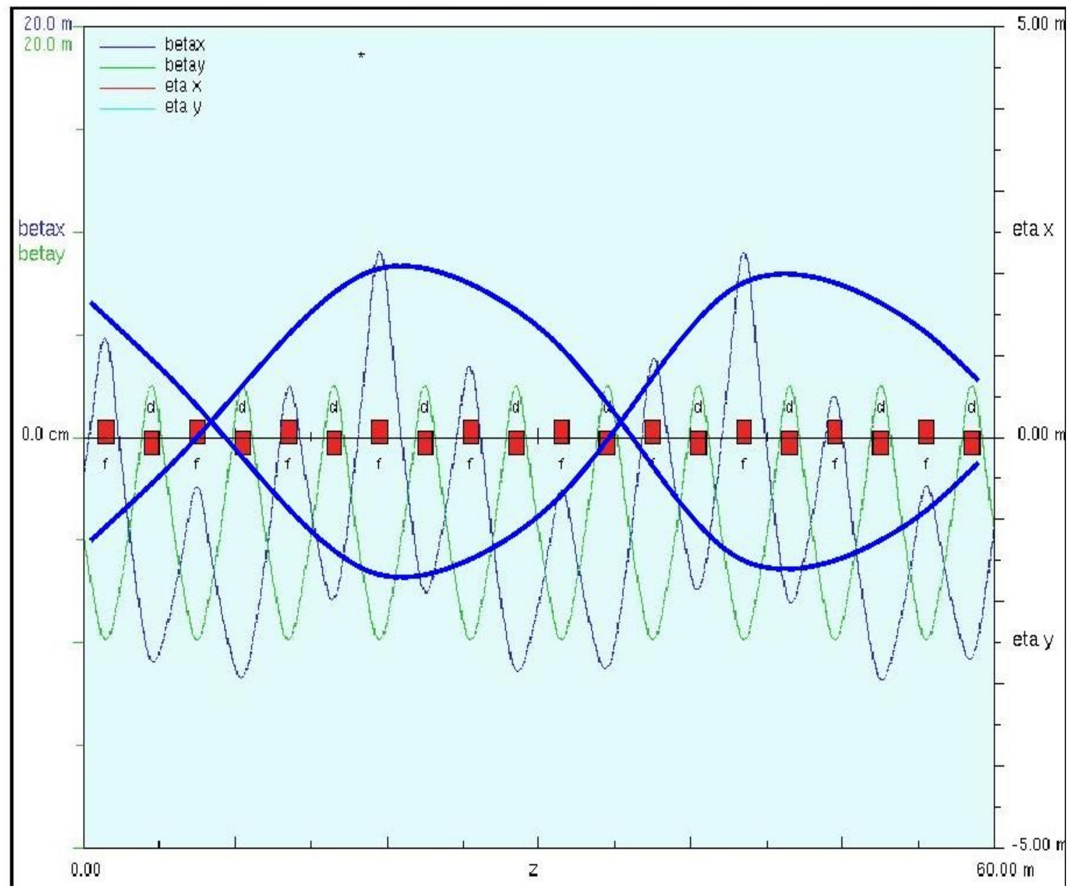
Why is the envelope breathing taking place at twice the phase advance?



Envelope Breathing of a Mismatched Beam

The reason that the envelope breathing seems to occur at twice the phase advance per cell is that the mismatched beam can be thought of launching two extreme rays with opposite signs. Each of these rays oscillate at a phase advance of 46 degrees per unit cell, but their combined peak amplitudes seem to breathe at twice the phase advance, or 92 degrees per cell.





the mismatched beam has rays outside of the matched emittance ellipse. They propagate with the same phase advance as the others, but add to the maximum envelope.

Quadrupole Position Errors

In a strong focusing (FODO) channel, quadrupole transverse alignment is important. In a linac, the drift tubes contain quadrupoles, which must be aligned to a very small tolerance.

If a quadrupole is displaced, there will be a dipole field on the axis, which will deflect the beam, which will then follow an oscillatory orbit.

$$B_{dipole\,equivalent} = B' \Delta x, \qquad \Delta\theta = \frac{B' \Delta x \, L_{quad}}{R}$$

This initial deflection angle Dq will cause the beam centroid to move around an ellipse with initial value at P_2 (previous slide).

Transformation of Betatron (Twiss) Parameters

We have seen how a beam vector transforms through a transport line. We can also specify the beam in terms of its Twiss parameters that transform through the line. *The Twiss parameters are a property of the transport line and are more fundamental than the properties of the beam being transported.*

Repeat definitions:

$$\beta\epsilon = \sigma_x^2$$

$$\gamma\epsilon - \sigma_{x'}^2$$

$$\alpha\epsilon = -\sigma_{xx'}, \qquad \alpha^2+1 = \beta\gamma$$

$$\epsilon^2 = \sigma_x^2\sigma_{x'}^2-(\sigma_{xx'})^2$$

If

$$\begin{bmatrix} x \\ x' \end{bmatrix}_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ x' \end{bmatrix}_i$$

then

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_f = \begin{bmatrix} a^2 & -2ab & b^2 \\ -ac & ad+bc & -bd \\ c^2 & -2cd & d^2 \end{bmatrix}\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}_i$$

and

$$\begin{bmatrix} \sigma_x^2 \\ \sigma_{xx'} \\ \sigma_{x'^2} \end{bmatrix}_f = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}\begin{bmatrix} \sigma_x^2 \\ \sigma_{xx'} \\ \sigma_{x'^2} \end{bmatrix}_i$$

For a drift

$$M_{drift} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$$

3x3 transformation matrix is

$$S_{drift} = \begin{bmatrix} 1 & -2L & L^2 \\ 0 & 1 & -L \\ 0 & 0 & 1 \end{bmatrix}$$

Homework Problems 2.2

Focusing with two dissimilar lenses.

Find the matched functions for an infinite FODO lattice

Quadrupole displacement error