Insertion Devices

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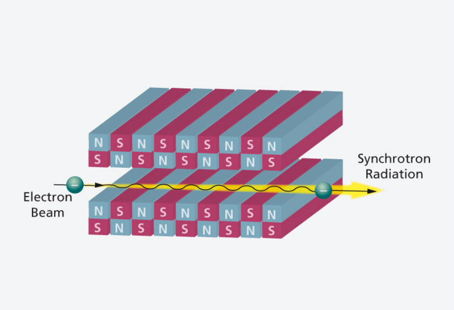
Overview

1. Undulators and wigglers
2. Impact on radiation damping and equillibrium beam sizes
3. Nonlinear Effects

Undulators and Wigglers

- Periodic series of dipole magnets with period $\lambda_w = \frac{2\pi}{k_w}$ with gap g



- Field is periodic along the beam axis (with $\tilde{B}$ being the peak field)

$$B_y = \tilde{B}\sin(k_w s) \tag{1}$$

Undulators and Wigglers (cont'd)

Electromagnetic wiggler at the ATF (left) and permanent magnet undulator at the ALS (right).

Trajectory in a wiggler

- Assuming $y = 0$ and $B_x = 0$ the equations of motion can be written as

$$\ddot{x} = -\dot{s}\,\frac{e}{m_e\gamma}\,B_z(s) \tag{2}$$

$$\ddot{s} = \dot{x}\,\frac{e}{m_e\gamma}\,B_z(s) \tag{3}$$

This can be approximated to (using $\dot{x} = v_x \ll x$ and $\dot{s} = v_s = \beta c =$const.):

$$\ddot{x} = -\frac{\beta c e B_w}{m_e\gamma}\,\cos k_w s \tag{4}$$

Using $\dot{x} = x'\beta c$ and $\ddot{x} = x''\beta^2 c^2$ this becomes

$$x'' = -\frac{eB_w}{m_e\beta c\gamma}\,\cos k_w s = -\frac{eB_w}{m_e\beta c\gamma}\,\cos\left(2\pi\frac{s}{\lambda_w}\right) \tag{5}$$

Trajectory in a wiggler (cont'd)

- Integration yields ($\beta = 1$):

$$x'(s) = \frac{\lambda_w e B_w}{2\pi m_e \gamma c} \sin k_w s \quad (6)$$

$$x(s) = \frac{\lambda_w^2 e B_w}{4\pi^2 m_e \gamma c} \cos k_w s \quad (7)$$

- The maximum angle of the trajectory and the wiggler axis is given by

$$\theta_w = x'_{max} = \frac{1}{\gamma} \frac{\lambda_w e B_w}{2\pi m_e c} \quad (8)$$

- If $\theta_w \leq \frac{1}{\gamma}$ the device is an undulator, otherwise it's a wiggler

Wiggler contribution to energy loss

- Increased energy loss from synchrotron radiation
- Ideally integrated field over the lenth is zero, therefore can be inserted into straight section without change to overall geometry
- Total energy loss per turn in a storage ring is given by ($C_\gamma = 8.846 \cdot 10^{-5} \, \frac{\mathrm{m}}{\mathrm{GeV}^3}$)

$$U_0 = \frac{C_\gamma}{2\pi} E_0^4 \, I_2 \qquad with \qquad I_2 = \oint \frac{1}{\rho^2} \, ds \tag{9}$$

- Need to add wiggler contribution

$$I_{2w} = \int_0^{Lw} \frac{1}{\rho^2} \, ds = \frac{1}{(B\rho)^2} \int_0^{L_w} B^2 \, ds = \frac{1}{(B\rho)^2} \, \frac{B_w^2 L_w}{2} \tag{10}$$

- I_{2w} does not depend on the wiggler period!

Wiggler contribution to energy loss (cont'd)

- Assume 5 GeV beam energy, circumference of 6.7 km and a desired damping time of 2.5 ms (ILC damping rings):

$$U_0 = 2E_0 \frac{T_0}{\tau} = 8.9\,\mathrm{MeV} \tag{11}$$

- Assuming 0.15 T for the dipoles, they contribute 500 keV per turn to the energy loss, so the wigglers have to provide 8.4 MeV:

$$\frac{C_\gamma}{2\pi} E_0^4 I_{2w} = 8.4\,\mathrm{MeV} \qquad \Rightarrow \qquad I_{2w} = 0.95\,\mathrm{m}^{-1} \tag{12}$$

- Using

$$\frac{1}{(B\rho)^2} \frac{B_w^2 L_w}{2} = 0.95\,\mathrm{m}^{-1} \tag{13}$$

  and assuming apeak field of 1.6 T, the total length of wigglers requires is $L_w \approx 210\,\mathrm{m}$

Wiggler contribution to the momentum compaction factor

- The momentum compaction factor α_C has an effect on other parameters like the synchrotron tune.

$$\alpha_C = \frac{1}{C_0} I_1 \qquad \textit{with} \qquad I_1 = \oint \frac{D_x}{\rho} ds \tag{14}$$

- In a FODO lattice α_C can be approximated using the horizontal tune:

$$\alpha_C \approx \frac{1}{Q_x^2} \qquad \textit{with} \qquad Q_x \approx \frac{1}{2\pi} \frac{C_0}{\beta_x} \tag{15}$$

- For the ILC damping rings without wigglers ($C_0 = 6.7\,\mathrm{km}$ and $\beta_x \approx 25\,\mathrm{m}$) one gets $\alpha_C \approx 5 \times 10^{-4}$

Wiggler contribution to the momentum compaction factor (cont'd)

- Need dispersion in wiggler to calculate contribution to momentum compaction factor.
- In a dipole with bending radius ρ and quadrupole gradient k_1, the dispersion is given by

$$\frac{d^2 D_x}{ds^2} + K\, D_x = \frac{1}{\rho} \qquad \textit{with} \qquad K = \frac{1}{\rho^2} + k_1 \tag{16}$$

- Assuming $k_1 = 0$ we get

$$\frac{d^2 D_x}{ds^2} + \frac{B_w^2}{(B\rho)^2} D_x \sin^2 k_w s = \frac{B_w}{B\rho} \sin k_w s \tag{17}$$

- Try $D_x \approx D_0 \sin k_w s$
- For $k_w \rho_w \gg 1$, one can neglect the second term on the left:

$$D_x \approx -\frac{\sin k_w s}{\rho_w k_w^2} \tag{18}$$

For the ILC damping wigglers $k_w \rho_w \approx 160$

Wiggler contribution to the momentum compaction factor (cont'd)

- Have assumed all contributions to dispersion are from bending in the wiggler. Things like misaligned quadrupole components etc. would add additional contributions which we neglect here.
- Dispersion in generated in the wiggler is small:

$$|D_0| \approx \frac{1}{\rho_w k_w^2} \qquad \textit{with} \qquad \rho_w = \frac{B\rho}{B_w} \tag{19}$$

- For the ILC damping wiggler we get $|D_0| \approx 0.39\,\mathrm{mm}$ compared to about 10 cm in the dipoles.

Wiggler contribution to the momentum compaction factor (cont'd)

- Now lets calculate the wiggler contribution to I_1:

$$I_{1w} = \int_0^{L_w} \frac{D_x}{\rho}\, ds \approx -\int_0^{L_w} \frac{\sin^2 k_w s}{\rho_w^2 k_w^2}\, ds = -\frac{L_w}{2\,\rho_w^2 k_w^2} \tag{20}$$

- I_{1w} is negative as higher energy particles have a shorter path length in the wiggler (which is the opposite from the path length in a storage ring).
- For the ILC damping wigglers ($\rho_w k_w \approx 160$ and $L_w \approx 210\,\mathrm{m}$) we get $I_{1w} \approx -0.004\,\mathrm{m}$ which is small compared to $I_1 \approx 3.4\,\mathrm{m}$ from the dipoles, so the contribution to the momentum compaction factor is negligible.

Wiggler contribution to the natural energy spread

- Natural energy spread ($C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.832 \times 10^{-13}\,\mathrm{m}$):

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{J_z I_2} \qquad \mathit{with} \qquad I_3 = \oint \frac{1}{|\rho|^3} \tag{21}$$

- I_3 does not depend on the dispersion, so the wiggler could possibly make a large contribution to the energy spread
- Bending radius in the wiggler:

$$\frac{1}{\rho} = \frac{B}{B\rho} = \frac{B_w}{B\rho} \sin k_w s = \frac{1}{\rho_w} \sin k_w s \tag{22}$$

- This yields

$$I_{3w} = \frac{1}{\rho_w^3} \int_0^{L_w} \left| \sin^3 k_w s \right| ds = \frac{4 L_w}{3\pi \rho_w^3} \tag{23}$$

- For the ILC damping wigglers ($L_w \approx 210\,\mathrm{m}$, $\rho_w \approx 10.4\,\mathrm{m}$), $I_{w3} \approx 0.079\,\mathrm{m}^{-2}$ which is large compared to the dipole contribution ($5.1 \times 10^{-4}\,\mathrm{m}^{-2}$).

Wiggler contribution to the natural energy spread (cont'd)

- In the ILC damping rings, the damping wiggler contribution to I_2 and I_3 is large compared to the contribution of the dipoles. Therefore the energy spread is largely determined by the wiggler:

$$\sigma_\delta^2 \approx \frac{4}{3\pi} C_q \frac{\gamma^2}{\rho_w} = \frac{4}{3\pi} \frac{e}{mc} C_q \gamma B_w \tag{24}$$

- For a damping ring, the energy spread of the extracted beam is an important parameter: The larger it is, the more difficult the downstream bunch compressors are to design
- With a beam energy of 5 GeV and a wiggler field of 1.6 T, the natural energy spread is about 0.13%. This is acceptable (upper limit is around 0.15%).

Wiggler contribution to the natural emittance

- The natural emittance depends on I_2 and I_5 ($C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.832 \times 10^{-13}\,\mathrm{m}$):

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{J_x I_2} \qquad with \qquad I_2 = \oint \frac{1}{\rho^2} ds \qquad and$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \qquad with \qquad \mathcal{H}_x = \gamma_x D_x^2 + 2\alpha_x D_x D_{px} + \beta_x D_{px}^2 \quad (25)$$

- The contribution of the wiggler to I_5 depends on the β-function in the wiggler. We assume $\alpha_x \approx 0$. Then we get

$$D_{px} \approx \frac{dD_x}{ds} = k_w D_0 \, cos k_w s \quad (26)$$

- Assuming $k_w \gg \frac{1}{\beta_x}$ we can approximate

$$\mathcal{H}_x \approx \frac{\beta_x}{\rho_w^2 k_w^2} \cos^2 k_w s \quad (27)$$

Wiggler contribution to the natural emittance (cont'd)

- With this we can write the wiggler contribution to I_5 as

$$I_{5w} \approx \frac{\langle \beta_x \rangle}{\rho_w^2 k_w^2} \int_0^{L_w} \frac{\cos^2 k_w s}{|\rho|^3}\, ds = \frac{\langle \beta_x \rangle}{\rho_w^5 k_w^2} \int_0^{L_w} |\sin^3 k_w s| \cos^2 k_w s\, ds \quad (28)$$

- Using $\langle |\sin^3 x| cos^2 x \rangle = \frac{4}{15\pi}$ we have

$$I_{5w} \approx \frac{4}{15\pi} \frac{\langle \beta_x \rangle L_w}{\rho_w^5 k_w^2} \quad (29)$$

- Assuming $\langle \beta_x \rangle \approx 10\,\mathrm{m}$ and the usual wiggler parameters ($k_w \approx 15.7\,\mathrm{m}^{-1}$), we get $I_{5w} \approx 5.9 \times 10^{-6}\,\mathrm{m}^{-1}$.

Wiggler contribution to the natural emittance (cont'd)

- Lets see how this compares to the contribution from the dipoles (assuming TME lattice tuned for minimum dispersion; θ is the bending angle in the dipoles, ρ the bending radius):

$$I_5 = \frac{\pi}{5\sqrt{15}} \frac{\theta^3}{\rho} \tag{30}$$

- Assuming 120 dipoles with a field of 0.15 T and 5 GeV one gets $I_{5D} \approx 1.7 \times 10^{-7}\,\mathrm{m}^{-1}$, so the wiggler contribution dominates. However often a less than ideal TME lattice is used where the wiggler contribution can be significant. In other lattices, like FODO, the dipole contribution can dominate.

Wiggler contribution to the natural emittance (cont'd)

- Combining I_{2w} and I_{5w} we get for the natural emittance

$$\epsilon_0 \approx \frac{8}{15\pi} C_q \gamma^2 \frac{\langle \beta_x \rangle}{\rho_w^3 k_w^2} \tag{31}$$

- Using the usual parameters this yields $\epsilon_0 \approx 0.22\,\mathrm{nm}$.
- If the dipole contribution is comparable to the wiggler contribution, the natural emittance will be larger than this by about a factor of two.
- The wiggler contribution can be reduced by
  - reducing the horizontal β-function
  - reducing the wiggler period, i.e. increasing k_w
  - reducing the wiggler field, i.e. increasing ρ_w

Dynamical effects of wigglers

- Aside from the effects on the natural emittance and the energy spread, the wigglers have two other effects:
  1. provide linear focusing which must be included in the lattice design
  2. non-linear field components that can affect particles at large amplitude and thus can limit the dynamic aperture

3D field in an ideal wiggler

- If the poles are infinitely wide, the horizontal field component vanishes:

$$B_x = 0 \quad (32)$$
$$B_y = B_w \sin k_z z \cosh k_z y \quad (33)$$
$$B_z = B_w \cos k_z z \sinh k_z y \quad (34)$$

- As B_z is non-zero for a vertical offset and the particle has a horizontal velocity thanks to the wiggler field, a particle with a horizontal offset will experience a vertical deflecting force which leads to vertical focusing in the wiggler.

Vertical focusing in a wiggler

- For simplicity we will assume that the trajectory of a particle is determined by the vertical field component of the wiggler. Other forces, e.g. vertical deflections, will be trated as perturbations.
- The horizontal equation of motion on the mid-plane is given by

$$\frac{d^2x}{ds^2} = \frac{B_y}{B\rho} = \frac{B_w}{B\rho} \sin k_z s \cosh k_z y \tag{35}$$

- with the solution

$$x = -\frac{B_w}{B\rho}\frac{1}{k_z^2} \sin k_z s \cosh k_z y \tag{36}$$

and

$$p_x = -\frac{B_w}{B\rho}\frac{1}{k_z} \cos k_z s \cosh k_z y \tag{37}$$

Vertical focusing in a wiggler (cont'd)

- The vertical equation of motion is

$$\frac{dp_y}{ds} = \frac{q}{p_0} p_x B_z = \frac{B_w}{B\rho} p_x \cos k_z s \sinh k_z y \tag{38}$$

- The total deflection per period is

$$\Delta p_y \approx \frac{B_w}{B\rho} \sinh k_z y \int_0^{\lambda_w} p_x \cos k_z s \, ds \tag{39}$$

- Using p_x as from above we find

$$\Delta p_y \approx -\left(\frac{B_w}{B\rho}\right)^2 \frac{1}{k_z} \sinh k_z y \cosh k_z y \int_0^{\lambda_w} \cos^2 k_z s \, ds \tag{40}$$

$$= -\frac{\pi}{2k_z^2} \left(\frac{B_w}{B\rho}\right)^2 \sinh 2k_z y \tag{41}$$

Vertical focusing in a wiggler (cont'd)

- Series expansion in y yields

$$\Delta p_y \approx -\frac{\pi}{k_z}\left(\frac{B_w}{B\rho}\right)^2\left(y+\frac{2}{3}k_z^2y^3+\ldots\right) \tag{42}$$

- Taking only the term linear in y into account, per wiggler period this is equivalent to a vertically focusing quadrupole with the integrated strength

$$k_1 l = -\frac{\pi}{k_z}\left(\frac{B_w}{B\rho}\right)^2 \tag{43}$$

- The cubic term contributes

$$\Delta p_y^{(3)} \approx -\frac{2\pi}{3}\left(\frac{B_w}{B\rho}\right)^2 k_z y^3 \tag{44}$$

which is often referred to as the "dynamic octupole" term.

Horizontal focusing in a wiggler

- The finite width of the magnet poles leads to a decrease of the field strength for large horizontal offsets.
- In a simple model the field can be written as

$$B_x = -\frac{k_x}{k_y} B_w \sin k_x x \sinh k_y y \sin k_z z \quad (45)$$

$$B_y = B_w \cos k_x x \cosh k_y y \sin k_z z \quad (46)$$

$$B_z = \frac{k_z}{k_y} B_w \cos k_x x \sinh k_y y \cos k_z z \quad (47)$$

with the condition (from Maxwell's equations)

$$k_x^2 + k_z^2 = k_y^2 \quad (48)$$

- A particle with a horizontal offset sees a weaker field in one set of poles and a stronger field in the other. The net effect is a horizontal deflection that appears as a horizontal defocusing force.

Horizontal focusing in a wiggler (cont'd)

- Consider a particle with the trajectory (x_0 is the inital horizontal offset)

$$x = x_0 + \hat{x} \sin k_z s \qquad \textit{with} \qquad \hat{x} = \frac{1}{k_z^2} \frac{B_w}{B\rho} \tag{49}$$

- Assuming $y = 0$ the horizontal kick per period is

$$\begin{aligned}
\Delta p_x &= -\frac{1}{B\rho} \int_0^{\lambda_w} B_y \, ds & (50)\\
&= -\frac{B_w}{B\rho} \int_0^{\lambda_w} \cos\left[k_x \left(x_0 + \hat{x} \sin k_z s\right)\right] \sin k_z s \, ds & (51)\\
&= \frac{B_w}{B\rho} \lambda_w J_1 \left(k_x \hat{x}\right) \sin(k_x x_0) & (52)\\
&\approx \frac{B_w}{B\rho} \lambda_w \frac{k_x \hat{x}}{2} k_x x_0 & (53)
\end{aligned}$$

(54)

Horizontal focusing in a wiggler (cont'd)

- The horizontal focusing can be written as

$$\Delta p_x \approx \frac{\lambda_w}{2} \left(\frac{B_w}{B\rho} \right)^2 \frac{k_x^2}{k_z^2} x_0 \tag{55}$$

- Compare to the vertical focusing

$$\Delta p_y \approx -\frac{\lambda_w}{2} \left(\frac{B_w}{B\rho} \right)^2 \frac{k_y^2}{k_z^2} y_0 \tag{56}$$

- For infinitely wide poles, $k_x \to 0$ which means there is no horizontal focusing. In this case one also gets $k_y = k_z$. For finite horizontal poles, the vertical focusing is enhanced due to $k_y^2 = k_x^2 + k_z^2$.

Nonlinear effects in wigglers

- At the center of a pole ($\sin k_z s = 1$) and with $y = 0$ the vertical field is given by

$$B_y = B_w \cos k_x x = B_w \left(1 - \frac{1}{2} k_x^2 x^2 + \frac{1}{24} k_x^4 x^4 - ...\right) \qquad (57)$$

- The quadratic term leads to horizontal defocusing, the sextupole field "feeds down" (when combined with the wiggling trajectory) to give a linear focusing effect.
- In the same manner, the decapole component feeds down to give a octupole component. So for finite pole width, we have a "dynamic octupole" in both planes.
- Wigglers can have a significant impact on the non-linear dynamics. They can potentially restrict the dynamic aperture. Therefore it is important to have a good model for analysing the nonlinear effects.

Modelling the nonlinear effects of wigglers

Modelling the nonlinear effects of wigglers is done in four steps:

1. Magnetostatic codes (e.g. Tosca, Radia) are used to calculate the magnetic field in one period.
2. An analytical model for the field (a mode decomposition) is fitted to the field obtained in step 1.
3. The analytical model is then used to create a dynamical map which described the motion of a particle through the wiggler. This is done with codes like MaryLie or COSY.
4. The dynamical map is then used in a tracking code to determine the impact of the wiggler on non-linear dynamics, e.g. tune shifts, resonances or dynamic aperture.

We will have a brief look at some of the steps.

Modelling the nonlinear effects of wigglers, step 2

Generalise the representation used so far to inlude a series of wiggler modes:

$$B_x = -B_w \sum_{m,n} c_{m,n} \frac{mk_x}{k_{y,mn}} \sin mk_x x \sinh k_{y,mn} y \sin nk_z z \quad (58)$$

$$B_y = B_w \sum_{m,n} c_{m,n} \cos mk_x x \, cosh k_{y,mn} y \sin nk_z z \quad (59)$$

$$B_z = B_w \sum_{m,n} c_{m,n} \frac{nk_z}{k_{y,mn}} \cos mk_x x \sinh k_{y,mn} y \cos nk_z z \quad (60)$$

$$k_{y,mn}^2 = m^2 k_x^2 + n^2 k_z^2 \quad (61)$$

Modelling the nonlinear effects of wigglers, step 2 (cont'd)

From the vertical field on the mid-plane ($y = 0$)

$$B_y = B_w \sum_{m,n} c_{m,n} \cos mk_x x \, \sin nk_z z \tag{62}$$

one can in principle determine the coefficients $c_{m,n}$ by using a 2D Fourier transform of the field data from step 1. In practice this does not work well. The hyperbolic dependence of the field on y means that any small errors from the fit increase exponentially away from the mid-plane.
A better technique is to fit the field on a surface enclosing the region of interest. The hyperbolic dependence of the field means that in this case any small errors actually decrease exponentially towards the axis of the wiggler.

Modelling the nonlinear effects of wigglers, step 2 (cont'd)

- Using a cylindrical surface within the wiggler aperture and standard cylindrical coordinates, we get:

$$B_\rho = \sum_{m,n} \alpha_{m,n} I'_m(nk_z\rho) \sin m\phi \sin nk_z z \qquad (63)$$

$$B_\phi = \sum_{m,n} \alpha_{m,n} \frac{m}{nk_z\rho} I_m(nk_z\rho) \cos m\phi \sin nk_z z \qquad (64)$$

$$B_z = \sum_{m,n} \alpha_{m,n} I_m(nk_z\rho) \sin m\phi \cos nk_z z \qquad (65)$$

- If we know the radial field component B_ρ at a fixed radius, we can obtain the mode coefficients $\alpha_{m,n}$ by a 2D Fourier transform.
- Usually done as close to the poles as data quality allows.
- Number of modes required depends on shape of field.
- Once $\alpha_{m,n}$ are known, we can construct the field components everywhere. The errors are small within the cylindrical surface.

Modelling the nonlinear effects of wigglers, step 3

- With the mode decomposition of the field, we can now use an "algebraic" code to construct a dynamical map.
- An "algebraic" code manipulates algebraic expressions instead of numbers. There are different types of algebraic codes:
  - Differential algebra codes like COSY
  - Lie algebra codes like MaryLie

  A differential algebra code can manipulate Taylor series. By incorporating an integrator to solve the equations of motion into a differential algebra code, we can construct a Taylor map representing the dynamics of a particle in the given field.

Modelling the nonlinear effects of wigglers, step 4

- The map constructed in step 3 now needs to be included in a tracking code to look at the impact on beam dynamics
- Could just track a set of particles at varying amplitudes to “measure” the dynamic aperture, however a frequency map analysis yields much more informtion