Coupling and Alignment

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Overview

1. Fundamental lower limit on vertical emittance from synchrotron radiation
2. Generation of vertical emittance from coupling and dispersion
3. Coupling and dispersion generated by alignment errors
4. Ground motion

Fundamental lower limit on vertical emittance from synchrotron radiation

- Natural horizontal emittance is given by

$$\varepsilon_0 = C_q \gamma^2 \frac{I_5}{I_2 - I_4} \tag{1}$$

with the synchrotron radiation integrals

$$I_2 = \oint \frac{1}{\rho^2}\, ds \tag{2}$$

$$I_4 = \oint \frac{D_x}{\rho}\left(\frac{1}{\rho^2} + 2k_1\right) ds \quad \textit{with} \quad k_1 = \frac{e}{p_0}\frac{\partial B_y}{\partial x} \tag{3}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3}\, ds \quad \textit{with} \quad \mathcal{H}_x = \gamma_x D_x^2 + 2\alpha_x D_x D_{px} + \beta_x D_{px}^2 \tag{4}$$

Vertical emittance in a storage ring

- Assume horizontal and vertical motion are independent of each other: If we could build a totally flat ring, i.e. $D_y = D_{py} \Rightarrow \mathcal{H}_y = 0 \Rightarrow I_{5y} = 0$
- This assumes that photons are always emitted exactly forward, which is not true
- Photons are actually emitted in a cone with opening angle $1/\gamma$
- This results in exciting vertical betatron oscillations leading to a non-zero vertical emittance

Fundamental Lower Limit

- Detailed analysis results in

$$\varepsilon_{y,min} = \frac{13}{55} \frac{C_q}{J_y I_2} \int \frac{\beta_y}{|\rho|^3} ds \quad (5)$$

  This can be approximated to

$$\varepsilon_{y,min} \approx \frac{1}{4} \frac{\langle \beta_y \rangle C_q}{J_y I_2} \int \frac{1}{|\rho|^3} ds = \frac{\langle \beta_y \rangle}{4} \frac{J_z}{J_y} \frac{\sigma_\delta^2}{\gamma^2} \quad (6)$$

- Assuming typical parameters for a damping ring ($\beta_y = 20\,\mathrm{m}$, $J_z = 2$, $J_y = 1$, $\sigma_\delta = 10^{-3}$ and $\gamma = 10^4$) one gets $\varepsilon_{y,min} \approx 0.1\,\mathrm{pm}$

Vertical emittance in practice

- In a storage ring, the vertical emittance is usually dominated by two effects:
  - Residual vertical dispersion
  - Betatron coupling

  Both of those are mainly caused by
  - tilts of dipole magnets around the beam axis
  - vertical alignment errors of quadrupoles
  - tilts of quadrupoles
  - vertical alignment errors of sextupoles

Steering Errors

- Lead to distortion of closed orbit which generates vertical dispersion. Orbit distortion leeds to beam being offset in sextupoles resulting in betatron coupling
- vertical steering can be created by:
  - a tilted dipole resulting in the magntic field not being exactly vertical
  - vertical misalignment of a quadrupole resulting at a horizontal magnetic field at the location of the reference trajectory

Rotated quadrupoles

- Tilted quadrupole results in the field being a mixture or a normal and a skew quadrupole field
- Skew quadrupole field provides a vertical kick the size of which depends on the horizontal offset in the quadrupole
- Some of the quantum excitation that creates the horizontal emittance feeds into the vertical plane, blowing up the vertical emittance

Misaligned sextupoles

- A vertical offset in a sextupole has the same effect as a skew quadrupole
- Sextupole field is given by

$$B_x = k_2 xy \tag{7}$$
$$B_y = \frac{1}{2}k_2(x^2 - y^2) \tag{8}$$

Vertical offset ($y \rightarrow y + \Delta y$) results in

$$B_x = k_2 \Delta y\, x + k_2 xy \tag{9}$$
$$B_y = -k_2 \Delta y\, y + \frac{1}{2}k_2(x^2 - y^2) - \frac{1}{2}k_2 \Delta y^2 \tag{10}$$

- The first term in each expression represents a skew quadrupole with a strength of $k_z = k_2 \Delta y$

Closed Orbit Distortion From Steering Errors

- Use action angle variables A_y and ϕ_y

$$y = \sqrt{2\beta_y A_y} \cos \phi_y \tag{11}$$

$$p_y = -\sqrt{\frac{2J_y}{\beta_y}} (\sin \phi_y + \alpha_y \cos \phi_y) \tag{12}$$

- A steering error at $s = s_0$ gives a vertical kick which results in a change $\Delta\theta$ of the vertical momentum
- The trajectory will result in a closed orbit when

$$\sqrt{2\beta_{y0} A_{y0}} \cos(\phi_{y0} + \mu_y) = \sqrt{2\beta_{y0} A_{y0}} \cos \phi_{y0} \tag{13}$$

and

$$-\sqrt{\frac{2J_{y0}}{\beta_{y0}}} (\sin(\phi_{y0} + \mu_y) + \alpha_{y0} \cos(\phi_{y0} + \mu_y)) \tag{14}$$

$$= -\sqrt{\frac{2J_{y0}}{\beta_{y0}}} (\sin \phi_{y0} + \alpha_{y0} \cos \phi_{y0}) - \Delta\theta \tag{15}$$

Closed Orbit Distortion From Steering Errors (cont'd)

- Solving the equations yields

$$A_{y0} = \frac{\beta_{y0}\Delta\theta^2}{8\sin^2\pi\nu_y} \tag{16}$$

$$\phi_{y0} = \pi\nu_y \tag{17}$$

where $\nu_y = \mu_y/2\pi$ is the vertical tune

- Note: If the tune is an integer, the smallest steering error will kick the beam out of the ring
- We can now write the closed orbit at any point in the ring as

$$y_{CO}(s) = \frac{\sqrt{\beta_y(s_0)\beta_y(s)}}{2\sin\pi\nu_y}\Delta\theta\cos(\pi\nu_y + \mu_y(s;s_0)) \tag{18}$$

Closed Orbit Distortion From Steering Errors (cont'd)

- Adding up all distortions around the ring, we get

$$y_{CO}(s) = \frac{\sqrt{\beta_y(s)}}{2 \sin \pi\nu_y} \int_{o}^{C} \sqrt{\beta_y(s')} \frac{d\theta}{ds'} \cos(\pi\nu_y + \mu_y(s; s'))\, ds \qquad (19)$$

Closed Orbit Distortions from Quadrupole Alignment Errors

- To estimate the effect of quadrupole misalignment on the closed error we can use the closed orbit equation. A quadrupole with integrated strength $k_1 L$ which is vertically misaligned by ΔY, the steering is

$$\frac{d\theta}{ds} = (k_1 L)\Delta Y \tag{20}$$

- Using equation 19 we can calculate the square of the distortion
- Averaging yields

$$\left\langle \frac{y_{CO}^2(s)}{\beta_y(s)} \right\rangle = \frac{\langle \Delta Y^2 \rangle}{8 \sin^2 \pi \nu_y} \sum_{quads} \beta_y (k_1 L)^2 \tag{21}$$

Closed Orbit Errors and Vertical Emittance

- To reach small vertical emittance, vertical closed orbit distortions are a concern for two reasons:
  1. vertical steering creates vertical dispersion
  2. vertical orbit errors lead to vertical offsets in sextupoles which then act as skew quadrupoles

Betatron coupling

- Betatron coupling describes the effects that can arise when the vertical motion depends on the horizontal motion and vice versa.
- Skew quadrupole: A particle passing through with a horizontal offset will get a vertical kick, thus coupling horizontal and vertical motion.
- Skew quadrupoles are often caused by tilted quadrupoles or misaligned sextupoles
- Full treatment of betatron coupling is quite complex and different formalisms can be used

Hamilton's equations

- Hamiltonian for the motion of a single particle:

$$H = H(\phi_x, A_x, \phi_y, A_y, s) \tag{22}$$

$$\frac{dA_x}{ds} = -\frac{\partial H}{\partial \phi_x} \qquad \frac{dA_y}{ds} = -\frac{\partial H}{\partial \phi_y}$$

$$\frac{d\phi_x}{ds} = \frac{\partial H}{\partial A_x} \qquad \frac{d\phi_y}{ds} = \frac{\partial H}{\partial A_y}$$

- Hamiltonian for a prticle moving along a linear, uncoupled beam line:

$$H = \frac{A_x}{\beta_x} + \frac{A_y}{\beta_y} \tag{23}$$

- Hamiltonian for a skew quadrupole:

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - k_s xy \tag{24}$$

Equations of motion in a coupled beam line

- The "focusing" effect of a skew quadrupole is represented by one term in the Hamiltonian

$$k_s xy = 2k_s\sqrt{\beta_x\beta_y}\sqrt{A_x A_y}\cos\phi_x\cos\phi_y \tag{25}$$

- Therefore the Hamiltonian for a beam line with distributed skew quads can be written as

$$H = \frac{A_x}{\beta_x} + \frac{A_y}{\beta_y} - 2k_s(s)\sqrt{\beta_x\beta_y}\sqrt{A_x A_y}\cos\phi_x\cos\phi_y \tag{26}$$

- It's difficult to solve the equation of motion as the β-functions and k_s are functions of s.

Equations of motion in a coupled beam line (cont'd)

- Can be simplified by "averaging" the Hamiltonian

$$H = \omega_x A_x + \omega_y A_y - 2\kappa\sqrt{A_x A_y}\cos\phi_x \cos\phi_y \tag{27}$$

  where ω_x, ω_y and κ are constants.
- ω_x and ω_y are the betatron frequencies

$$\omega_{x,y} = \frac{1}{C}\int_0^C \frac{ds}{\beta_{x,y}} \tag{28}$$

- Rewrite the coupling term

$$H = \omega_x A_x + \omega_y A_y - \kappa_- \sqrt{A_x A_y}\cos(\phi_x - \phi_y) - \kappa_+ \sqrt{A_x A_y}\cos(\phi_x + \phi_y) \tag{29}$$

Equations of motion in a coupled beam line (cont'd)

- The constants $\kappa_\pm$ represent the skew quadrupole strength averaged around the ring. Taking into account that the kick depends on the betatron phase:

$$\kappa_\pm e^{i\chi} = \frac{1}{C} \int_0^C e^{i(\mu_x \pm \mu_y)} k_s \sqrt{\beta_x \beta_y}\, ds \tag{30}$$

- Assuming $\kappa_- \ll \kappa_+$ we can drop one term from the Hamiltonian:

$$H = \omega_x A_x + \omega_y A_y - \kappa_- \sqrt{A_x A_y} \cos(\phi_x - \phi_y) \tag{31}$$

Equations of motion in a coupled beam line (cont'd)

- With this we get for the equations of motion

$$\frac{dA_x}{ds} = -\frac{\partial H}{\partial \phi_x} = \kappa_- \sqrt{A_x A_y} \sin(\phi_x - \phi_y) \tag{32}$$

$$\frac{dA_y}{ds} = -\frac{\partial H}{\partial \phi_y} = \kappa_- \sqrt{A_x A_y} \sin(\phi_x - \phi_y) \tag{33}$$

$$\frac{d\phi_x}{ds} = \frac{\partial H}{\partial A_x} = \omega_x + \frac{\kappa_-}{2} \sqrt{\frac{A_y}{A_x}} \cos(\phi_x - \phi_y) \tag{34}$$

$$\frac{d\phi_y}{ds} = \frac{\partial H}{\partial A_y} = \omega_y + \frac{\kappa_-}{2} \sqrt{\frac{A_x}{A_y}} \cos(\phi_x - \phi_y) \tag{35}$$

Equations of motion in a coupled beam line (cont'd)

- The equations of motion are rather difficult to solve. However we are not interested in a general solution, just in some properties of some special cases.
- Note that the sum of the actions is constant (in all cases):

$$\frac{dA_x}{ds} + \frac{dA_y}{ds} = 0 \qquad A_x + A_y = constant \tag{36}$$

  If $\phi_x = \phi_y$, the rate of change of each action is zero:

$$If \qquad \phi_x = \phi_y \qquad then \qquad \frac{dA_x}{ds} = \frac{dA_y}{ds} = 0 \tag{37}$$

- If we can find a solution with $\phi_x = \phi_y$ for all s, then the actions will remain constant.

Fixed point solutions to the equations of motion in a coupled beam line

- From the equations of motion we find that if

$$\phi + x = \phi_y \qquad and \qquad \frac{d\phi_x}{ds} = \frac{d\phi_y}{ds} \tag{38}$$

then

$$\frac{A_y}{A_x} = \frac{\sqrt{1 + \kappa_-^2/\Delta\omega^2} - 1}{\sqrt{1 + \kappa_-^2/\Delta\omega^2} + 1} \tag{39}$$

where $\Delta\omega = \omega_x - \omega_y$

Fixed point solutions to the equations of motion in a coupled beam line (cont'd)

- If we use $A_x + A_y = A_0$ where A_0 is a constant, then we have the fixed point solution

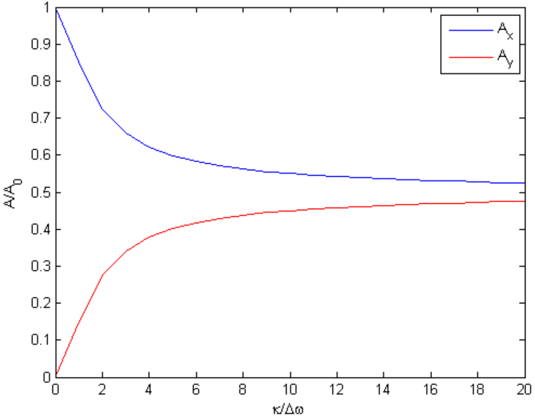
$$A_x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 + \kappa_-^2/\Delta\omega^2}}\right) A_0 \quad (40)$$

$$A_y = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 - \kappa_-^2/\Delta\omega^2}}\right) A_0 \quad (41)$$

- Fixed point actions are well separated for $\kappa_- \ll \Delta\omega$ but approach each other for $\kappa_- \gg \Delta\omega$

Fixed point solutions to the equations of motion in a coupled beam line (cont'd)

The condition at which the tunes are equal (or differ by an exact integer) is known as the difference coupling resonance.



Equillibrium Emittances

- The emittance may be defined as the betatron action averaged over all particles in the beam

$$\varepsilon_x = \langle A_x \rangle \qquad \textit{and} \qquad \varepsilon_y = \langle A_y \rangle \tag{42}$$

- Due to synchrotron radiation damping the betatron actions of the particles will only change slowly, i.e. on the timescale of the radiation damping, which is slow compared to the timescale of betatron oscillations

Equillibrium Emittances (cont'd)

- In that case, the actions of most particles must be in the correct ratio for a fixed point solution to the equations of motion. As a result if we assume $\varepsilon_x + \varepsilon_y = \varepsilon_0$ being the natural emittance of the storage ring, the equillibrium emittances are

$$\varepsilon_x = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 - \kappa_-^2/\Delta\omega^2}}\right)\varepsilon_0 \qquad (43)$$

$$\varepsilon_y = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 - \kappa_-^2/\Delta\omega^2}}\right)\varepsilon_0 \qquad (44)$$

- The model appears to be rather simplistic and we did gloss over many subtle issues. However if you do a tune scan in a lattice code (or a real accelerator for that matter) the emittances behave as expected.

Measuring the Coupling Strength

- There is an elegant technique for measuring the coupling strength κ. Using the fixed point solution, the Hamiltonian becomes

$$H = \omega_x A_x + \omega_y A_y - \kappa_- \sqrt{A_x A_y} \tag{45}$$

$$= \begin{pmatrix} \sqrt{A_x} & \sqrt{A_y} \end{pmatrix} \cdot \begin{pmatrix} \omega_x & -\frac{1}{2}\kappa_- \\ -\frac{1}{2}\kappa_- & \omega_y \end{pmatrix} \cdot \begin{pmatrix} \sqrt{A_x} \\ \sqrt{A_y} \end{pmatrix} \tag{46}$$

- The normal modes can be constructed using eigenvectors of the matrix in the final expression. The frequencies of these normal modes are the eigenvalues of this matrix. These are the frequencies at which a particle (or beam) will resonate if driven by an external oscillator
- The normal mode frequencies are

$$\omega \pm = \frac{1}{2}\left(\omega_x + \omega_y \pm \sqrt{\kappa_-^2 + \Delta\omega^2}\right) \tag{47}$$

- Notice that as $\Delta\omega \to 0$ the measured tunes are separated by κ_-

Difference and Sum Coupling Resonances

- Recall that the skew quadrupupoles introduced two terms into the Hamiltonian:

$$H = \omega_x A_x + \omega_y A_y - \kappa_- \sqrt{A_x A_y} \cos(\phi_x - \phi_y) - \kappa_+ \sqrt{A_x A_y} \cos(\phi_x + \phi_y) \tag{48}$$

  We have assumed that the third term dominates over the fourth. In this case we have the difference resonance:
  - Sum of the actions is constant
  - At a fixed point solution, the angle variables remain equal, and the actions are in a fixed ratio determined by $\kappa_- \Delta\omega$
- If the fourth term dominates, the behaviour is totally different, as can be seen by writing down the equations of motion. There are no fixed point solutions and the actions can grow indefinitely.
- The fourth term can have a strong effect if the sum of the tunes is close to an integer: sum resonances are to be avoided

Coupling, vertical emittance and magnet alignment

- Major sources of coupling in storage rings are quadrupole tilts and sextupole misalingment
- Look at alignment tolerance to achieve a specified vertical emittance in a given ring
- It is sufficient to find an expression for $\kappa_-\ \Delta\omega$ in terms of optical functions, magnet parameters and rms misalignment
- We start with

$$\frac{\kappa e^{i\chi}}{\Delta\omega} = \frac{1}{2\pi\Delta\nu}\int_0^C e^{i(\mu_x-\mu_y)} k_s \sqrt{\beta_x\beta_y}\, ds \qquad (49)$$

- Taking the modulus squared, and using (for sextupoles) $k_s = k_2\Delta y$:

$$\left(\frac{\kappa}{\Delta\omega}\right)^2 \approx \frac{\langle \Delta Y_S^2\rangle}{4\pi^2\Delta\nu^2} \sum_{sextupoles} \beta_x\beta_y (k_2 l)^2 \qquad (50)$$

Coupling, vertical emittance and magnet alignment (cont'd)

- The value obtained this way is much larger than the real allowed alignment tolerances as it does not take into account a few things:
- The closed orbit distortions will contribute to offsets in sextupoles; orbit amplification factors can significantly reduce the alignment tolerances by a factor of 10 or 20
- We have not made allowances for contributions of the vertical dispersion to the vertical emittance

Vertical dispersion

- Equation of motion for the trajectory of a particle with momentum P:

$$\frac{d^2y}{ds^2} = \frac{e}{P} B_x \tag{51}$$

- For small energy deviation δ:

$$P \approx (1+\delta) P_0 \tag{52}$$

The horizontal field to the first order in derivatives:

$$B_x \approx B_{0x} + y\frac{\partial B_x}{\partial y} + x\frac{\partial B_x}{\partial x} \tag{53}$$

- Consider a particle following an off-momentum orbit:

$$y = D_y\delta \qquad and \qquad x = D_x\delta \tag{54}$$

Vertical dispersion

- Combining equations, we find

$$\frac{d^2 D_y}{ds^2} - k_1 D_y \approx -k_{0s} + k_{1s} D_x \tag{55}$$

- This is similar to the "equation of motion" for the closed orbit

$$\frac{d^2 y_{CO}}{ds^2} - k_1 y_{CO} \approx -k_{0s} + k_{1s} x_{CO} \tag{56}$$

- We can therefore generalise the relationship between the closed orbit and the quadrupole misalignments, to apply to the dispersion:

$$\left\langle \frac{D_y^2(s)}{\beta_y(s)} \right\rangle = \frac{\langle \Delta Y_Q^2 \rangle}{8 \sin^2 \pi \nu_y} \sum_{quads} \beta_y (k_1 L)^2 + \frac{\langle \Delta Y_S^2 \rangle}{8 \sin^2 \pi \nu_y} \sum_{sext} D_x^2 \beta_y (k_2 L)^2 \tag{57}$$

Vertical Dispersion and Vertical Emittance

- Now we need to relate the vertical dispersion to the vertical emittance. Using the same equation as for the horizontal emittance:

$$\varepsilon_y = C_q \gamma^2 \frac{I_{5y}}{J_y I_2} \tag{58}$$

with the synchrotron radiation integrals

$$I_2 = \oint \frac{1}{\rho^2} ds \tag{59}$$

$$I_5 = \oint \frac{\mathcal{H}_x}{|\rho|^3} ds \quad with \quad \mathcal{H}_x = \gamma_x D_x^2 + 2\alpha_x D_x D_{px} + \beta_x D_{px}^2 \tag{60}$$

If the vertical disperion is generated randomly, then it will, in general, not be correlated with the curvature $1/\rho$ of the reference trajectory. Therefore we can write

$$I_{5y} \approx \langle \mathcal{H}_y \rangle \oint \frac{1}{|\rho|^3} ds = \langle \mathcal{H}_y \rangle I_3 \tag{61}$$

Vertical Dispersion and Vertical Emittance

- We now get for the vertical emittance

$$\varepsilon_Y \approx C_q \gamma^2 \langle \mathcal{H}_y \rangle \frac{I_3}{J_y I_2} \tag{62}$$

- Using the natural energy spread

$$\sigma_\delta^2 = C_q \gamma^2 \frac{I_3}{J_z I_2} \tag{63}$$

we get

$$\varepsilon_y \approx \frac{J_z}{J_y} \langle \mathcal{H}_y \rangle \sigma_\delta^2 \tag{64}$$

Vertical Dispersion and Vertical Emittance (cont'd)

- Comparing the action

$$2A_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2 \tag{65}$$

and the $\mathcal{H}$-function

$$\mathcal{H}_y = \gamma_y D_y^2 + 2\alpha_y D_y D_{py} + \beta_y D_{py}^2 \tag{66}$$

- This implies that we can write

$$D_y \sqrt{\beta_y \mathcal{H}_y} \cos\phi_{D_y} \qquad \left\langle \frac{D_y^2}{\beta_y} \right\rangle = \frac{1}{2} \langle \mathcal{H}_y \rangle \tag{67}$$

- Combining with equation 64 we get

$$\varepsilon_y \approx 2 \frac{J_z}{J_y} \left\langle \frac{D_y^2}{\beta_y} \right\rangle \sigma_\delta^2 \tag{68}$$

Dispersion and Coupling Contributions to the Vertical Emittance

- Together with our previous results (including quadrupole tilts) we get

$$\left\langle \frac{D_y^2}{\beta_y} \right\rangle \approx \frac{\langle \Delta Y_Q^2 \rangle}{8 \sin^2 \pi\nu_y} \sum_{quads} \beta_y (k_1 L)^2 + \frac{\langle \Delta\Theta_Q^2 \rangle}{8 \sin^2 \pi\nu_y} D_x^2 \beta_y (k_1 L)^2 + \frac{\langle \Delta Y_S^2 \rangle}{8 \sin^2 \pi\nu_y} \sum_{sexts} D_x^2 \beta_y (k_2 L)^2 \qquad (69)$$

  to relate the expected vertical emittance generated by dispersion with the rms quadrupole and sextupole alingment errors.

- To this we need to add the contribution from vertical coupling

$$\varepsilon_y \approx \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \kappa^2/\Delta\omega^2}} \right) \varepsilon_0 \qquad (70)$$

  where

$$\left(\frac{\kappa}{\Delta\omega} \right)^2 \approx \frac{\langle \Theta_Q^2 \rangle}{4\pi^2 \Delta\nu^2} \sum \beta_x \beta_y (k_1 l)^2 + \frac{\langle \Delta Y_S^2 \rangle}{4\pi^2 \Delta\nu^2} \sum \beta_x \beta_y (k_2 l) \qquad (71)$$

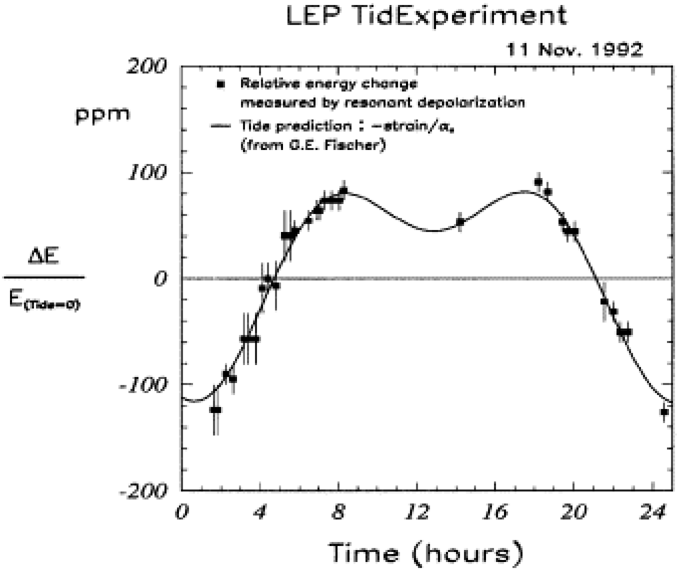
Ground motion

- Ground moving means magnets in the ring move which can negatively impact the performance
- Two main types:
  1. Slow movements, e.g. tides
  2. Fast movements, e.g. vibrations due to nearby highway

Slow movements

- Water table and earth tides can cause distortions of the ring or circumference changes
- Usually slow and easily corrected by automated feedback
- Although earth tides are minuscule, because of the large momentum compaction factor they can result in a measurable change of the beam energy

Tides measured at LEP



Fast movements