1 Harmonic number

The accelerating structures in a circular accelerator may be either distributed around the ring, or grouped together at a single location in the ring. One potential advantage of having the RF systems in one place is that all the associated electronics to produce the low level signals and amplified voltage can then be at a single location. In any case, the frequency of the voltage waveform in the accelerating structures, f_{RF}, must be an integer multiple of the revolution frequency, f_{rev} (the number of times per second a particle orbits the accelerator).

$$f_{RF} = hf_{rev}$$

where the integer multiple h is called the harmonic number. The harmonic number is also the maximum number of beam bunches it is possible to load into the accelerator. For example, if the harmonic number were one, then the RF frequency would be equal to the revolution frequency. Suppose there were one accelerating cavity for simplicity. Only once per revolution period, T_{rev}, would the RF voltage be at the proper level to accelerate an ideal particle. The beam bunch would drift around the accelerator while the RF voltage was evolving, arriving at the accelerating gap just when the voltage at the gap was once again at the design value. If the RF frequency were doubled, two bunches could revolve around the machine, since twice per revolution the RF voltage would be at the design value (and so on).

2 Difference equations for longitudinal motion

The longitudinal phase space coordinates for a given arbitrary particle may be specified as the phase error and the energy error from an ideal particle. The ideal particle is perfectly synchronous with the RF voltage wave, always arriving at the center of each RF gap at the same phase with respect to the applied voltage. This constant phase is called the synchronous phase, ϕ_s. The ideal particle arrives at the gap at the perfect time (or phase) to receive the energy kick needed to maintain this same phase at the next gap. The phase space errors of an arbitrary particle can be described by difference equations, which evolve the phase space errors from one gap to the next.

$$\phi_{error} = \quad \omega\Delta\tau$$

$$E_{error} = \quad e\Delta V$$

Here ω is the RF angular frequency, τ is the error in transit time with respect to the synchronous particle from the previous gap center to current gap center, e is the elementary charge, and ΔV is the difference between the voltage kick given to the particle and the synchronous particle at the previous gap. Or, putting it another way:

$$\phi_{n+1} = \phi_n + \omega_{rf}(t - t_s)$$

$$\Delta E_{n+1} = \Delta E_n + e(V_n - V_s)$$

The first equation describes the phase error at the $(n+1)^{th}$ accelerating gap compared to the ideal particle. This is the same as the phase error at the previous gap, plus the additional phase slip between gaps due to the difference in transit time of the particle compared to an ideal particle. The second equation describes the energy error at the $(n+1)^{th}$ gap compared to the ideal particle. This is the same as the energy error at the previous gap, plus the additional energy difference from the error in kick strength at the previous gap.

The form of the difference equations becomes ornamented with more symbols when $\Delta\tau = t - t_s$ is expressed in terms of the particle energy error. It is desirable to do this so the equations are clearly seen as two coupled equations. Once put into differential form, they can be combined into a single second-order differential equation. To convert from an error in time to an error in energy, Eq. 3 is used to get difference equations in the form found in Edwards and Syphers,

$$\phi_{n+1} = \phi_n + \frac{\omega_{rf}\tau\eta c^2}{v^2 E_s}\Delta E_{n+1} \tag{1}$$

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s) \tag{2}$$

Let's begin by discussing the energy difference equation. The energy kick experienced by a particle in an accelerating gap is equal to the charge of the particle (denoted e) multiplied by the voltage level at the gap. If variation of voltage across the accelerating gap can be neglected, then this is just the voltage present at the phase of the particle. The level of the voltage phaser at the time of particle crossing will be $V\sin\phi$ or $V\cos\phi$ depending on the reference angle chosen. Here, V is the peak RF voltage. If the reference phase is defined to be when the voltage level is zero, then the correct voltage is $V\sin\phi$. If the reference phase is when the voltage is at the peak level, then the correct voltage is $V\cos\phi$. So, the meaning of Eq. 2 is that the energy difference of a particle from the synchronous particle at the entrance to gap $n+1$ is whatever the energy difference was at the entrance to gap n, plus the difference in energy kick at gap n due to its error in phase at gap n.

In a circular machine, the time difference of arrival at successive RF gaps between an arbitrary particle and a synchronous particle, Δt, depends both on their velocity difference and and on the difference in path length traveled between gaps. While in a linac it is a straight shot from gap to gap, in a circular machine the dispersion of particles by magnetic fields also plays a role. The kick experienced by a particle in a magnetic field is inversely proportional to its momentum. In a ring made up mostly of bending dipoles (for example) the net effect of dispersion is that higher momenta particles will travel in a circular path with a larger circumference. So, a higher momentum particle will have a larger velocity than a lower momentum particle, but it will also have further to go. The relative importance of each of these factors depends on the energy of the particles, velocity differences being less important at higher energies as the particles approach the speed of light.

The revolution frequency of a particle is given by the machine circumference divided by the particle velocity.

$$T_{rev} = \frac{C}{v}$$

The revolution frequency can change if either the pathlength around the machine or the particle velocity changes.

$$\begin{aligned}
\frac{\Delta T_{rev}}{T_{rev}} &= \frac{\Delta(\frac{C}{v})}{\frac{C}{v}} \\
&= \frac{v}{C}\left[\frac{\Delta C}{v} - \frac{C\Delta v}{v^2}\right] \\
&= \frac{\Delta C}{C} - \frac{\Delta v}{v}
\end{aligned}$$

Particles of higher momentum have a longer path around the machine. This causes those particles to have a positive error in revolution period (longer time). However, higher momenta particles may also have a greater velocity, producing a negative error in revolution period (shorter time). Ultra- relativistic particles have velocities close to the speed of light, the dependence of velocity on momentum is not very strong in this regime. Writing $\frac{\Delta C}{C}$ and $\frac{\Delta v}{v}$ in terms of momentum error will allow a direct comparison of the magnitude of the terms. Begin with the velocity term,

$$\frac{dp}{dv} = \frac{d(\gamma m v)}{dv}$$

$$= \quad \gamma m \frac{dv}{dv} + mv\frac{d\gamma}{dv} = \gamma m + \frac{mv^2\gamma^3}{c^2}$$

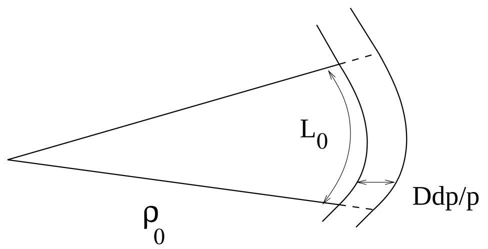
$$= \quad (\gamma mv)\left(\frac{1}{v}\right)\left[1+\gamma^2\beta^2\right] = (\gamma mv)\left(\frac{1}{v}\right)\gamma^2$$

Dividing both sides of the equation by $p = \gamma mv$, and multiplying by dv,

$$\frac{dv}{v} = \frac{1}{\gamma^2}\frac{dp}{p}$$

Now we have,

$$\frac{dT_{rev}}{T_{rev}} = \frac{dC}{C} - \frac{1}{\gamma^2}\frac{dp}{p}$$



If you change the particle momentum, the orbit changes by a proportionality factor which depends on the lattice of the machine. The path length along the off-momentum orbit changes due to the dispersion of the magnets. If a particle were otherwise on the design orbit, it would have an error in position given by,

$$\Delta x = D\frac{\Delta p}{p}$$

where the off-momentum particle has momentum error Δp, the momentum of the ideal particle is p, and D is the dispersion function. The path length of the off-momentum particle is,

$$L \quad = \quad L_0\left(\frac{\rho_0 + D\frac{\Delta p}{p}}{\rho_0}\right)$$

$$= \quad L_0 + D\left(\frac{\Delta p}{p}\right)\frac{L_0}{\rho_0}$$

The total change in the off-momentum particle is,

$$\Delta C = \frac{dp}{p}\oint\frac{Dds}{\rho_0}$$

Giving a path length error in the transit around the accelerator,

$$\frac{\Delta C}{C} = \left(\frac{1}{C}\oint \frac{Dds}{\rho}\right)\frac{dp}{p} = \frac{1}{\gamma_t^2}\frac{dp}{p}$$

For convenience, the proportionality factor scaling $\frac{dp}{p}$ is called $\frac{1}{\gamma_t^2}$, where γ_t is called the transition gamma of the ring. Then,

$$\begin{aligned}\frac{dT_{rev}}{T_{rev}} &= \frac{d\tau}{\tau} = \frac{1}{\gamma_t^2}\frac{dp}{p} - \frac{1}{\gamma^2}\frac{dp}{p} \\ &= \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)\frac{dp}{p} = \eta\frac{dp}{p} = \frac{\eta}{\beta^2}\frac{dE}{E} \qquad (3)\end{aligned}$$

where η is defined by the above relation, and is called the 'slip factor'. Notice that when $\gamma_t = \gamma$, then the slip factor goes to zero. When this happens, the travel time from one accelerating gap to another does not depend on the particle momentum! There is no phase stability at this particular energy. This dangerous energy is called the 'transition energy'. Remember, this is a machine dependent parameter, linacs do not have a transition energy, and for many circular accelerators the transition energy lies outside its range of operation.

3 Differential forms for the difference equations

In order to examine the properties of longitudinal motion, it is worthwhile to write the difference equations in differential form.

$$\phi_{n+1} - \phi_n \quad \rightarrow \quad \frac{d\phi}{dn}$$

$$\Delta E_{n+1} - \Delta E_n \quad \rightarrow \quad \frac{d\Delta E}{dn}$$

Then the difference equations become,

$$\frac{d\phi}{dn} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s}\Delta E \qquad (4)$$

$$\frac{d\Delta E}{dn} = eV(\sin\phi - \sin\phi_s) \qquad (5)$$

Differentiating eq. 4 with respect to n, and substituting eq. 5 into eq. 4, we get a second-order differential equation.

$$\frac{d^2\phi}{dn^2} - \frac{2\pi h\eta}{\beta^2 E_s} eV(\sin\phi - \sin\phi_s) = 0 \tag{6}$$

This is a nonlinear differential equation which looks similar to the nonlinear pendulum equation. Some phase space trajectories for a pendulum are shown in Fig. 1. Small oscillations of the pendulum are the nearly circular trajectories around the (0,0) point. Some initial conditions result in trajectories that don't close, moving from one side of the figure to the other. The trajectories shown in Fig. 1 resemble the phase space trajectories in an accelerator when the beam is bunched but not accelerating. The area in longitudinal phase space where particle trajectories close and beam remains captured in trajectories around the synchronous phase is referred to as a 'bucket'.

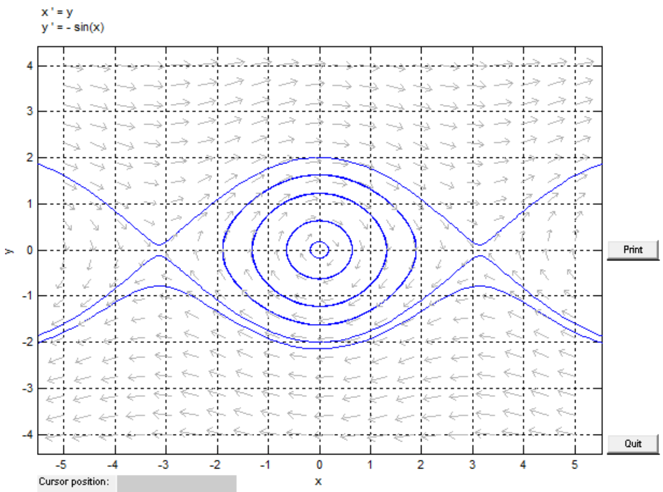


Figure 1: Phase space trajectories for a pendulum bob. This was made by Amichay Perry using a Matlab code from John C. Polking at Rice University.

For small oscillations, Eq. 6 reduces to an equation of harmonic motion. In order to have small oscillations, the phase error, $\Delta\phi = \phi - \phi_s$, must be small. The phase in eq. 6 can be re-written in terms of the phase error, $\phi = \phi_s + \Delta\phi$, the trigonometric function expanded, and the small angle approximation applied.

$$\sin(\phi_s + \Delta\phi) = \sin\phi_s \cos\Delta\phi + \sin\Delta\phi\cos\phi_s \approx \sin\phi_s + \Delta\phi\cos\phi_s$$

Then,

$$\sin\phi - \sin\phi_s \approx \Delta\phi\cos\phi_s$$

and the second-order differential equation becomes,

$$\frac{d^2\Delta\phi}{dn^2} + \left(-\frac{2\pi h\eta eV}{\beta^2 E_s}\cos\phi_s\right)\Delta\phi = 0 \tag{7}$$

Equation 7 has the form of a harmonic oscillator equation if $\eta\cos\phi_s$ is less than zero. Since in a circular machine η switches sign at the transition energy, the synchronous phase angle, ϕ_s must be changed then as well in order to maintain the stability of small oscillations. Since we have a harmonic oscillator equation for small oscillations, but a pendulum-like equation in the general case, we expect that the phase space trajectories for the longitudinal plane will look like circles or ellipses for small phase space errors, but take on the characteristic shape of the separatrix for the phase space trajectories of the pendulum at large errors.

Since Eq. 7 has the form of a harmonic oscillator equation, the characteristic frequency of longitudinal oscillation, called the synchrotron frequency, can be identified.

$$\Omega_s = \sqrt{-\frac{2\pi h\eta eV}{\beta^2 E_s}\cos\phi_s} \tag{8}$$

References

[1] Thomas Wangler, 'RF Linear Accelerators', Wiley series in beam physics and accelerator technology, Wiley, New York, 1998. ISBN 0-471-16814-9

[2] D.A. Edwards, M.J. Syphers, 'An introduction to the physics of high energy accelerators', Wiley, 1993. ISBN 0-471-55163-5

[3] S. Ohnuma, 'The beam and the bucket', Fermilab TM-1381, 1986.