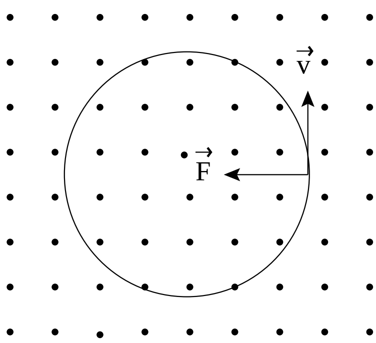
1 Overview on Magnetic fields

When accelerating particles in a linear accelerator, the machine must get longer to reach higher final particle energies. At high enough energies, the linac becomes awkwardly long. One solution is to take particles exiting an acceleration section and bend them back around to the entrance of the acceleration section so that the accelerator can be used repeatedly.

The magnetic fields of dipole magnets are used to bend particles. Dipole magnets can be used to keep particles moving in a circular trajectory, either to store them (storage ring), or to make them repeatedly pass through an acceleration section inserted into the circular path (circular accelerator). Circular accelerators or storage rings do not always have a perfectly circular shape, other shapes such as a rounded triangle can be used since the beam trajectory still closes on itself. However, a perfect circle has more uniform bending and so may be the easiest example with which to start.

In order for a particle to move in a circle, there must be a centripetal force. To move a negatively charged particle in counterclockwise circular motion as shown below, the magnetic field must be directed out of the plane of the figure. The magnetic force on the particle is given by $\vec{F} = q\vec{v} \times \vec{B}$, where q is the charge of the particle (including the sign of the charge), $\vec{v}$ is the particle velocity, and $\vec{B}$ is the magnetic field through which the particle travels. The direction of the particle velocity at any point in the circular trajectory is tangent to the circle. The force, velocity and magnetic field vectors are mutually orthogonal. Given that the force is directed radially inward, and that the velocity is tangent to the trajectory, the magnetic field must be directed upward, perpendicular to the plane containing the circular particle trajectory.



As an example, consider an antiproton (same charge as an electron) with a total energy of 9 GeV. Suppose we want to bend it around a circle (storage ring) of circumference 500 m. Calculating how much bend per length is needed;

$$\frac{\text{deflection}}{\text{meter}} = \frac{2\pi \text{ radian}}{500\,\text{m}} = 0.013\,\text{rad/m}$$

2 Field required to Bend particles

A calculation of basic importance in the manipulation of charged particle beams is the determination of the deflection of a particle by a magnetic or electric field. The force on a particle must have a component perpendicular to the direction of motion in order to bend its trajectory.

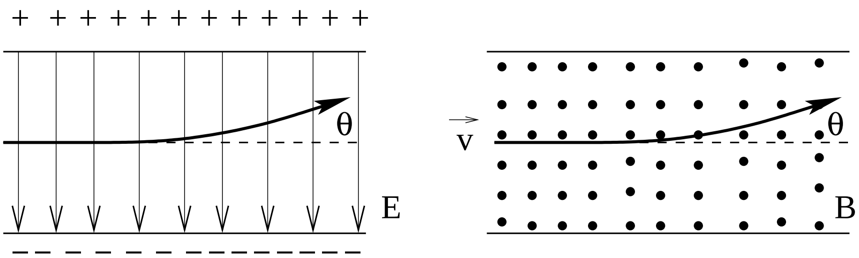


Figure 1: The left figure shows a negatively charged particle getting bent in an electrostatic field. The right figure shows a negatively charged particle getting bent in a magnetic field. The magnetic field points out of the page.

Particles in a beam typically have large momenta in the direction motion, $p_{\|}$. As a high energy particle travels through a field, the transverse momentum, $p_{\perp}$, imparted by the field to turn the particle is much smaller than the momentum in the direction of motion.

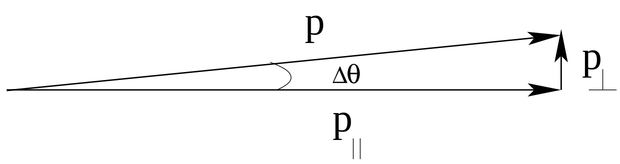


Figure 2: An sketch showing parallel and perpendicular components of particle motion (scale of perpendicular momentum exaggerated for clarity).

In other words, in traveling through the bending field, the particle is turned through a small angle. So, the small angle approximation may be used to simplify calculations.

$$\tan(\Delta\theta) \simeq \sin(\Delta\theta) \simeq \Delta\theta \simeq \frac{p_\perp}{p_\parallel} \simeq \frac{p_\perp}{p}$$

Writing the change in the transverse momentum as Δp, the longitudinal momentum as p, and the angle change imparted by the field as $\Delta\theta$:

$$\begin{aligned}
\Delta\theta &= \frac{\Delta p}{p} \\
&= \frac{1}{p}\frac{\Delta p}{\Delta t}\frac{\Delta t}{\Delta z}\Delta z \\
&= \frac{1}{p\left(\frac{\Delta z}{\Delta t}\right)}\frac{\Delta p}{\Delta t}\,\Delta z
\end{aligned}$$

where Δz is the short distance traveled through the field in a time Δt. Going to differential notation:

$$\begin{aligned}
\Delta\theta &= \frac{1}{p\left(\frac{dz}{dt}\right)}\frac{dp}{dt}\,dz \\
&= \frac{1}{pv}\,F\,dz
\end{aligned}$$

The total angle change through a field region of length L:

$$\theta = \frac{1}{pv}\int_0^L F\,dz$$

Suppose we want to bend a negatively charged particle with total energy 9 GeV and charge of magnitude e through an angle of 0.013 rad as it travels one meter in the direction of motion. This particle has a beta of $\beta = .995$, and its energy over charge is $\frac{9GeV}{e} = 9 \times 10^9$ volts. Let's find the strength needed from an electric field to bend the particle. Let the direction of beam motion be $\hat{z}$, and the desired deflection be in the $\hat{x}$

direction. If a particle is to be bent by an electric field, the field must be directed in the $-\hat{x}$ direction, or, $F_E\hat{x} = -eE_x(-\hat{x}) = eE_x\hat{x}$.

$$\theta_{total} = \frac{1}{pv}\int_0^L eE_x dz$$

Solving for the integrated electric field,

$$\int_0^L E_x dz = \frac{\theta pv}{e} = \frac{\theta(\gamma mc^2)\beta^2}{e}$$

$$E_x L = \frac{(13 \times 10^{-3}\text{rad})(9 \times 10^9\text{eV})(.99)}{e}$$

where $\gamma mv = \gamma m\beta c$ is the momentum of the particle, γmc^2 is the energy of the particle, $\beta = \frac{v}{c}$, and in this case, $\beta^2 = .99$. For a one meter long electrostatic device, the following field is needed:

$$E_x = 116\ \frac{\text{MV}}{\text{m}}$$

If a particle is to be bent by a magnetic field instead, the field must be directed in the $\hat{y}$ direction, or, $F_B\hat{x} = -ev\hat{z} \times B_y\hat{y} = evB_y\hat{x}$.

$$\begin{aligned}\theta_{total} &= \frac{1}{pv}\int_0^L evB_y dz \\ &= \frac{e}{p}\int_0^L B_y dz\end{aligned}$$

Note that the angular deflection a particle will experience is proportional to its charge and inversely proportional to its momentum. A single bending dipole may be used as a spectrometer, spreading out the particles in a beam according to their momenta. Solving the previous equation for the integrated magnetic field,

$$\int_0^L B_y dz = \frac{\theta p}{e} = \frac{\theta(\gamma mc^2)\beta}{ec}$$

$$B_yL \quad = \quad \frac{(.95)(13\times 10^{-3}\mathrm{rad})(9\times 10^{9})\mathrm{V}}{3\times 10^{8}\mathrm{m/s}}$$

For a one meter dipole magnet, the following field is needed:

$$B_y = 0.37 \ \ \mathrm{T}$$

The extra factor of $v = \beta c$ in the magnetic force makes a large difference in the scale of the needed magnetic field compared to electric field.

3 Magnetic rigidity

The 'magnetic rigidity' of a beam is a convenient parameter defined as the following:

$$\frac{p}{e} = B\rho$$

where p is the magnitude of the particle momentum, e is the charge of the particle, B is magnetic field, and ρ is the bending radius of a particle immersed in a magnetic field B. The ratio of p to e describes the 'stiffness' of a beam, it can be considered as a measure of how much angular deflection results when a particle travels through a given magnetic field. For a specific magnetic field, the greater the momentum of a particle, the less it will be bent as it travels through that field. The greater the charge of a particle, the more it will be bent as it travels through a given magnetic field.

This ratio can be put in a convenient form if both p and e are multiplied by c, the speed of light. Then, $B\rho = \frac{p}{e} = \frac{pc}{ec}$. In SI units this is:

$$(B\rho)[\mathrm{Tesla\ m}] = \frac{cp}{ce}\frac{[\mathrm{joule}]}{[\mathrm{C\ m/s}]} = \frac{\sqrt{E_{total}^2 - E_{rest}^2}}{(3\times 10^8)(1.6\times 10^{-19})}\frac{[\mathrm{joule}]}{[\mathrm{C\ m/s}]}$$

since $cp = \sqrt{E_{total}^2 - E_{rest}^2}$. It is convenient to specify cp in units of MeV. To keep the expression in SI units, it must be multiplied by the joule to MeV conversion factor. Then we have (still in SI units):

$$(B\rho)[\mathrm{Tesla\ m}] = \frac{cp[MeV]}{(3\times 10^8)[m/s](1.6\times 10^{-19}[\mathrm{coul}])}\frac{(1.6\times 10^{-19}[\mathrm{joule}])}{10^{-6}[MeV]}$$

Upon simplification this is:

$$(B\rho)[\text{Tesla m}] = \frac{cp[\text{MeV}]}{300[\text{MeV}]}\frac{[\text{joule s}]}{[\text{m coul}]} = \frac{cp[\text{MeV}]}{300[\text{MeV}]}[\text{Tesla m}]$$

Checking the units:

$$\frac{[\text{joule s}]}{[\text{m coul}]} = \frac{[\text{nt m s}]}{[\text{m coul}]} = \frac{[\text{nt}]}{[\text{coul}\frac{\text{m}}{\text{s}}]} = [\text{Tesla m}]$$

Example 1: An electron beam with total energy 6 GeV. If the rest energy of the particles is small compared to the total energy, the total energy may be used for cp to good approximation. Then it is easy to calculate $B\rho$; in this case, $B\rho = 6000/300$ Tesla m= 20 Tesla m. A 2 Tesla magnet would then have a bending radius of 0.1 meter. Alternatively, if a ring were made up completely of dipoles, for a ring radius of 1000 m, the field in the dipoles would have to be $\frac{B\rho}{\rho} = 20/1000 = .02$ Tesla.

Example 2: Fermilab Booster at injection (protons). The total energy cannot be used, the particles are not relativistic enough. The kinetic energy of the particles at injection is 400 MeV. So, the numerator is given by $pc = \sqrt{(1338)^2 - 938^2}$ MeV. The denominator is just c, the e cancels with the e in MeV to put the ratio in SI units.

$$\frac{pc}{ec} = \frac{9.54 \times 10^8}{3 \times 10^8} = 3.18 \text{ T-m}$$

4 Finding an equation for transverse motion

The design of an accelerator or beamline defines a desired trajectory for the particle motion. (From now on, I'll use the term 'machine' to mean accelerator, storage ring, or beamline.) This desired trajectory is called the 'reference trajectory' or 'ideal trajectory'. In the case of a circular path it is often called the 'reference orbit' of the particle. Since each particle is supposed to follow this path, the parameters of interest for the motion become the deviation of a particle from this path, rather than its absolute coordinates with respect to a fixed reference. Generally, convenient particle coordinates describe errors in the motion; deviations of position and direction from the ideal trajectory. A particle also potentially has a deviation of its momentum from the ideal momentum, and an error in longitudinal location. However, for the purpose of understanding transverse motion, for now it will be assumed that all particles have the design momentum, and ideal longitudinal position.

Finding an equation of motion is often done by writing a force equation. The harmonic oscillator is a familiar example. For the case of a spring with spring constant k this is $F = ma = -kx$, or $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$.

Let's find the equation of transverse motion in a simple case for a particle undergoing motion in a magnetic field at a constant energy. We make a bunch of assumptions, which usually reflect the situation of a high energy particle in a ring or transport line. First, the energy is constant (or varying slowly, and so we don't worry about it). Secondly, the magnetic field only has components transverse to the direction of motion, $\vec{B} = B_x\hat{x} + B_y\hat{y}$. Thirdly, the component of the particle velocity along the direction of motion, v_s, is much greater than the transverse components of particle velocity, $v_s >> v_x, v_y$. Since the only force is magnetic, the force equation is,

$$F = \frac{dp}{dt} = \gamma m\ddot{\vec{R}} = e(v \times B)$$

where v is the velocity of the particle, m is the particle mass, $\vec{R}$ is the transverse position vector which can also be written as $\vec{R} = (\rho + x)\hat{x} + y\hat{y}$ with ρ the radius of curvature. We are assuming the charge of a proton here, $+e$. Letting $v \sim v_s$, the cross-product $v \times B$ is:

$$v \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & v \\ B_x & B_y & 0 \end{vmatrix} = -vB_y\hat{x} + vB_x\hat{y}$$

If ρ is constant, then we have:

$$\gamma m\frac{d\vec{R}^2}{d^2t} = \gamma m\left(\frac{dx^2}{d^2t}\hat{x} + \frac{dy^2}{d^2t}\hat{y}\right) = -evB_y\hat{x} + evB_x\hat{y}$$

Now let's convert from a time derivative to a derivative with respect to the beam direction, s. Use $s = vt$, then $\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v\frac{d}{ds}$. Then:

$$\begin{aligned} v^2\frac{dx^2}{d^2s} &= \frac{-1}{\gamma m}evB_y \\ v^2\frac{dy^2}{d^2s} &= \frac{1}{\gamma m}evB_x \end{aligned}$$

These differential equations can be written:

$$\begin{aligned} \frac{dx^2}{d^2s} + \frac{e}{p}B_y &= 0 \\ \frac{dy^2}{d^2s} - \frac{e}{p}B_x &= 0 \end{aligned}$$

The case of motion through an ideal quadrupole magnet will be considered. The ideal particle goes through the centers of the dipole and quadrupole magnets of the beamline. The dipole field strength is required to bend particles to follow the reference trajectory, and affects all particles the same way. Since we are considering only deviations from the reference trajectory, bending magnets and sections with no magnets can be neglected in this simplest description of particle motion. On the other hand, the quadrupole magnets have strength that increases linearly with position from the center of the magnet, providing a linear restoring force to particles with position and direction errors. The magnetic quadrupole field components have the form:

$$\begin{aligned} B_x &= \frac{\partial B_x}{\partial y}y + \frac{\partial B_x}{\partial x}x \\ B_y &= \frac{\partial B_y}{\partial x}x + \frac{\partial B_y}{\partial y}y \end{aligned}$$

In the absence of coupling,

$$\begin{aligned} B_x &= \frac{\partial B_x}{\partial y}y \\ B_y &= \frac{\partial B_y}{\partial x}x \end{aligned}$$

Using the curl equation, $\nabla \times B = 0$, we find that $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$, and we can abbreviate the magnetic field gradient as $\frac{\partial B_y}{\partial x} = B^{'}$.

The equations describing motion through a quadrupole may be written:

$$\frac{dx^2}{d^2s} + \left(\frac{e}{p}B^{'}\right)x = 0$$

$$\frac{dy^2}{d^2s} - \left(\frac{e}{p}B^{'}\right)y \quad = \quad 0$$

In a high energy particle accelerator, a particle passes from one magnetic element to another, and so the magnetic force on the particle changes with longitudinal location, s, around the ring (or through the beamline). So, a simple model of linear motion (no magnetic elements of higher order than a quadrupole) through a beamline or accelerator is Hill's equation;

$$x^{''} + K(s)x = 0$$

where the derivative is with respect to longitudinal position, not time. There is a similar equation for motion in the y direction. The restoring force is a function of longitudinal position, and varies according to the quadrupole magnets through which the particle traverses. The general solution to Hill's equation is the following:

$$x(s) = A\sqrt{\beta(s)}\cos(\psi(s)) + B\sqrt{\beta(s)}\sin(\psi(s))$$

where A and B are arbitrary coefficients that depend on initial conditions. The function $\beta(s)$ describes how the amplitude of motion varies with longitudinal position, and it depends on the distribution and strength of the quadrupoles. The function $\psi(s)$ describes how the phase of the motion evolves along the particle trajectory, and also depends on the distribution and strength of the quadrupoles. Note that since there is transverse motion in both the x and y directions, there are two sets of beta functions, β_x and β_y, and separate phase evolutions ψ_x and ψ_y for each direction.

Hill's equation looks like an oscillator equation, although the amplitude and phase evolution of the 'oscillation' vary with longitudinal position. The transverse motion of a particle through a series of quadrupoles and other machine components is called a betatron oscillation. It is the quadrupole magnets that determine the character of the betatron oscillations.

5 Machine tune and the Twiss parameters

The beta function $\beta(s)$ and the phase advance of the betatron oscillation $\psi(s)$ are not independent. Their relationship can be found by stuffing the solution to Hill's equation back into Hill's equation. The result is terms that go as either $\cos(\psi(s))$ or $\sin(\psi(s))$. Once the terms are grouped, the coefficients of $\cos(\psi(s))$ and $\sin(\psi(s))$ must independently vanish since they are orthogonal functions. Applying this requirement to the coefficient of the sine term leads to a relationship between the phase advance, $\psi(s)$, and the beta function, $\beta(s)$. Application to the cosine term leads to an equation governing the evolution of the beta function (or beam envelope). For simplicity, write the solution to Hill's equation as

$$x(s) = A\sqrt{\beta(s)}\cos(\psi(s))$$

Take the derivative of the position $x(s)$ twice:

$$\begin{aligned}
x^{''}(s) &= -\frac{A}{4(\beta(s))^{\frac{3}{2}}}(\beta^{'})^2\cos(\psi(s)) + \frac{A}{2\sqrt{\beta(s)}}(\beta^{''})\cos(\psi(s)) \\
&- \frac{A}{\sqrt{\beta(s)}}\beta^{'}\psi^{'}\sin(\psi(s)) - A\sqrt{\beta(s))}(\psi^{'})^2\cos(\psi(s)) \\
&- A\sqrt{\beta(s))}(\psi^{''})\sin(\psi(s))
\end{aligned}$$

Substitute $x(s)$ and $x^{'}(s)$ into Hill's equation, $x^{''} + K(s)x = 0$, and group the cosine and sine terms,

$$\begin{aligned}
& A\left[-\tfrac{1}{4(\beta(s))^{\frac{3}{2}}}(\beta^{'})^2 + \tfrac{1}{2\sqrt{\beta(s)}}\beta^{''} - \sqrt{\beta(s)}(\psi^{'})^2 + K_x\sqrt{\beta(s)}\right]\cos(\psi(s)) \\
+ \quad & A\left[-\tfrac{1}{\sqrt{\beta}}\beta^{'}\psi^{'} - \sqrt{\beta}\psi^{''}\right]\sin(\psi(s)) = 0
\end{aligned}$$

Set the coefficient of the sine term to zero,

$$\begin{aligned}
\beta^{'}\psi^{'} - \beta\psi^{''} &= 0 \\
\frac{d(\beta\psi^{'})}{ds} &= 0 \\
\beta\psi^{'} &= \text{constant}
\end{aligned}$$

Let the constant be one (it is unconstrained, so we can choose it), then,

$$\psi(s) = \int \frac{ds}{\beta}$$

If the machine is a ring, the total phase advance for one revolution is called the 'tune', ν:

$$\nu = \frac{1}{2\pi}\oint \frac{ds}{\beta}$$

The tune, ν, is the total number of betatron oscillations per turn around the machine. There is a betatron tune for each transverse plane, ν_x and ν_y. The horizontal and vertical betatron oscillations in a (now decommissioned) accelerator called the 'Main Ring' are shown in Fig. 3. The horizontal (vertical) beam position is plotted as a function of location in the ring. For viewing purposes, the ring is 'cut' at a chosen location and unfolded into a linear picture, ending once again at the initial location.

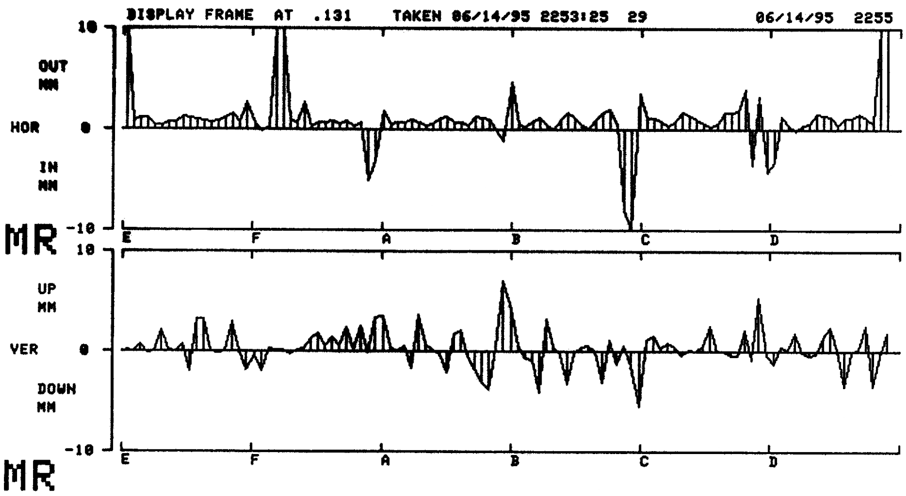


Figure 3: Horizontal and vertical closed orbits in the (now extinct) Main Ring at Fermilab.

The alpha function, $\alpha(s)$, is defined to be proportional to the slope of the beta function;

$$\alpha(s) = -\frac{1}{2}\frac{d\beta(s)}{ds}$$

The gamma function, $\gamma(s)$, is a combination of α and β that simplifies the notation in some common expressions;

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

The three machine functions $\beta(s), \alpha(s), \gamma(s)$ are referred to as the 'Twiss parameters'.

6 Transfer maps

Suppose that a particle has an initial position x_0 and angle $x_0^{'}$ at a machine location with Twiss parameters β_0, α_0, and γ_0. When the particle goes through an orbit of a storage ring (or other repeated group of magnets), its new position x and angle $x^{'}$ may be found using the following transfer map;

$$M = \begin{pmatrix} \cos(\Delta\psi) + \alpha_0 \sin(\Delta\psi) & \beta_0 \sin(\Delta\psi) \\ -\gamma_0 \sin(\Delta\psi) & \cos(\Delta\psi) - \alpha_0 \sin(\Delta\psi) \end{pmatrix}$$

where $\Delta\psi$ is the change in phase from the initial location to the final location at the end of the repeated period. If the repeated period is an orbit, then $\Delta\psi$ is $2\pi\nu$, where ν is the betatron tune.

$$M = \begin{pmatrix} \cos(2\pi\nu) + \alpha_0 \sin(2\pi\nu) & \beta_0 \sin(2\pi\nu) \\ -\gamma_0 \sin(2\pi\nu) & \cos(2\pi\nu) - \alpha_0 \sin(2\pi\nu) \end{pmatrix}$$

When a particle propagates from an arbitrary initial location with Twiss parameters β_1, α_1, and γ_1 to another arbitrary location with Twiss parameters β_2, α_2, and γ_2, then the evolution of the position and angle coordinates of a particle can be found using the following transfer map;

$$M_{12} = \begin{pmatrix} \frac{\beta_2}{\beta_1}[\cos(\psi_{12}) + \alpha_1 \sin(\psi_{12})] & (\beta_1\beta_2)^{\frac{1}{2}} \sin(\psi_{12}) \\ -\frac{1+\alpha_1\alpha 2}{(\beta_1\beta_2)^{\frac{1}{2}}} \sin(\psi_{12}) + \frac{\alpha_1-\alpha_2}{(\beta_1\beta_2)^{\frac{1}{2}}} \cos(\psi_{12}) & \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}} \cos(\psi_{12}) - \alpha_2 \sin(\psi_{12}) \end{pmatrix}$$

where ψ_{12} is the phase advance in going from location one to location two.

6.1 FODO example

The Twiss parameters, β, α, and γ may be related to magnet element parameters such as focal length, f, and drift length, L by equating equivalent transfer maps. Each magnetic element has a transfer map that allows the calculation of the position and angle of an exiting particle given its incoming position and angle. When magnetic elements are connected together to make a string of different elements (transport line), the transfer map of the entire string may be found by matrix multiplication of the individual transfer maps. Finally, this may be equated to the corresponding transfer map written using the Twiss parameters. A classic calculation for a FODO cell will be done as an example. The pattern of magnets is denoted by the abbreviation 'FODO'. This stands for Focusing quadrupole (F), drift space (or equivalently a bending magnet, O), Defocusing quadrupole (D), drift or bend (O). A 'lattice' is the pattern of magnets in an entire machine. A simple lattice would be to repeat the FODO sequence again and again. For calculation purposes, the repeated cell would start and end in the middle of an element to insure matching of the beta and alpha functions at the cell boundary. This way, the cell can fairly be considered a repeated section. The FODO cell will start in the middle of the focusing quadrupole for this example, as shown in Fig. 4. Half a focusing quadrupole has half the strength of the original, and so twice the focal length.

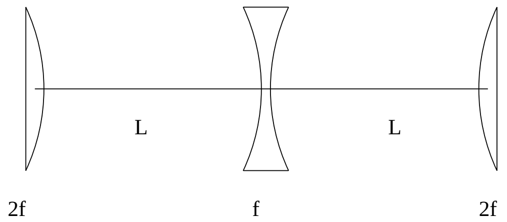


Figure 4: FODO cell, starting and ending in the center of the focusing quadrupole.

The maximum size of the beta function, β_{max}, will be in the middle of the focusing quadrupole, while the minimum size of the beta function, β_{min} will be in the middle of the defocusing quadrupole. The alpha function, α, will be zero at the centers of the quadrupoles. Since α is given by the slope of the beta function, it will be zero at the locations where the beta function has a minimum or maximum. The matrix for a repeated section written in terms of the Twiss parameters at the start (end) of a repeated section is

$$\begin{pmatrix} \cos(\Delta\psi) + \alpha\sin(\Delta\psi) & \beta\sin(\Delta\psi) \\ -\gamma\sin(\Delta\psi) & \cos(\Delta\psi) - \alpha\sin(\Delta\psi) \end{pmatrix} = \begin{pmatrix} \cos(\Delta\psi) & \beta_{max}\sin(\Delta\psi) \\ -\frac{1}{\beta_{max}}\sin(\Delta\psi) & \cos(\Delta\psi) \end{pmatrix} \quad (1)$$

The matrix for the repeated section may also be calculated based on the thin lens elements in that section, multiplying the matrices from the upstream to downstream ends.

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L + \frac{L^2}{f} \\ -\frac{L}{2f^2} + \frac{L^2}{4f^3} & 1 - \frac{L^2}{2f^2} \end{pmatrix} \quad (2)$$

To find the phase advance, $\Delta\psi$, equate the traces of the matrices from Eq. 1 and Eq. 2.

$$2\cos(\Delta\psi) = 2 - \frac{L^2}{f^2}$$

$$= \quad 2\left(1-\frac{\sqrt{2}L}{2f}\right)\left(1+\frac{\sqrt{2}L}{2f}\right)$$

Use the identity,

$$\begin{aligned}
\cos(\Delta\psi) &= 1-2\sin^2\left(\frac{\Delta\psi}{2}\right) \\
&= \left(1+\sqrt{2}\sin\left(\frac{\Delta\psi}{2}\right)\right)\left(1-\sqrt{2}\sin\left(\frac{\Delta\psi}{2}\right)\right)
\end{aligned}$$

So that,

$$\sin\left(\frac{\Delta\psi}{2}\right) = \frac{L}{2f}$$

Once the phase advance is known, β_{max} can be found by equating matix elements M_{12}:

$$\begin{aligned}
\beta_{max}\sin(\Delta\psi) &= 2L\left(1+\frac{L}{2f}\right) \\
&= 2L\left(1+\sin\left(\frac{\Delta\psi}{2}\right)\right)
\end{aligned}$$

and with the help of some algebra and trigonometric identities,

$$\begin{aligned}
\beta_{max} &= 2L\left[\frac{1+\sin(\frac{\Delta\psi}{2})}{\sin(\Delta\psi)}\right] \\
&= 2f\left[\frac{1+\sin(\frac{\Delta\psi}{2})}{1-\sin(\frac{\Delta\psi}{2})}\right]^{\frac{1}{2}}
\end{aligned}$$

One way to find β_{min} would be to follow this procedure again, but start the cell in the middle of the defocusing quadrupole, rather than the focusing quadrupole. Then,

$$\beta_{min} = 2L\left[\frac{1-\sin(\frac{\Delta\psi}{2})}{\sin(\Delta\psi)}\right]$$

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