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# Design of Electron Storage and Damping Rings

## Homework Problems 4

In the following problems, we will derive a criterion for the stability of a bunch in a storage ring in the presence of a transverse wake field, by developing a model of the bunch as two macroparticles, performing betatron and synchrotron oscillations.

Consider a bunch with a total population of $N_0$ particles. We construct a highly simplified model of the bunch as two macroparticles, each containing half the total population of the bunch, and performing betatron oscillations with frequency $\omega_\beta$ and synchrotron oscillations with frequency $\omega_s$ (= $2\pi/T_s$) and (longitudinal coordinate) amplitude $\sigma_z$. At $t = 0$, the first macroparticle has longitudinal coordinate:

$$z_1 = 0 \qquad \dot{z}_1 > 0$$

and the second macroparticle has:

$$z_2 = 0 \qquad \dot{z}_2 < 0$$

with positive $z$ towards the head of the bunch.

The transverse wake field $W_0$ is defined so that for two macroparticles with transverse and longitudinal coordinates ($y_1$,$z_1$) and ($y_2$,$z_2$), the change in normalized transverse momentum of particle 2 (trailing behind particle 1) over distance $L$ is given by:

$$\Delta p_{y2} = -\frac{r_e}{\gamma} N_1 y_1 W_0 \frac{(z_2 - z_1)}{2\sigma_z} \frac{L}{C_0}$$

where $N_1 e$ is the charge of macroparticle 1, $\gamma$ is the relativistic factor, and $C_0$ is the ring circumference.

1. Show that for $0 < t < \frac{1}{2}T_s$ the equations of motion of the macroparticles may be written:

$$\ddot{y}_1 + \omega_\beta^2 y_1 = 0$$

$$\ddot{y}_2 + \omega_\beta^2 y_2 = -iAy_1 e^{-i\omega_s t}$$

and write an expression for the constant $A$.

2. If $\omega_\beta >> \omega_s$, show that an approximate solution to the equations of motion may be written, for $0 < t < \frac{1}{2}T_s$:

$$y_1 \approx y_1(0)\, e^{-i\omega_\beta t}$$

$$y_2 \approx y_2(0)\, e^{-i\omega_\beta t} - i\frac{A}{2\omega_s \omega_\beta} y_1(0)\, e^{-i\omega_\beta t}\left(1 - e^{-i\omega_s t}\right)$$

3. Hence show that we can write:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=\frac{1}{2}T_s} \approx e^{-\frac{1}{2}i\omega_\beta T_s} \begin{pmatrix} 1 & 0 \\ ia & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=0}$$

and find an expression for the constant $a$.

4. By considering the motion in the second half of the synchrotron period, show that:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=T_s} \approx e^{-i\omega_\beta T_s} \begin{pmatrix} 1-a^2 & ia \\ ia & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{t=0}$$

5. Write a constraint on the constant $a$ for the motion of the macroparticles to remain stable. Hence, write a stability condition for the bunch as an upper limit for the bunch population $N_0$ in terms of the wake field, the synchrotron frequency, the betatron frequency, and the bunch length.

6. Describe the motion of the macroparticles in the case of (i) beam stability (i.e. the bunch population is below the instability threshold), and (ii) beam instability (i.e. the bunch population exceeds the stability limit). Explain physically the appearance of the instability threshold.