Lecture 5
RF-accelerators:
Synchronism conditions

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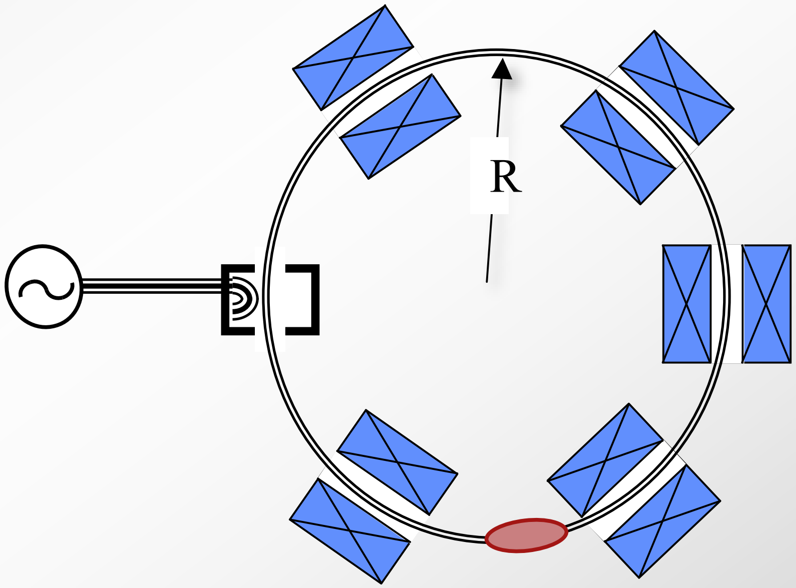
The synchrotron introduces two new ideas: change B_{dipole} & change ω_{rf}

- For low energy ions, f_{rev} increases as E_{ion} increases
- ==> Increase ω_{rf} to maintain synchronism
- For any E_{ion} circumference must be an integral number of rf wavelengths

$$L = h\,\lambda_{rf}$$

- h is the harmonic number



$$L = 2\pi R$$

$$f_{rev} = 1/\tau = v/L$$

Ideal closed orbit in the synchrotron

- Beam particles will not have identical orbital positions & velocities
- In practice, they will have transverse oscillatory motion (betatron oscillations) set by radial restoring forces
- An ideal particle has zero amplitude motion on a closed orbit along the axis of the synchrotron

Ideal closed orbit & synchronous particle

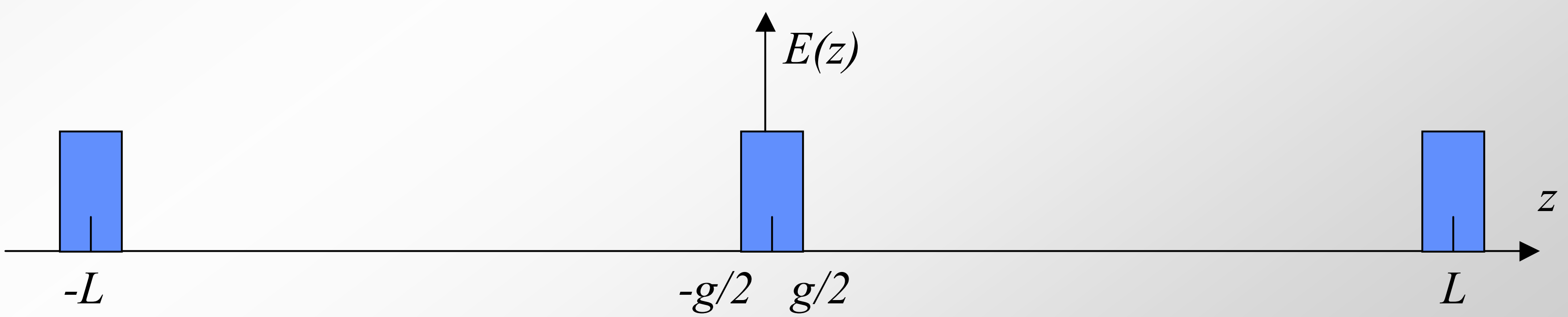
- The ideal synchronous particle always passes through the rf-cavity when the field is at the same phase

Synchrotron acceleration

- The rf cavity maintains an electric field at $\omega_{rf} = h\, \omega_{rev} = h\, 2\pi v / L$
- Around the ring, describe the field as $E(z,t) = E_1(z) E_2(t)$
- $E_1(z)$ is periodic with a period of L

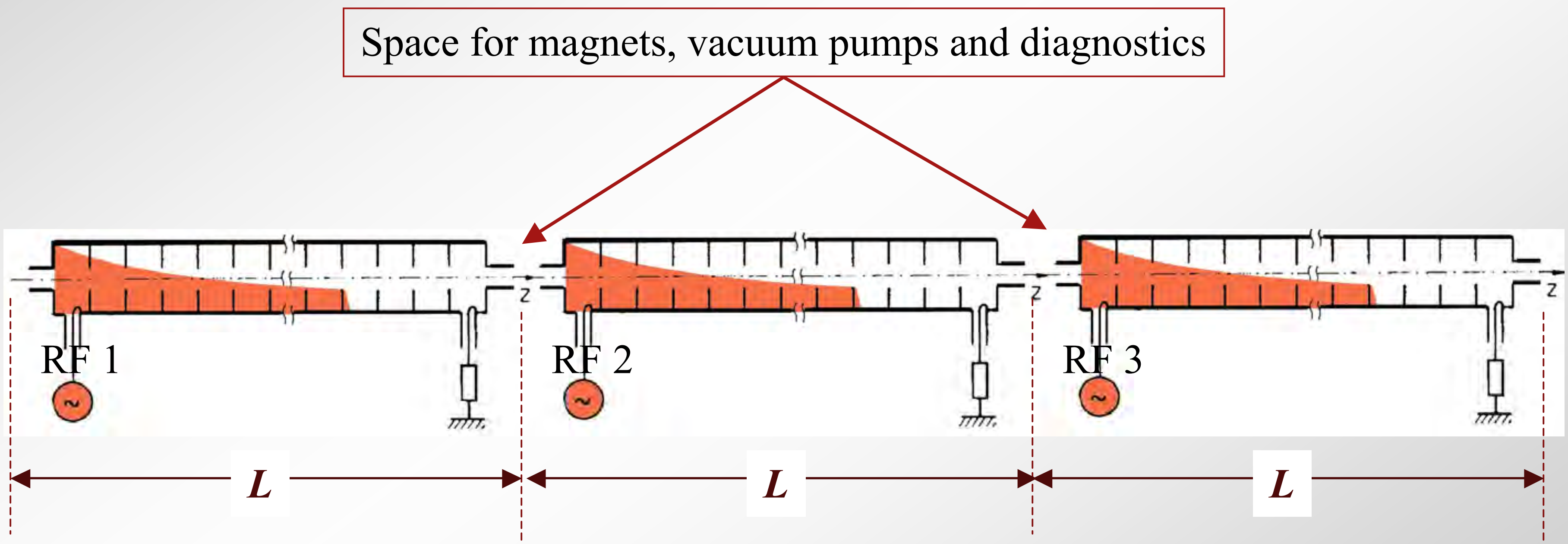
$$E_2(t) = E_o \sin\left(\int_{t_o}^{t} \omega_{rf}\, dt + \varphi_o \right)$$

- The particle position is $z(t) = z_o + \int_{t_o}^{t} v\, dt$



Phasing in a linac

❖ In the linac we must control the rf-phase so that the particle enters each section at the same phase.



Energy gain

- The energy gain for a particle that moves from 0 to L is given by:

$$W = q\int_0^L E(z,t)\cdot dz = q\int_{-g/2}^{+g/2} E_1(z)E_2(t)dz =$$

$$= qgE_2(t) = qE_o \sin\left(\int_{t_o}^{t} \omega_{rf}dt + \varphi_o\right) = qV$$

- V is the voltage gain for the particle.
  - depends only on the particle trajectory
  - includes contributions from all electric fields present
    - (RF, space charge, interaction with the vacuum chamber, …)

- Particles can experience energy variations $U(E)$ that depend on energy
  - synchrotron radiation emitted by a particle under acceleration

$$\Delta E_{Total} = qV + U(E)$$

Energy gain -II

- The synchronism conditions for the synchronous particle
  - condition on rf- frequency,
  - relation between rf voltage & field ramp rate

- The rate of energy gain for the synchronous particle is

$$\frac{dE_s}{dt} = \frac{\beta_s c}{L} eV \sin\varphi_s = \frac{c}{h\lambda_{rf}} eV \sin\varphi_s$$

- Its rate of change of momentum is

$$\frac{dp_s}{dt} = eE_o \sin\varphi_s = \frac{eV}{L} \sin\varphi_s$$

Beam rigidity links B, p and ρ

- Recall that $p_s = e\rho B_o$
- Therefore,

$$\frac{dB_o}{dt} = \frac{V\sin\varphi_s}{\rho L}$$

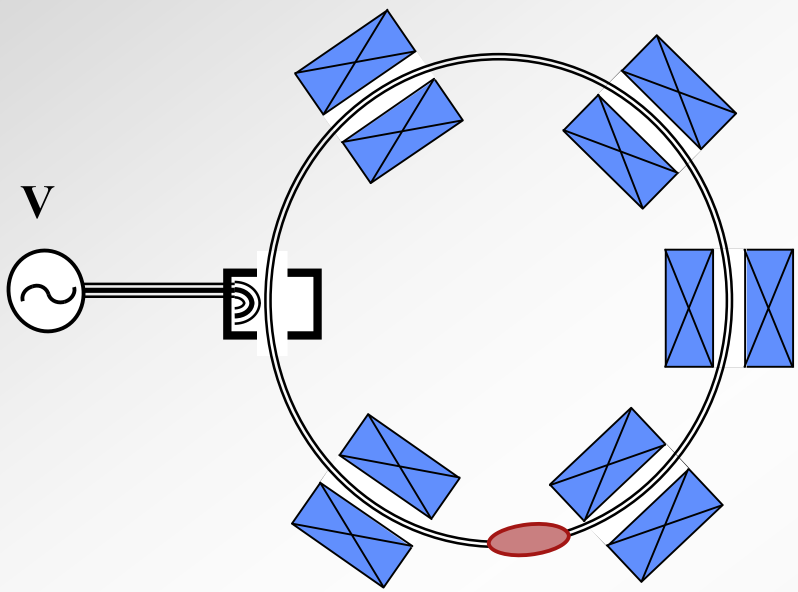
- If the ramp rate is uniform then $V\sin\phi_s = constant$
- In rapid cycling machines like the Tevatron booster

$$B_o(t) = B_{\min} + \frac{B_{\max} - B_{\min}}{2}\left(1 - \cos 2\pi f_{cycle} t\right)$$

- Therefore $V\sin\phi_s$ varies sinusoidally

Phase stability
&
Longitudinal phase space

Phase stability: Will bunch of finite length stay together & be accelerated?

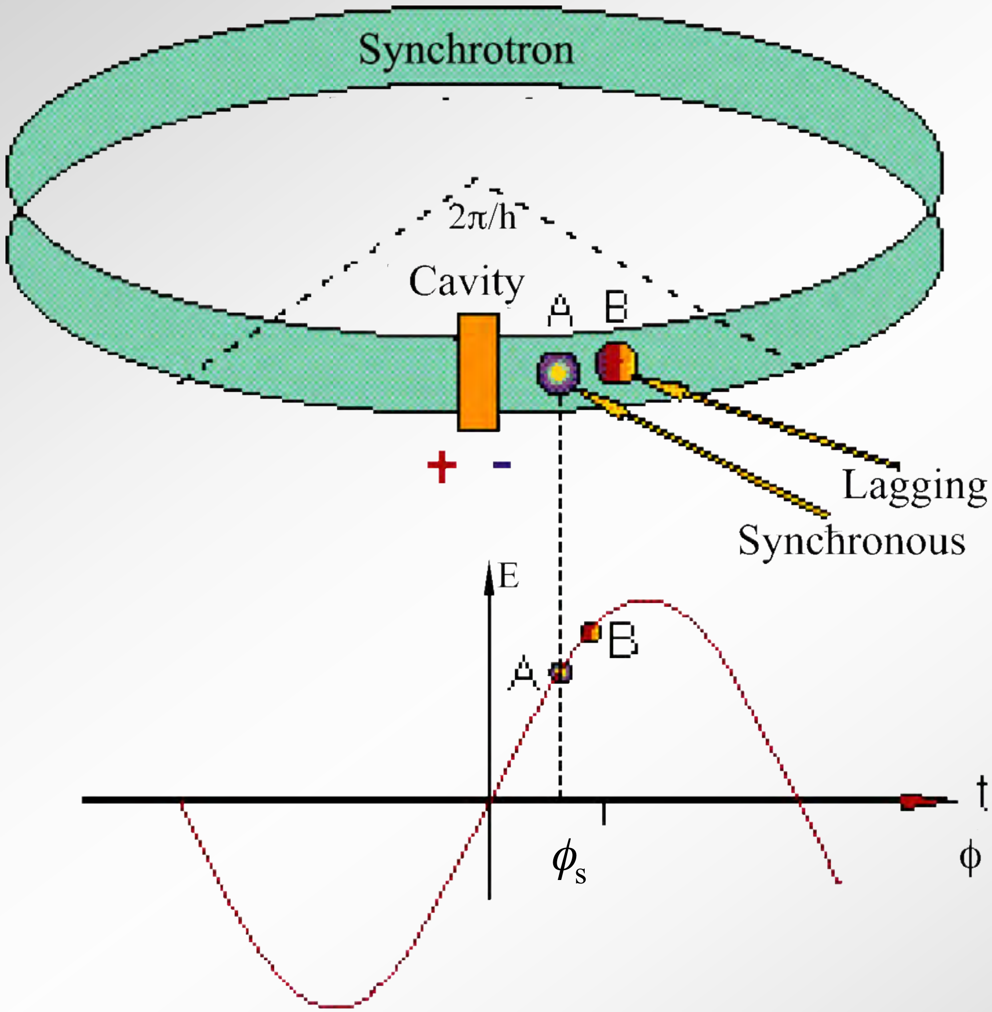


Let's say that the synchronous particle makes the ith revolution in time: T_i

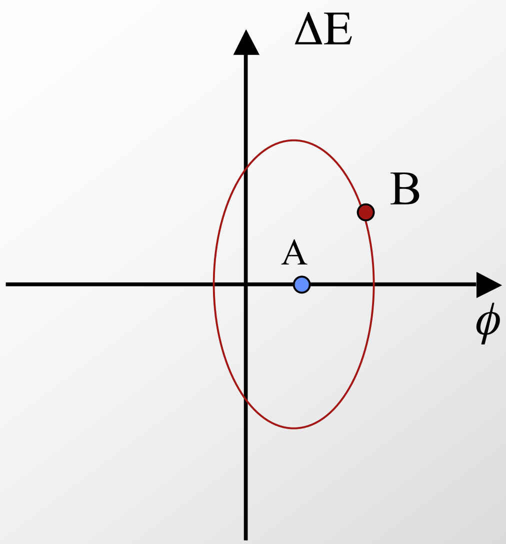
Will particles close to the synchronous particle in phase stay close in phase?

Discovered by MacMillan & by Veksler

What do we mean by phase?
Let's consider non-relativistic ions



From E. J. N. Wilson CAS lecture



How does the ellipse change as B lags further behind A?

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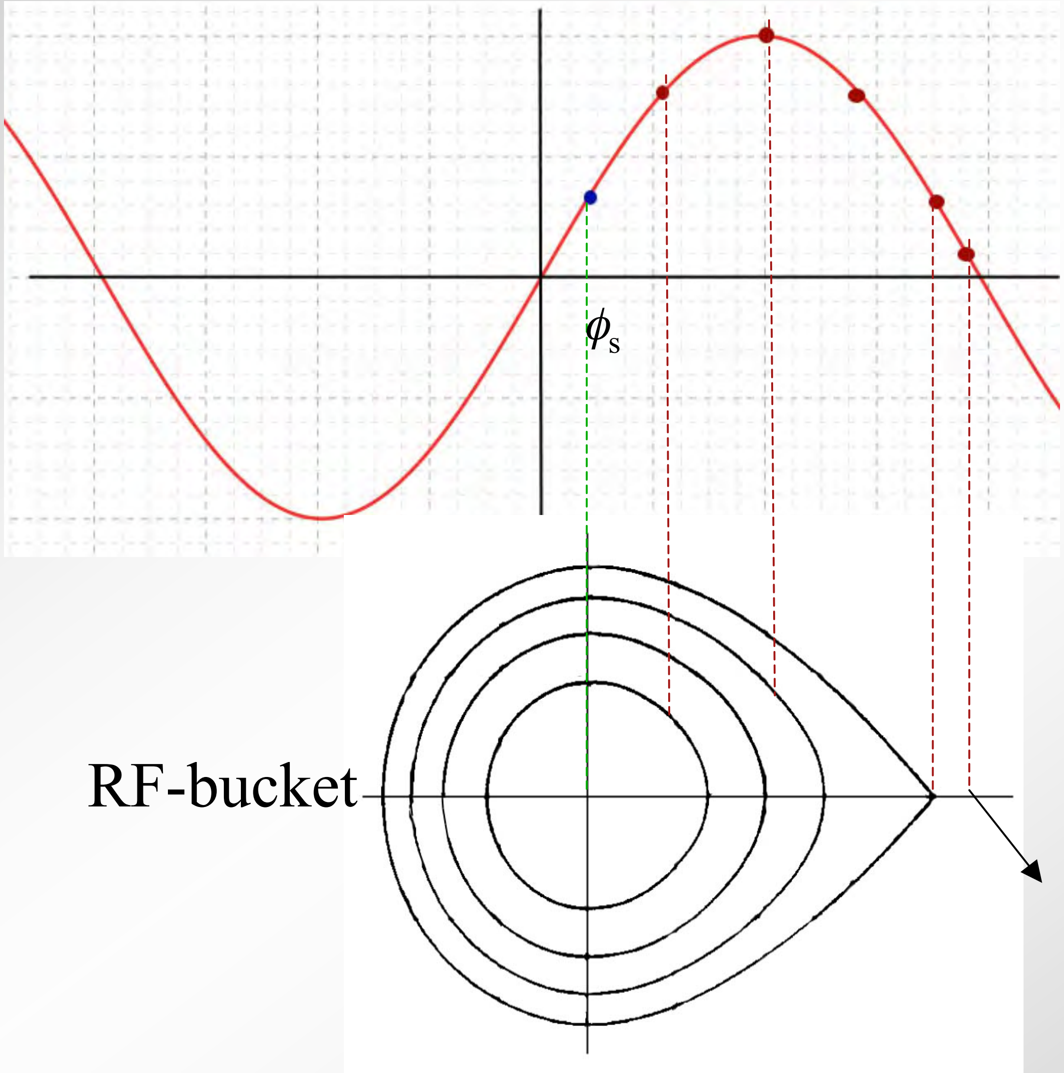
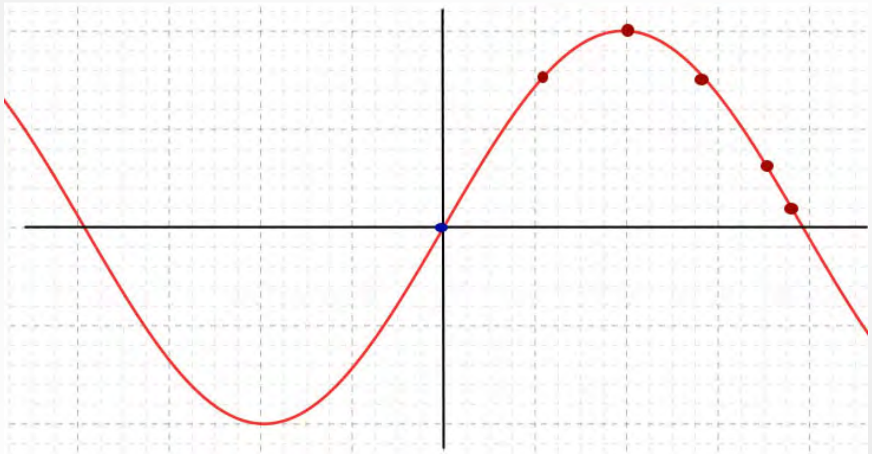


How does the size of the bucket change with ϕ_s ?

This behavior can be though of as
phase or longitudinal focusing

- Stationary bucket: A special case obtains when $\phi_s = 0$
  - The synchronous particle does not change energy
  - All phases are trapped

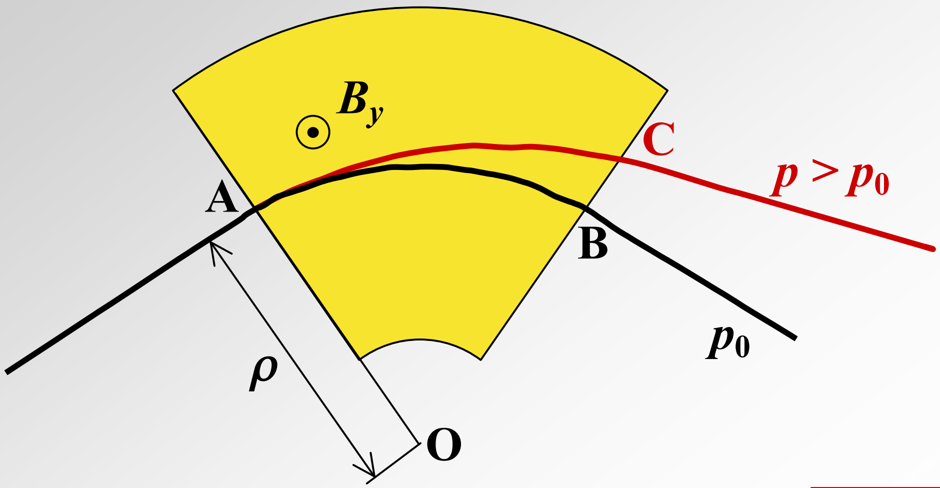


- We can expect an equation of motion in ϕ of the form

$$\frac{d^2\varphi}{ds^2} + \Omega^2 \sin\varphi = 0$$

Pendulum equation

Length of orbits in a bending magnet



$$\rho = \frac{p}{qB_z} = \frac{\beta\gamma m_0 c}{q B_z}$$

L_0 = Trajectory length between A and B

L = Trajectory length between A and C

$$\frac{L - L_0}{L_0} \propto \frac{p - p_0}{p_0} \Rightarrow \frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0}$$

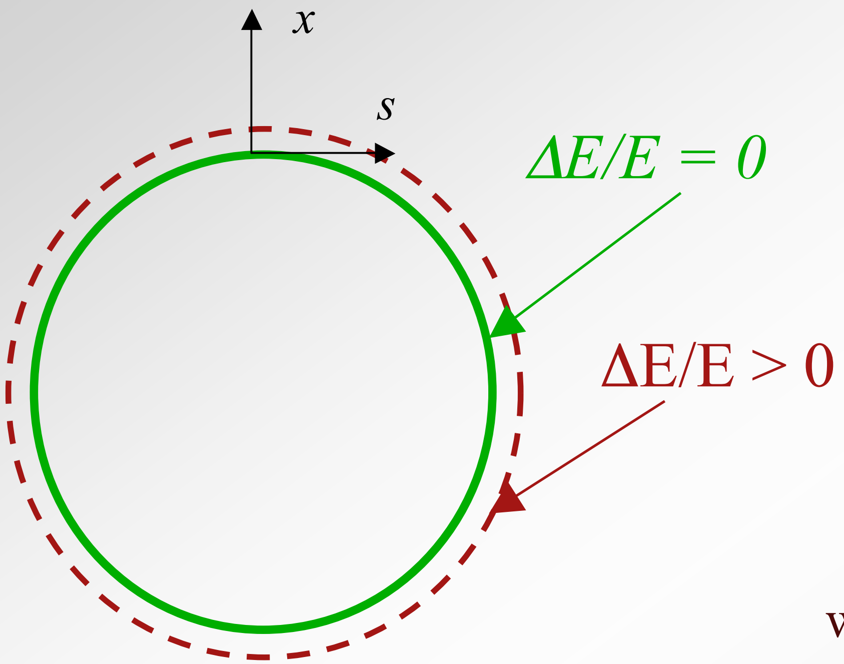
where α is constant

$$For \;\; \gamma >> 1 \;\; \Rightarrow \frac{\Delta L}{L_0} = \alpha \frac{\Delta p}{p_0} \cong \alpha \frac{\Delta E}{E_0}$$

In the sector bending magnet $L > L_0$ so that $a > 0$

Higher energy particles will leave the magnet later.

Definition: Momentum compaction



$$\frac{\Delta L}{L} = \alpha \frac{\Delta p}{p}$$

$$\alpha = \int_0^{L_o} \frac{D_x}{\rho} ds$$

where dispersion, D_x, is the change in the closed orbit as a function of energy

Momentum compaction, α, is the change in the closed orbit length as a function of momentum.

Phase stability: Basics

- Distance along the particle orbit between rf-stations is *L*
- Time between stations for a particle with velocity *v* is

$$\tau = L/v$$

- Then

$$\frac{\Delta\tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

- Note that

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$

(Exercise)

- For circular machines, L can vary with p
- For linacs L is independent of p

Phase stability: Slip factor & transition

- Introduce γ_t such that

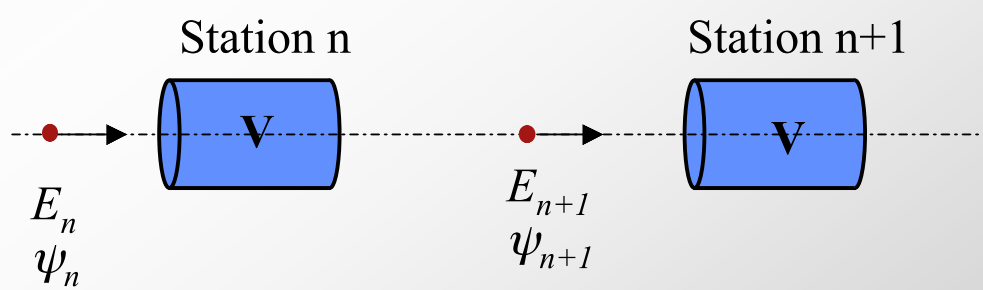
$$\frac{\Delta L}{L} = \frac{1}{\gamma_t^2}\frac{\Delta p}{p}$$

- Define a slip factor

$$\eta \equiv \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

- At some *transition energy* η changes sign

- Now consider a particle with energy E_n and phase ψ_n w.r.t. the rf that enters station n at time T_n



Equation of motion for particle phase

- The phase at station *n+1* is

$$\psi_{n+1} = \psi_n + \omega_{rf}(\tau + \Delta\tau)_{n+1}$$

$$= \psi_n + \omega_{rf}\tau_{n+1} + \omega_{rf}\tau_{n+1}\left(\frac{\Delta\tau}{\tau}\right)_{n+1}$$

- By definition the synchronous particle stays in phase (mod 2π)
- Refine the phase mod 2π

$$\phi_n = \psi_n - \omega_{rf} T_n$$

$$\phi_{n+1} = \phi_n + \omega_{rf}\tau_{n+1}\left(\frac{\Delta\tau}{\tau}\right)_{n+1} = \phi_n + \eta\underbrace{\omega_{rf}\tau_{n+1}}\left(\frac{\Delta p}{p}\right)_{n+1}$$

harmonic number = 2πN

Equation of motion in energy

$$(E_s)_{n+1} = (E_s)_n + eV\sin\phi_s$$ and in general $$E_{n+1} = E_n + eV\sin\phi_n$$

Define $\Delta E = E - E_s$

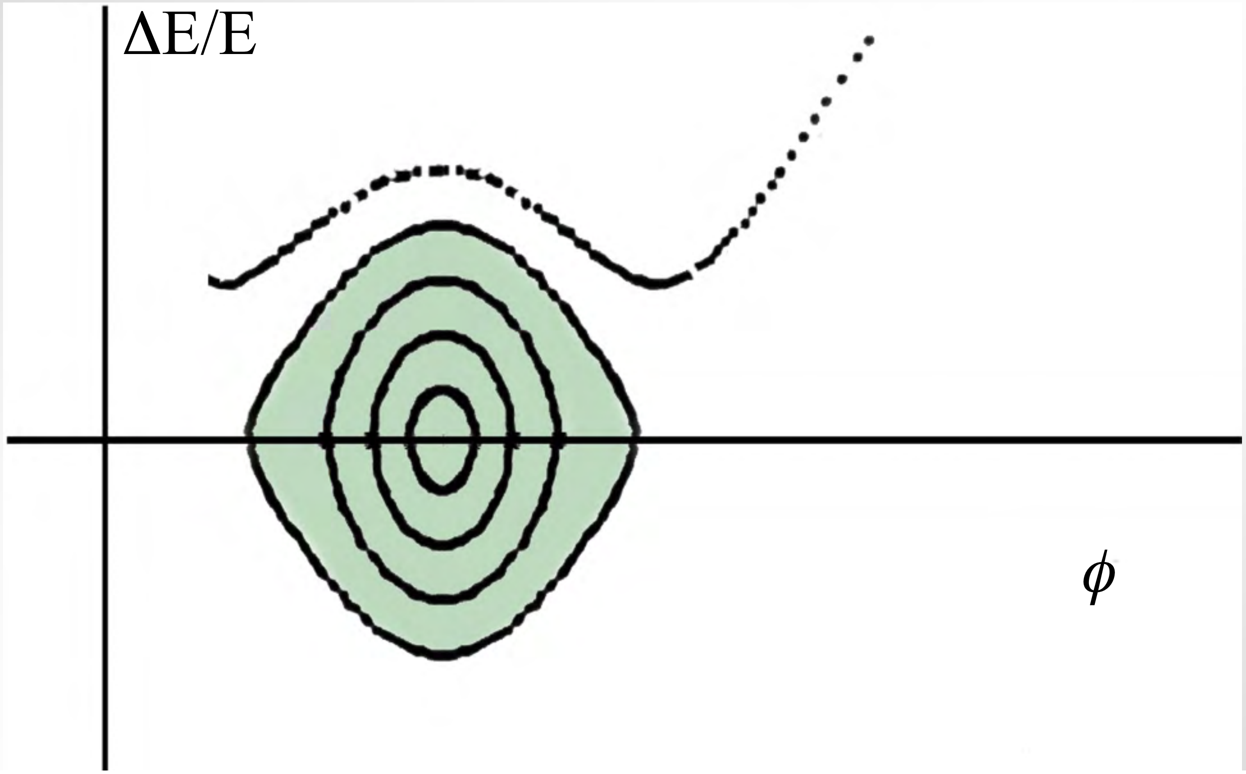
$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s)$$

Exercise: Show that $$\frac{\Delta p}{p} = \frac{c^2}{v^2}\frac{\Delta E}{E}$$

Then

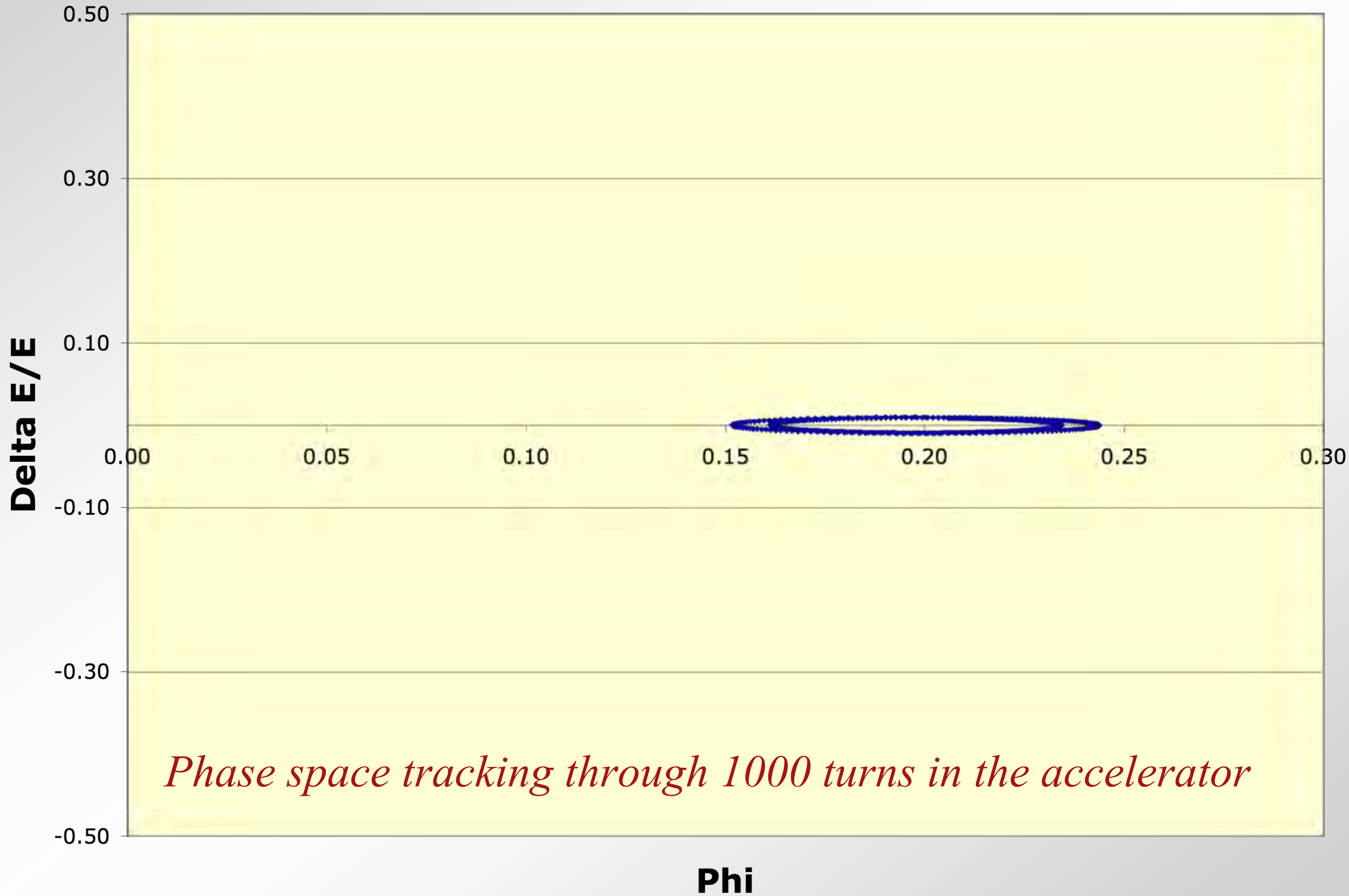
$$\phi_{n+1} = \phi_n + \frac{\omega_{rf}\tau\eta c^2}{E_s v^2}\Delta E_{n+1}$$

Longitudinal phase space of beam

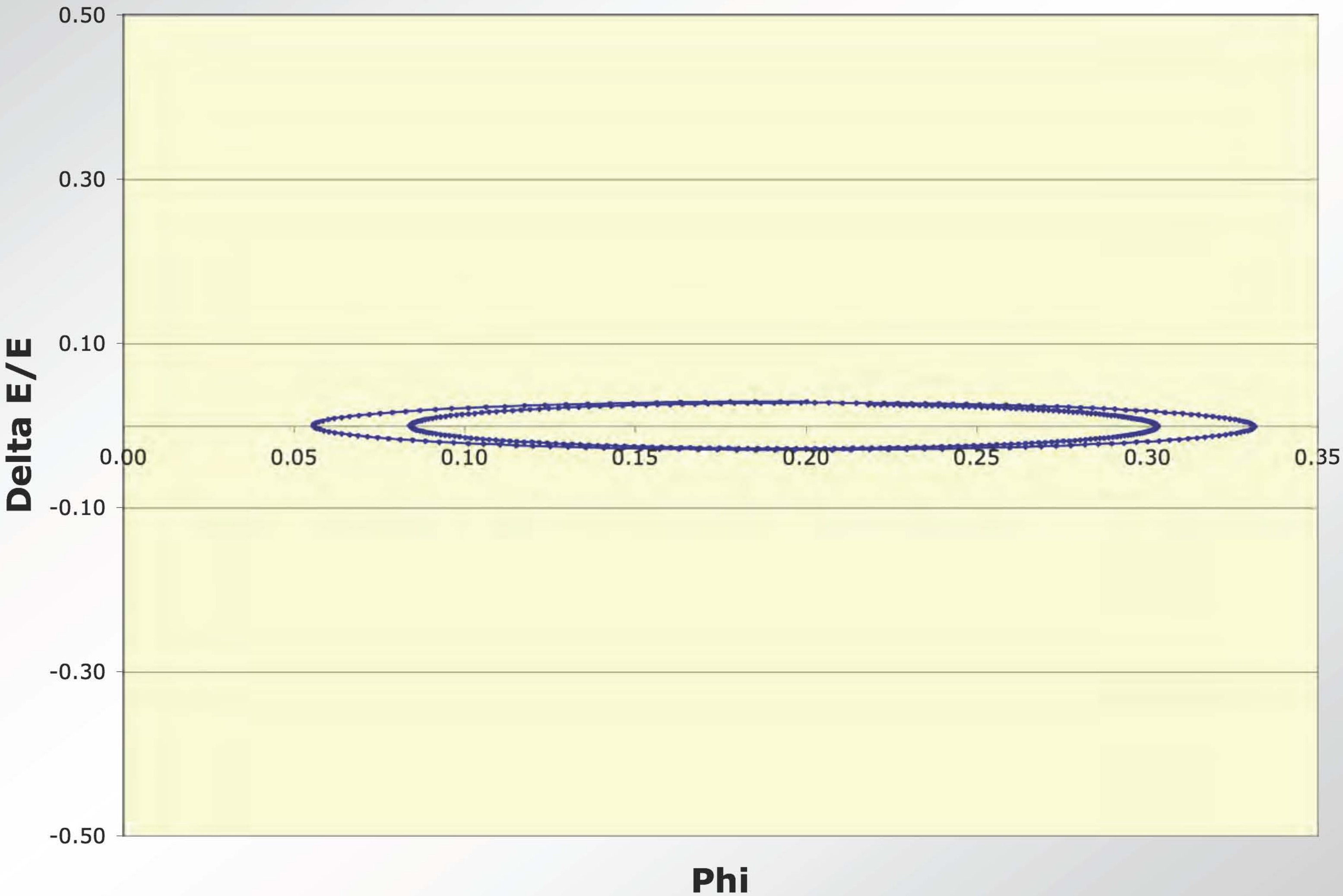


Solving the difference equations will show if there are areas of stability in the (ΔE/E, ϕ) longitudinal phase space of the beam

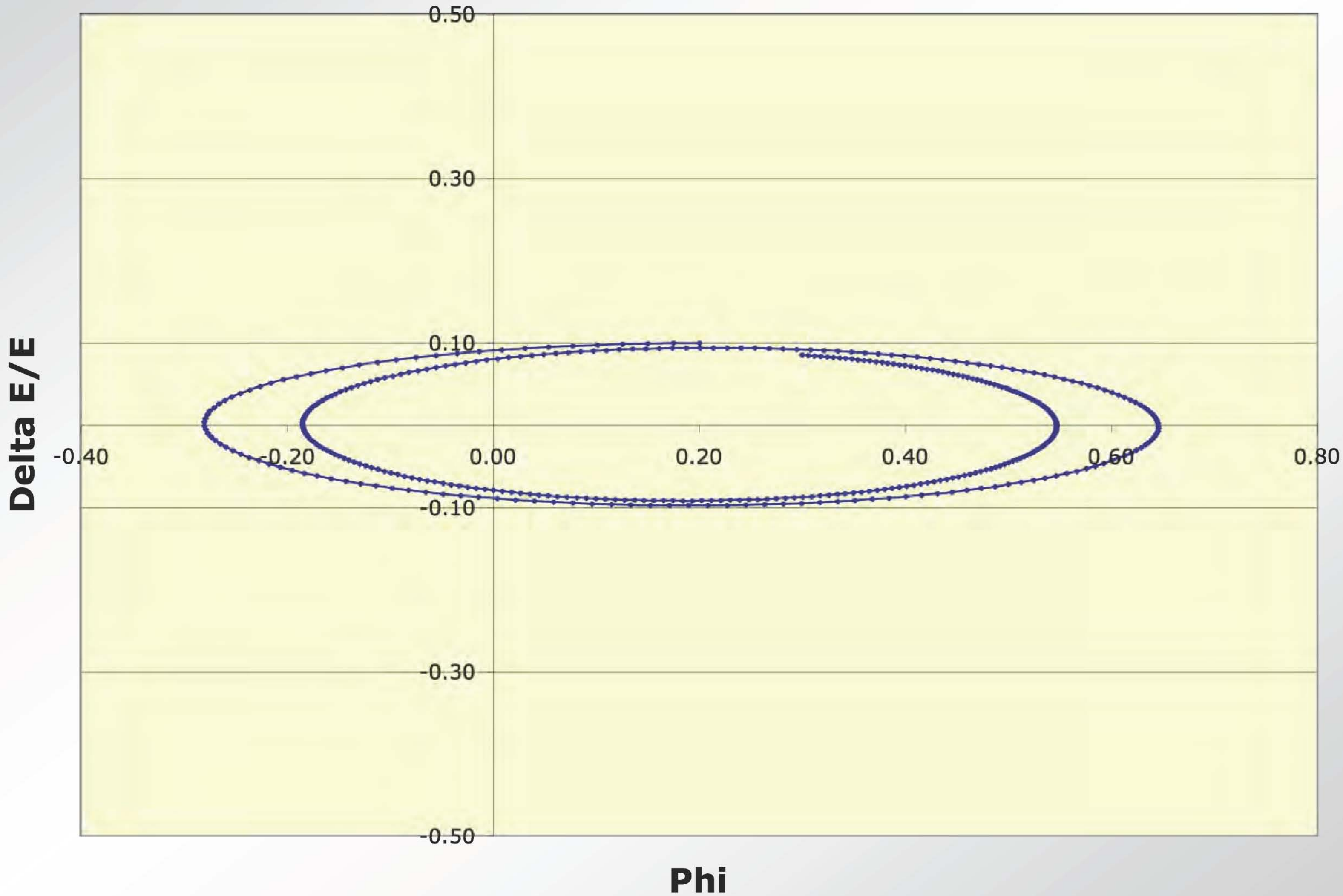
Phase stability, ΔE/E = 0.03, $\phi_n = \phi_s$



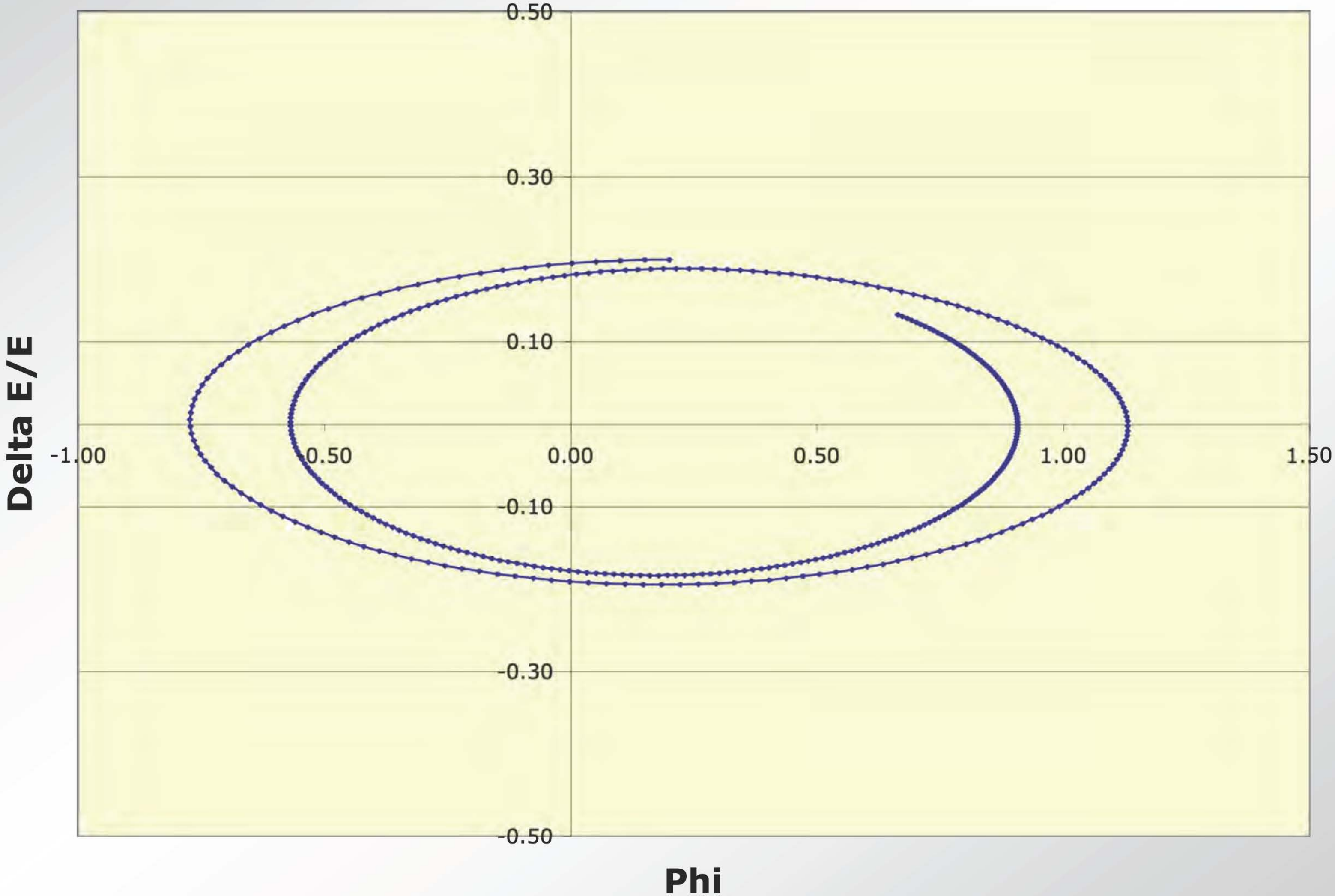
Phase stability, $\Delta E/E = 0.05$, $\phi_n = \phi_s$



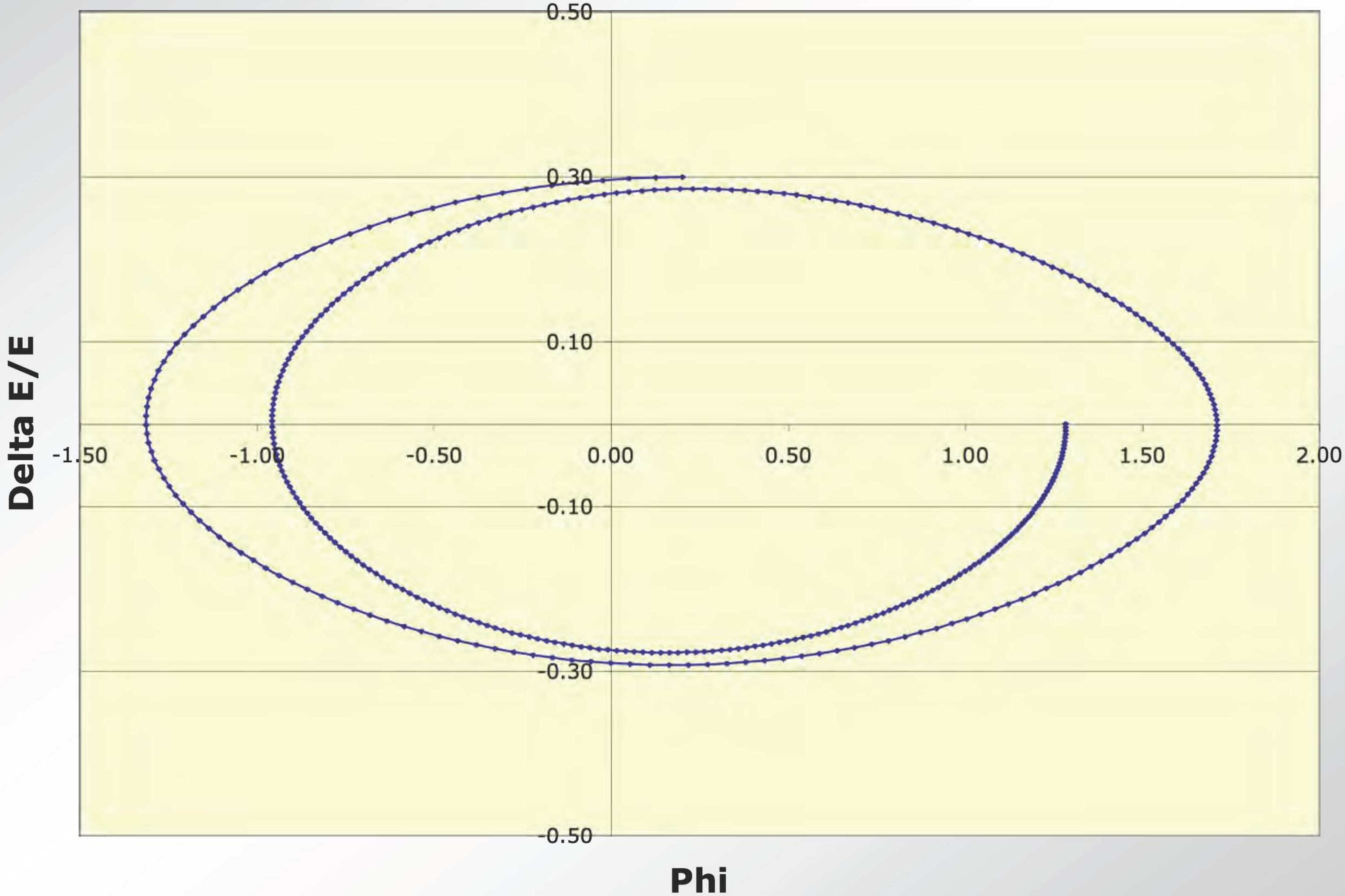
Phase stability, ΔE/E = 0.1, $\phi_n=\phi_s$



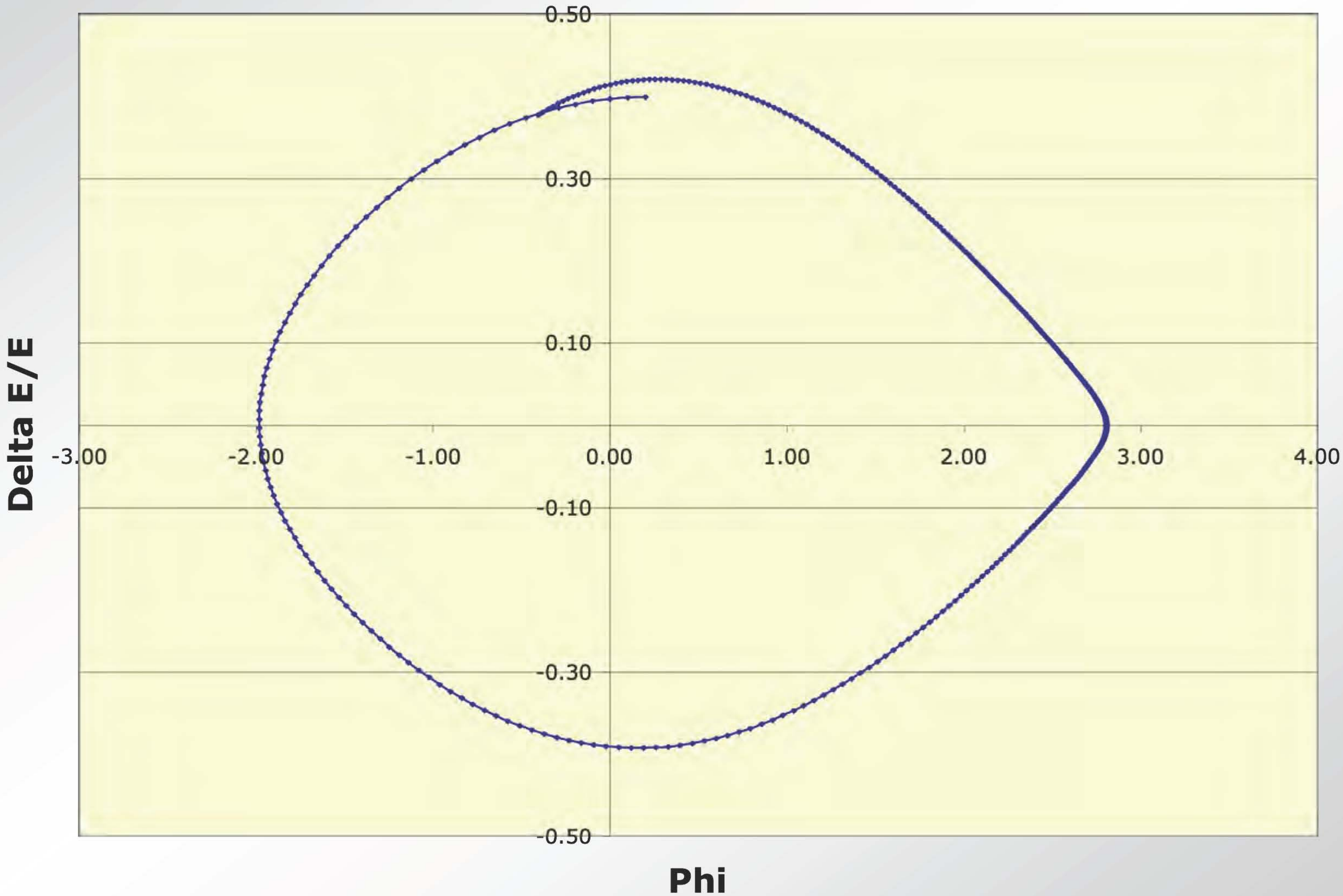
Phase stability, $\Delta E/E = 0.2$, $\phi_n = \phi_s$



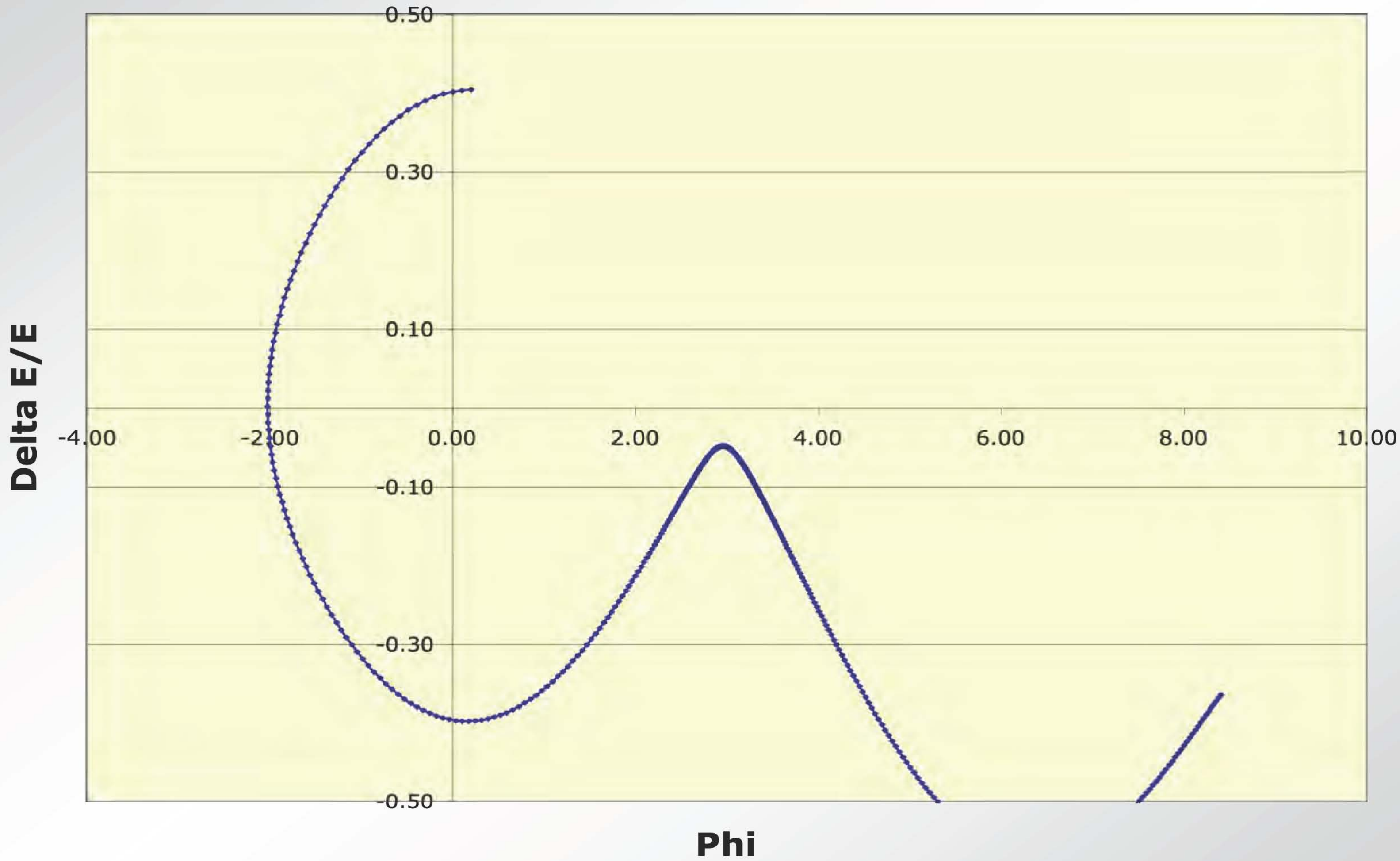
Phase stability, ΔE/E = 0.3, $\phi_n = \phi_s$



Phase stability, ΔE/E = 0.4, $\phi_n=\phi_s$

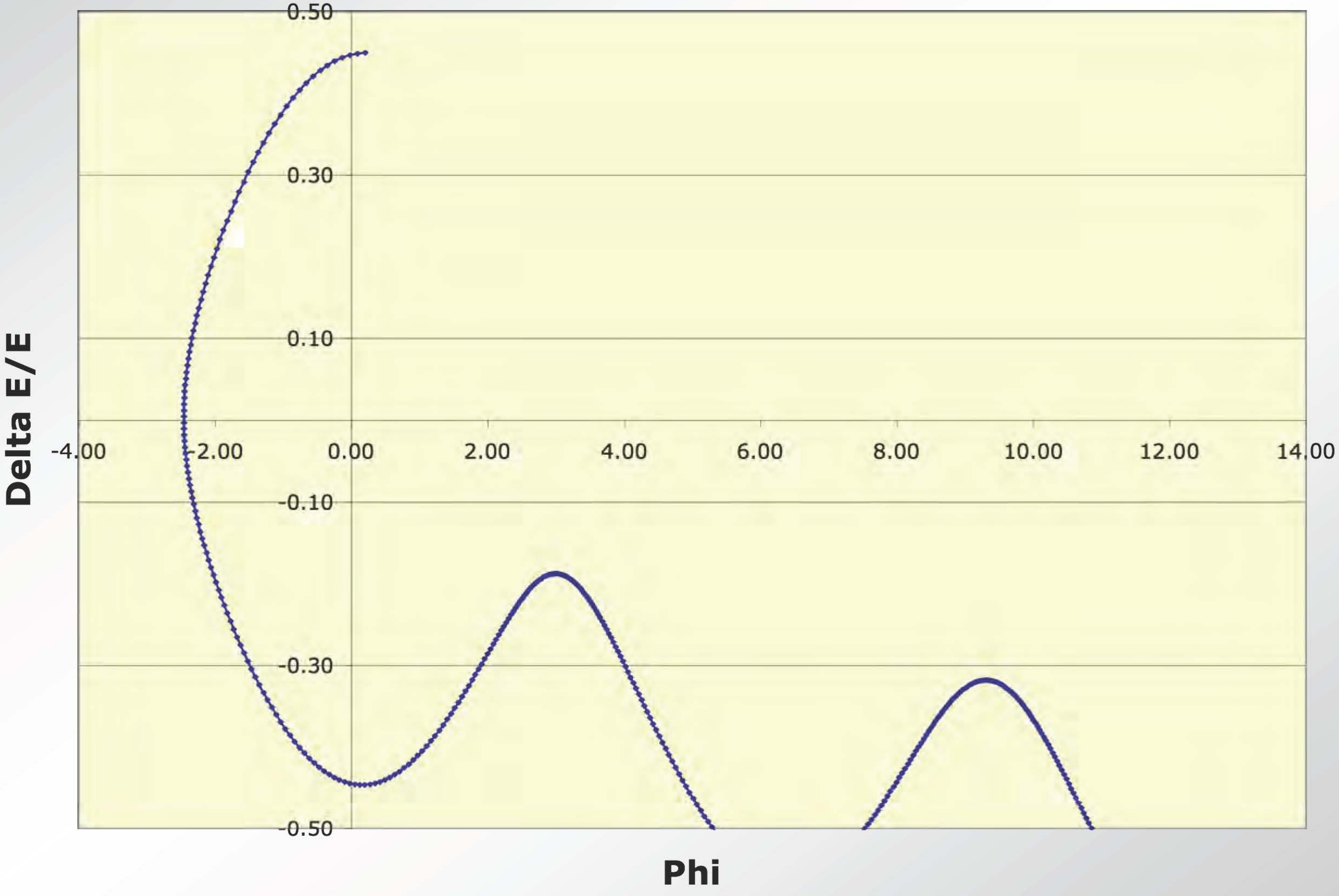


Phase stability, $\Delta E/E = 0.405$, $\phi_n=\phi_s$

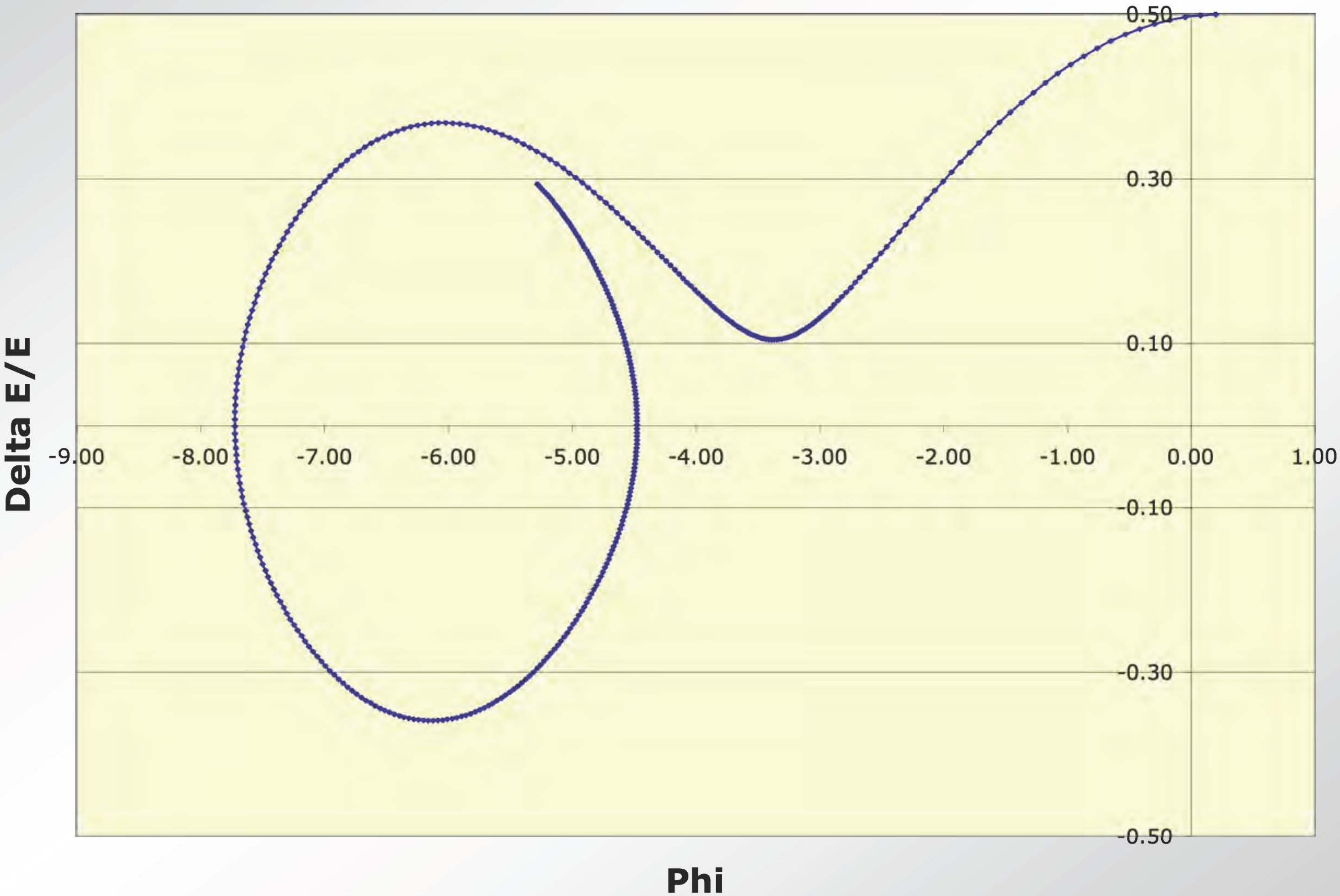


Regions of stability and instability are sharply divided

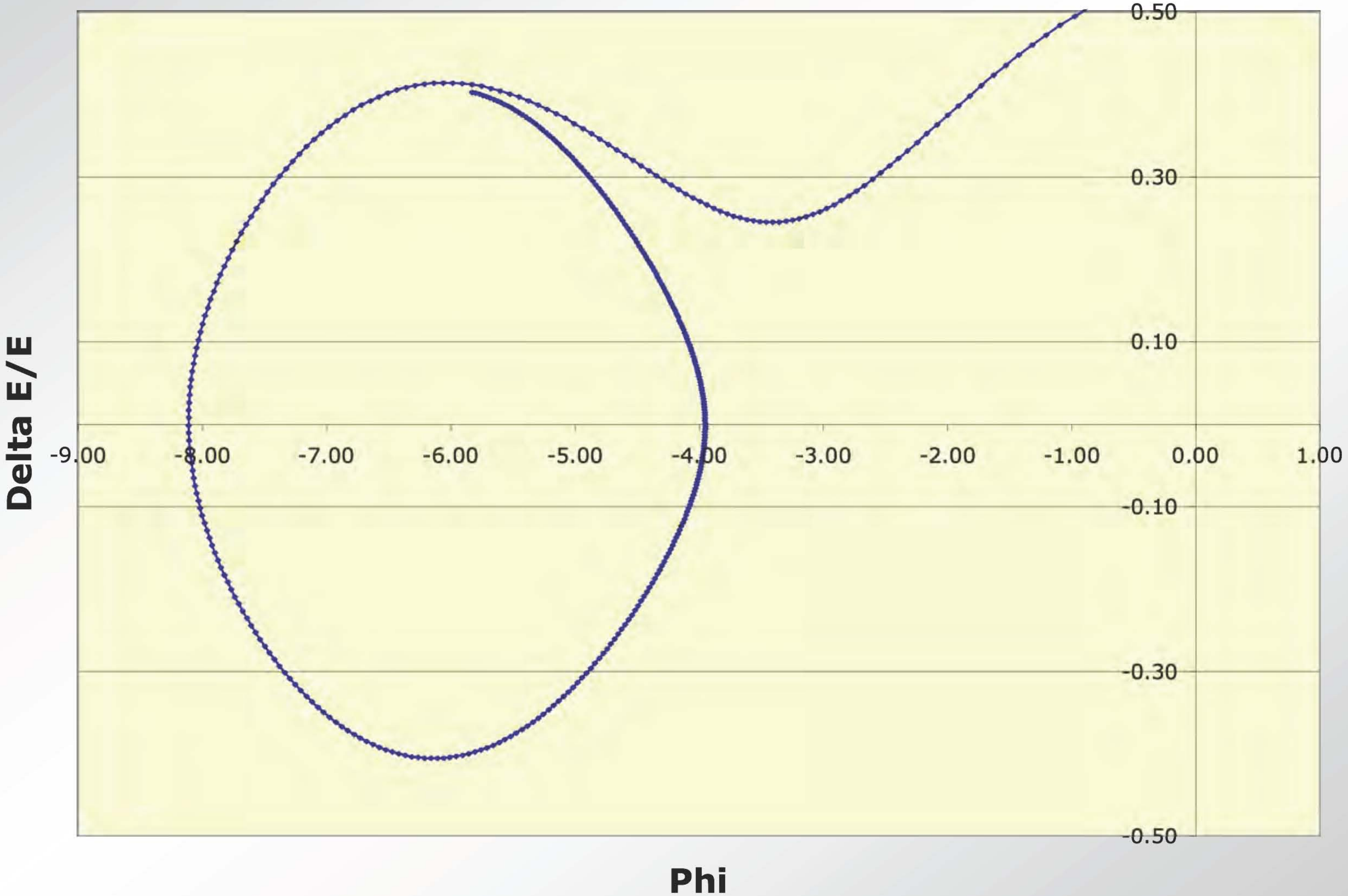
Phase stability, $\Delta E/E = 0.45$, $\phi_n = \phi_s$



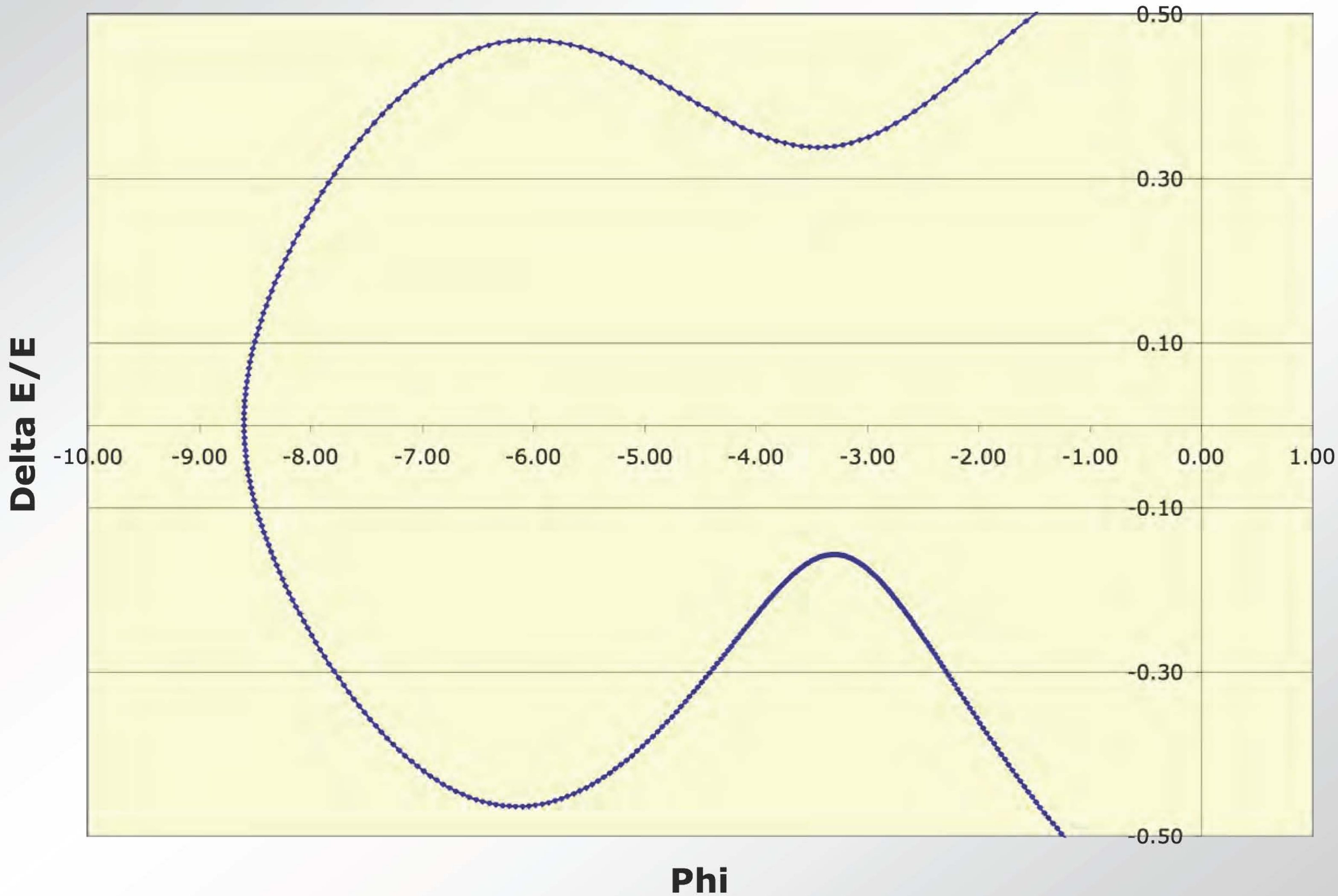
Phase stability, $\Delta E/E = 0.5$, $\phi_n = \phi_s$



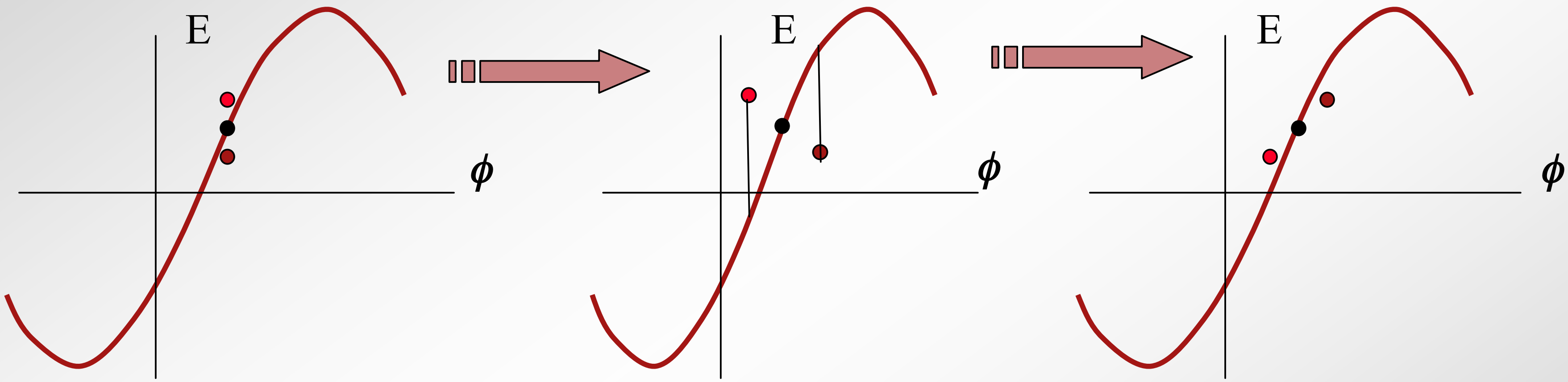
Phase stability, $\Delta E/E = 0.55$, $\phi_n = \phi_s$



Phase stability, $\Delta E/E = 0.6$, $\phi_n = \phi_s$



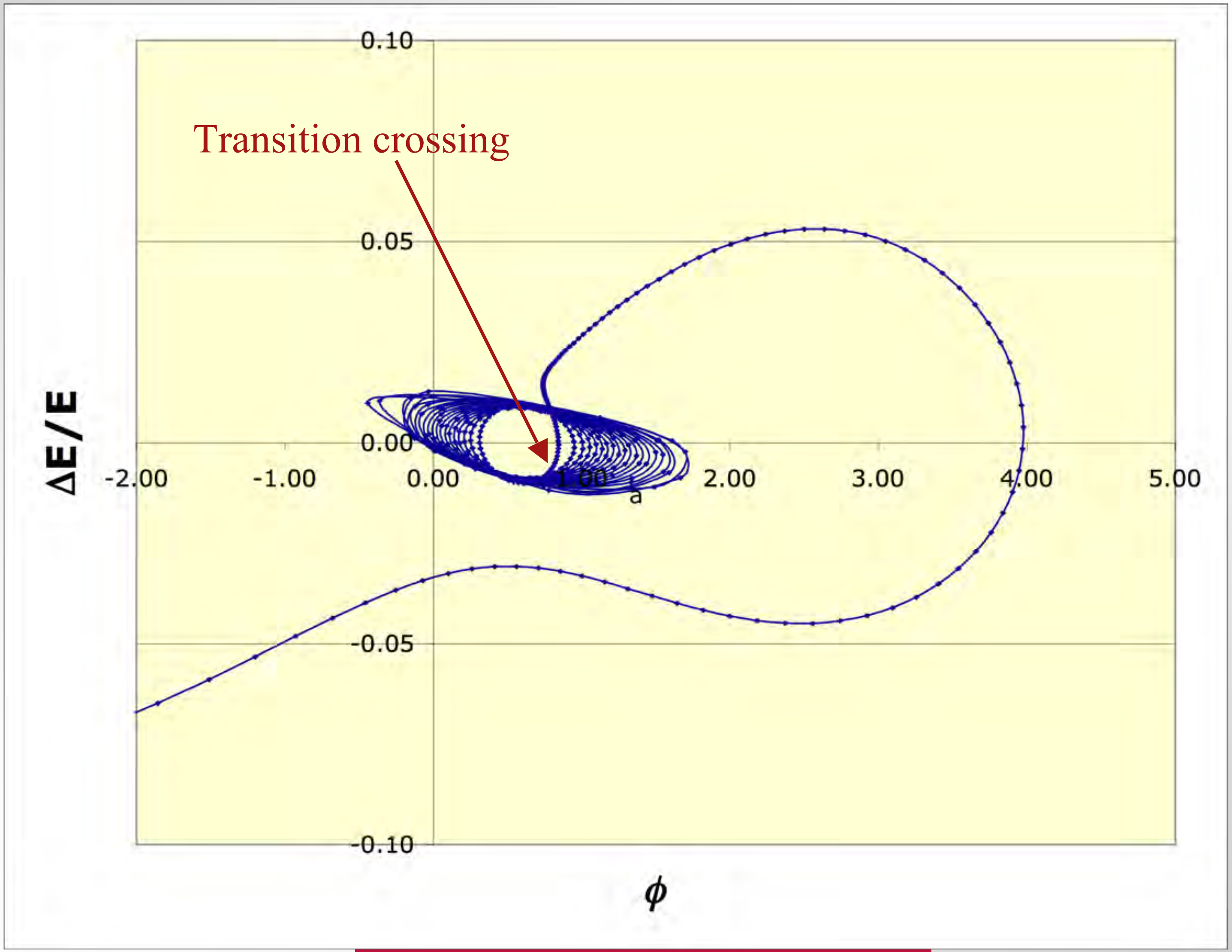
Physical picture of phase stability



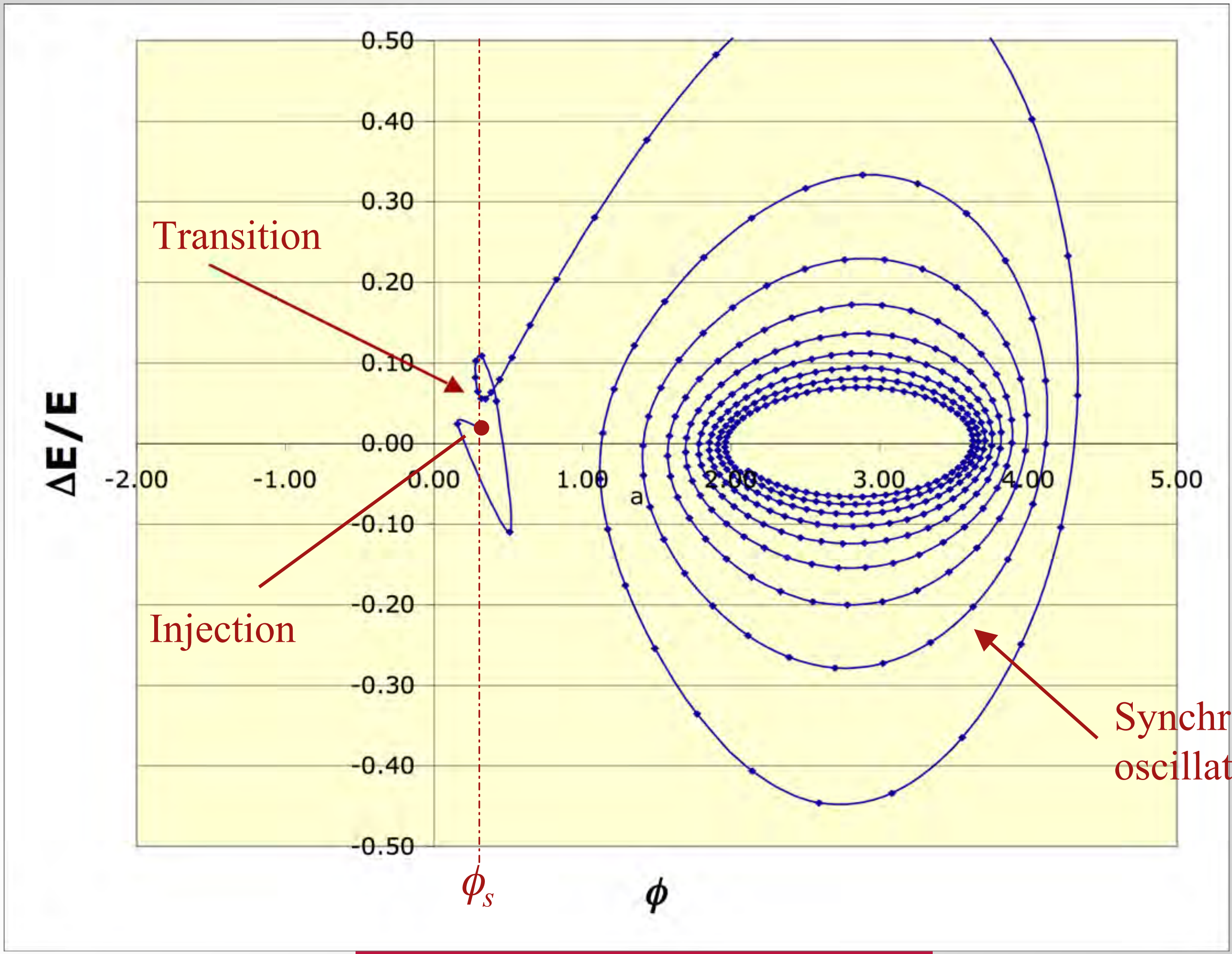
Here we've picked the case in which we are above the transition energy

(typically the case for electrons)

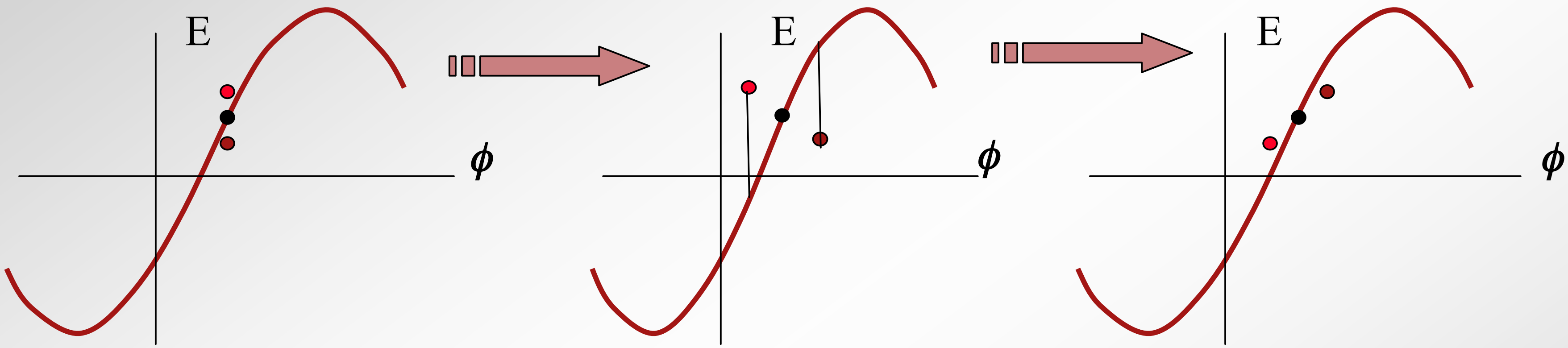
Consider this case for a proton accelerator



Case of favorable transition crossing in an electron ring



Frequency of synchrotron oscillations



- Phase-energy oscillations mix particles longitudinally within the beam
- What is the time scale over which this mixing takes place?
- If ΔE and ϕ change slowly, approximate difference equations by differential equations with n as independent variable

Two first order equations ==> one second order equation

$$\frac{d\varphi}{dn} = \frac{\eta \omega_{rf} \tau}{\beta^2 E_s} \Delta E$$

and

$$\frac{d\Delta E}{dn} = eV(\sin\varphi - \sin\varphi_s)$$

yield

$$\frac{d^2\varphi}{dn^2} = \frac{\eta \omega_{rf} \tau}{\beta^2 E_s} eV(\sin\varphi - \sin\varphi_s)$$

(Pendulum equation)

if

V = constant and $\frac{dE_s}{dn}$ is sufficiently small

Multiply by dϕ/dn & integrate

$$\int \frac{d^2\varphi}{dn^2}\frac{d\varphi}{dn}dn = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV \int \frac{d\varphi}{dn}(\sin\varphi - \sin\varphi_s)dn$$

==>

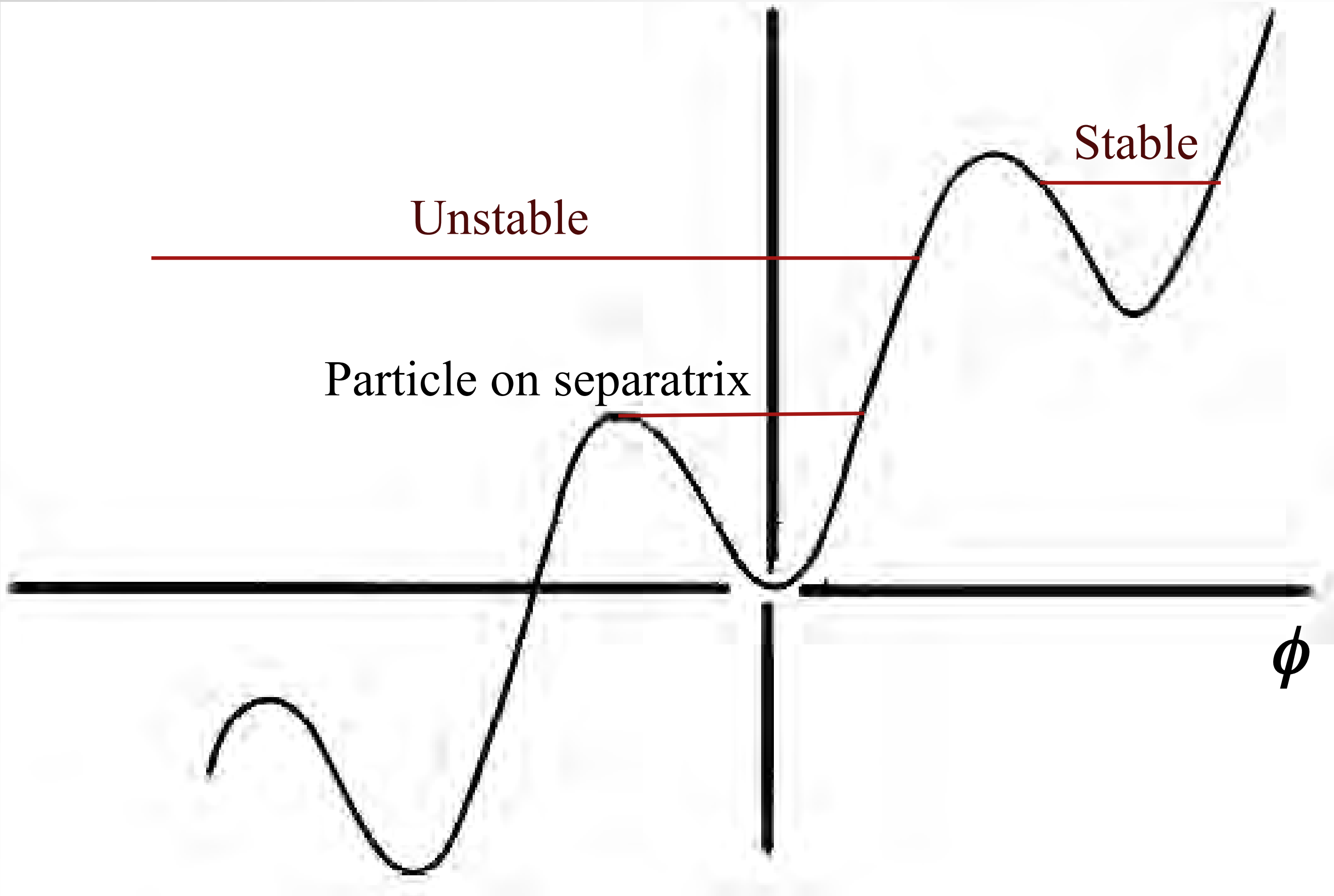
$$\frac{1}{2}\left(\frac{d\varphi}{dn}\right)^2 = -\frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\cos\varphi - \sin\varphi_s) + const$$

Rearranging

$$\frac{1}{2}\left(\frac{d\varphi}{dn}\right)^2 + \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\cos\varphi - \sin\varphi_s) = const$$

"K.E." + "P.E" = Total

“Energy” diagram for $\cos\phi + \phi \sin\phi_s$



Stable contours in phase space

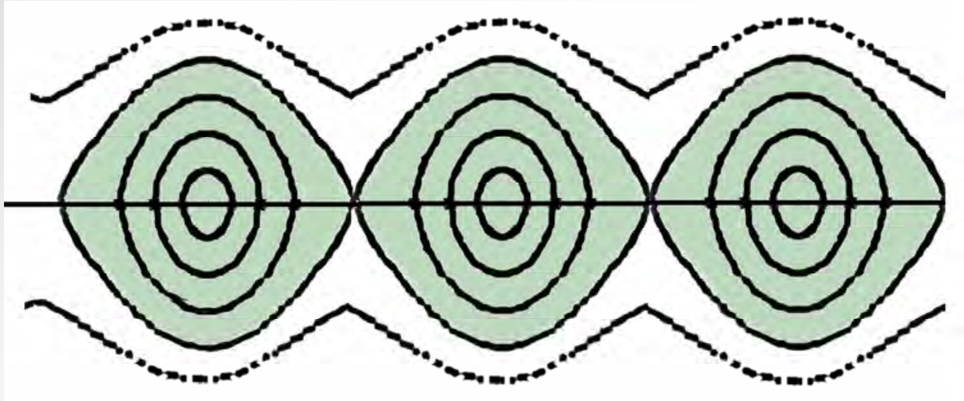
Insert $$\frac{d\varphi}{dn} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s}\Delta E$$

into $$\frac{1}{2}\left(\frac{d\varphi}{dn}\right)^2 + \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\cos\varphi - \sin\varphi_s) = const$$

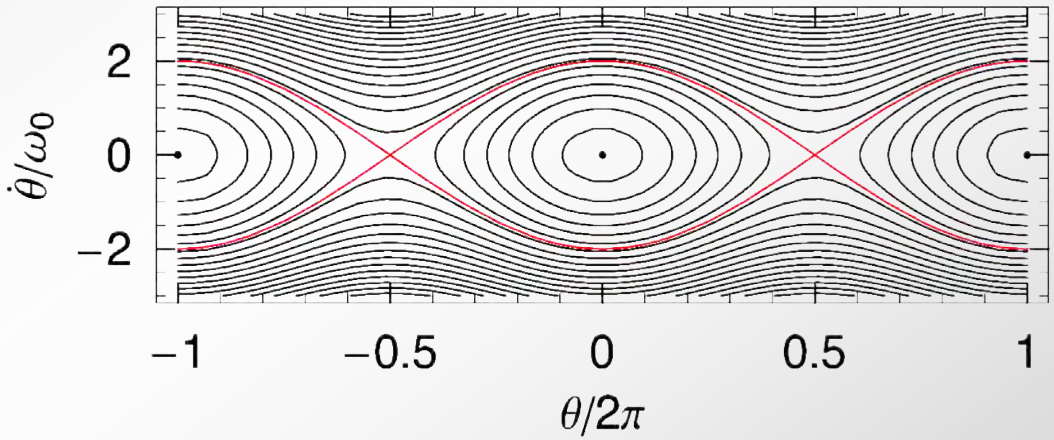
$$(\Delta E)^2 + 2eV\frac{\beta^2 E_s}{\eta\omega_{rf}\tau}(\cos\varphi - \sin\varphi_s) = const$$

for all parameters held constant

For $\phi_\sigma = 0$ we have



We've seen this behavior for the pendulum



Now let's return to the question of frequency

For *small* phase differences, $\Delta\phi = \phi - \phi_s$, we can linearize our equations

$$\frac{d^2\varphi}{dn^2} = \frac{d^2\Delta\varphi}{dn^2} = \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV(\sin\varphi - \sin\varphi_s)$$

$$= \frac{\eta\omega_{rf}\tau}{\beta^2 E_s} eV\left(\sin(\varphi_s + \Delta\varphi) - \sin\varphi_s\right)$$

$$\approx 4\pi^2 \left(\frac{\eta\omega_{rf}\tau}{4\pi^2\beta^2 E_s} eV\cos\varphi_s \right) \Delta\varphi$$

(harmonic oscillator in $\Delta\phi$)

The bracketed term equals $-\nu_s^2$ — *Synchrotron tune*

$$\Omega_s = \frac{2\pi\nu_s}{\tau} = \sqrt{-\frac{\eta\omega_{rf}}{\tau\beta^2 E_s} eV\cos\varphi_s}$$ = synchrotron angular frequency

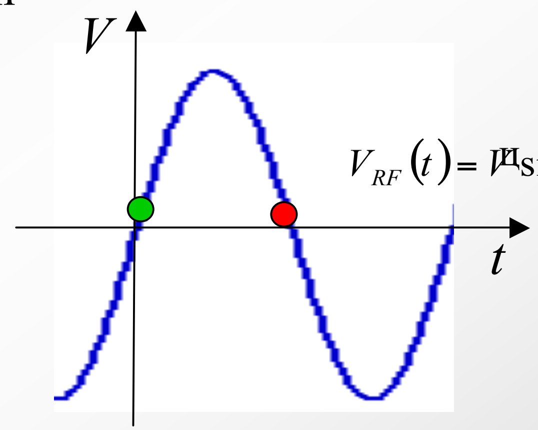
Choice of stable phase depends on η

$$\Omega_s = \sqrt{-\frac{\eta \omega_{rf}}{\tau \beta^2 E_s} eV \cos\varphi_s}$$

- Below transition ($\gamma < \gamma_t$),
  - $\eta < 0$, therefore $\cos \phi_s$ must be > 0
- Above transition ($\gamma > \gamma_t$),
  - $\eta > 0$, therefore $\cos \phi_s$ must be < 0
- At transition $\Omega_s = 0$; there is no phase stability
- Circular accelerators that must cross transition shift the synchronous phase at $\gamma > \gamma_t$
- Linacs have no path length difference, $\eta = 1/\gamma^2$; particles stay locked in phase and $\Omega_s = 0$

Two synchronous phases:
one stable, one unstable

$$\sin\varphi_S = \frac{U_0}{q\hat{V}}$$ where U_o is the desired energy gain/turn



For negative charge particles all the phases are shifted by π.

But

$$\frac{\Delta\tau}{\tau} = \frac{\Delta s}{L} = \alpha\frac{\Delta p}{p}$$

For particles with positive charge:

$$\textit{For } \alpha > 0 \quad \Rightarrow \quad \varphi_S^1 \ \textit{stable}, \quad \varphi_S^2 \ \textit{unstable}$$

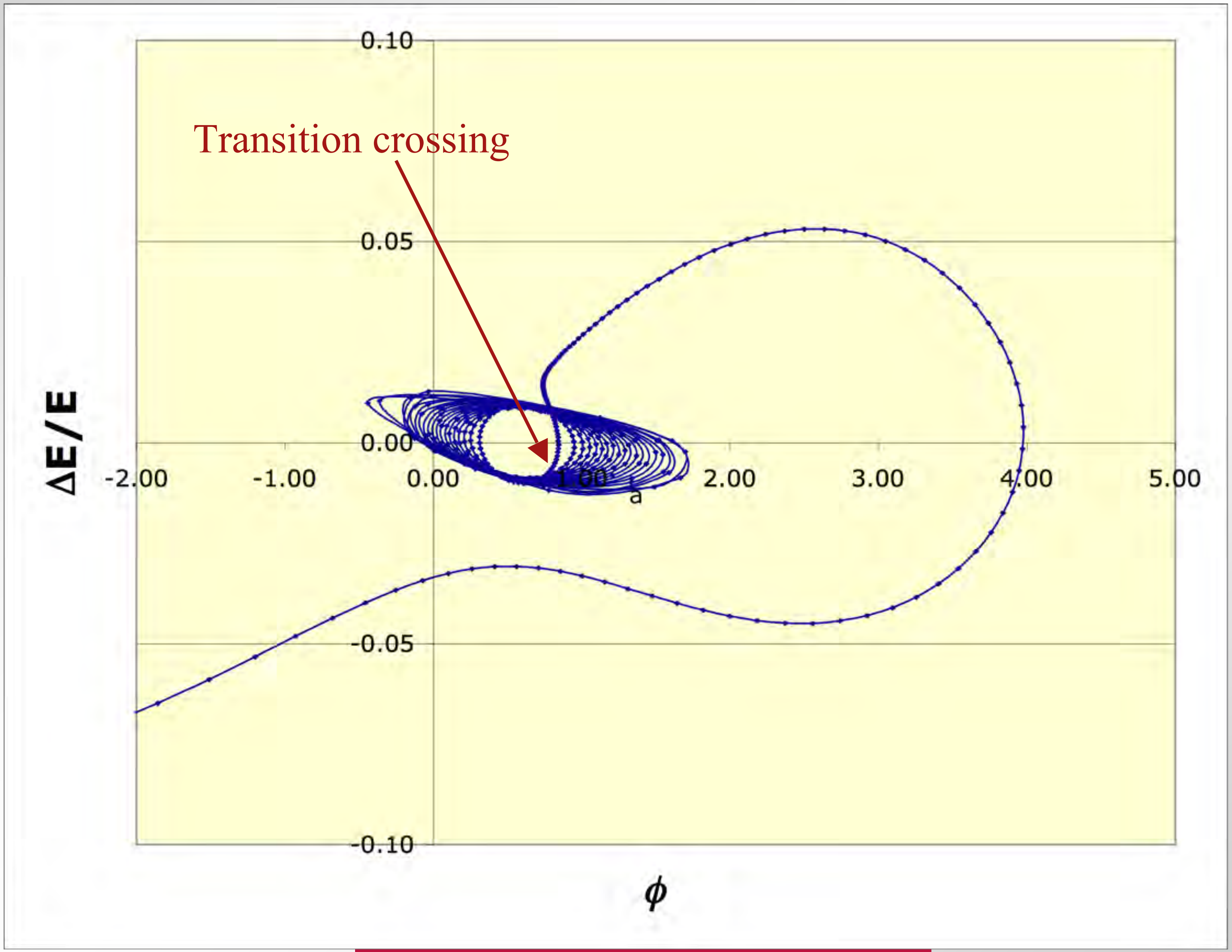
$$\textit{For } \alpha < 0 \quad \Rightarrow \quad \varphi_S^1 \ \textit{unstable}, \quad \varphi_S^2 \ \textit{stable}$$

Transition = energy at which α changes sign

Crossing transition during energy ramping ==> phase jump of ~ π

From: D. Robin USPAS lectures

Consider this case for a proton accelerator



Longitudinal phase space

- Absent a (synchro-betatron) coupling between the transverse & longitudinal motion, longitudinal phase area of a beam is conserved

- If the longitudinal coordinates are canonical conjugates, the area is invariant even under acceleration
  - Example: E & t

- For $\Delta\phi$ and E, the product of amplitudes (^) varies as 1/τ

- The area of a phase space ellipse will be

$$\pi \Delta\hat{\phi}\Delta\hat{E} = \pi \frac{AB}{\tau}$$

Using the canonical pair, E & Δt, we have

Using $\Delta\hat{\phi} = \omega_{rf}\Delta t$

$$\Rightarrow \quad \pi\Delta\hat{t}\Delta\hat{E} = \frac{\pi AB}{\omega_{rf}\tau} = \text{constant}$$

- The area in phase space that contains the particles is called the longitudinal emittance
  - Should be smaller than the bucket area, $\mathcal{A}$
    - Maximum for $\phi = 0°$ or $180°$

$$\mathcal{A}_{max} = \frac{16(v/c)}{\omega_{rf}}\sqrt{\frac{eV \cdot E_s}{2\pi h\eta}}$$

This equation for small phase oscillations represents an harmonic oscillator

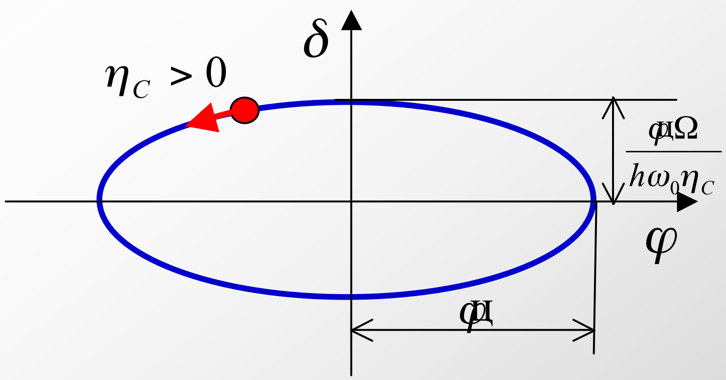
- Therefore the phase varies as

$$\varphi = \hat{\varphi}\cos(\Omega t + \psi)$$

- As we saw in the simulations the energy variation, δ = ΔE/E also varies

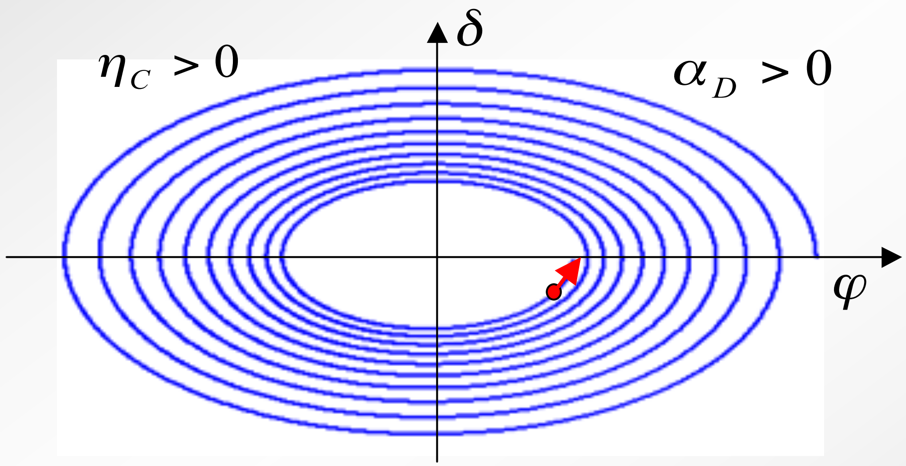
$$\delta = \frac{\hat{\varphi}\Omega}{h\omega_0\eta_C}\sin(\Omega t + \psi)$$

- ==> particle trace an ellipse in longitudinal phase space



Acceleration damps the (δ,ϕ) phase motion

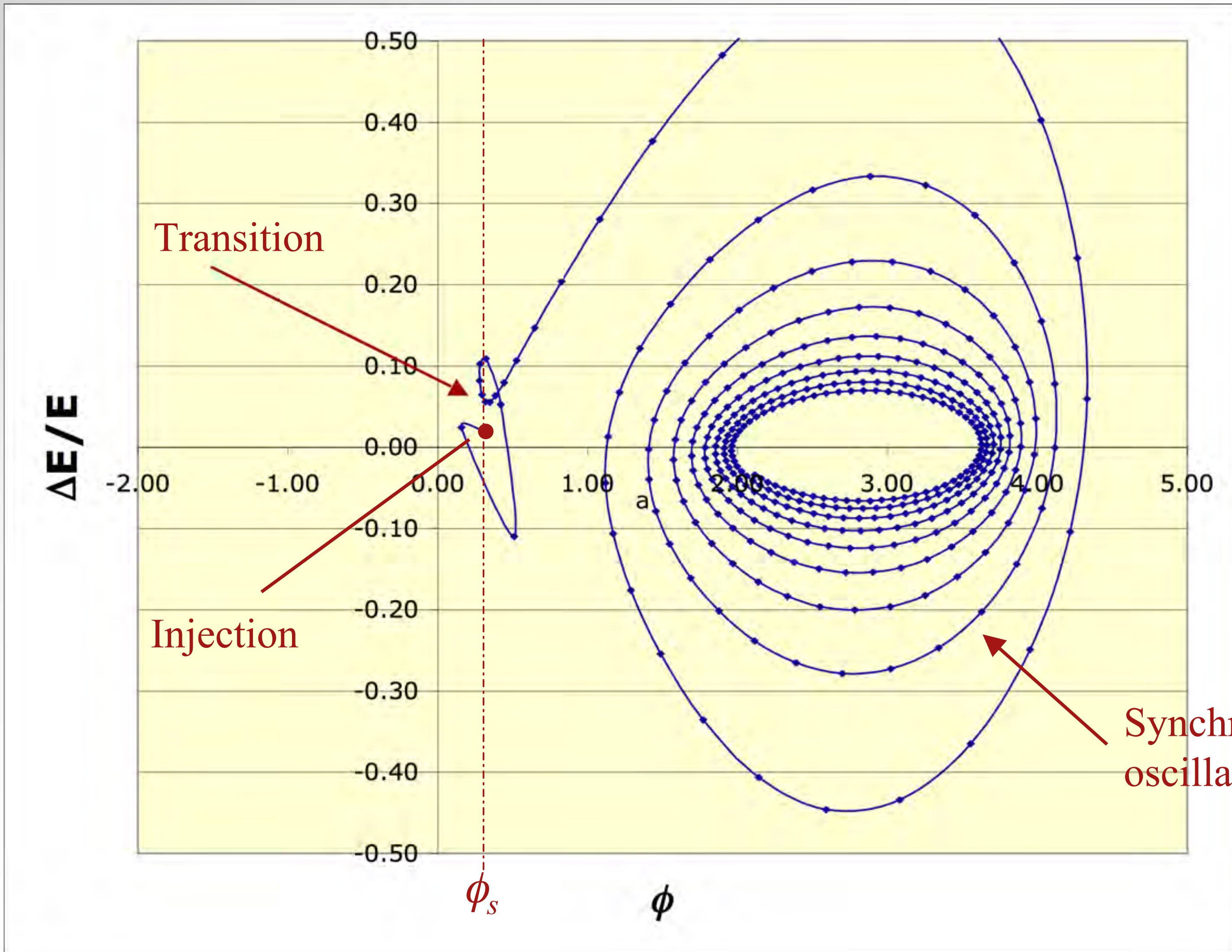
- With adiabatic damping:



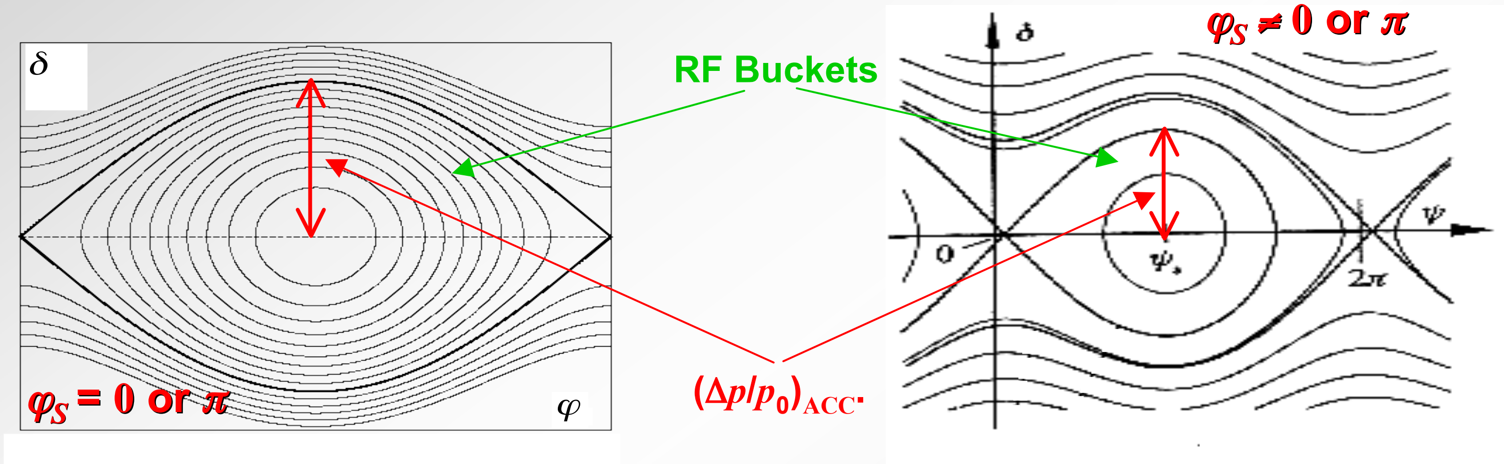
$$\varphi = \hat{\varphi} e^{-\alpha_D t} \cos(\Omega t + \psi) \qquad \delta = \frac{\hat{\varphi}\Omega}{h\omega_0 \eta_C} e^{-\alpha_D t} \sin(\Omega t + \psi)$$

In rings with negligible synchrotron radiation (or with negligible non-Hamiltonian forces, the invariant longitudinal emittance is conserved.

Example of adiabatic phase damping



Momentum acceptance: maximum momentum of any particle on a stable orbit



$$\left(\frac{\Delta p}{p_0}\right)^2_{ACC} = \frac{2|q|\hat{V}}{\pi\, h |\eta_C| \beta\, c\, p_0}$$

$$\left(\frac{\Delta p}{p_0}\right)^2_{ACC} = \frac{F(Q)}{2Q}\,\frac{2|q|\hat{V}}{\pi\, h |\eta_C| \beta\, c\, p_0}$$

$$F(Q) = 2\left(\sqrt{Q^2 - 1} - \arccos\frac{1}{Q}\right)$$

$$Q = \frac{1}{\sin\varphi_s} = \frac{q\hat{V}}{U_0}$$

Over voltage factor

Bunch length

- In electron storage rings, statistical emission of synchrotron radiation photons generates gaussian bunches

- The over voltage Q is usually large
  - Bunch "lives" in the small oscillation region of the bucket.
  - Motion in the phase space is elliptical

$$\frac{\varphi^2}{\hat{\varphi}^2} + \delta^2 \left(\frac{h\omega_0 \eta_C}{\hat{\varphi}\Omega} \right)^2 = 1 \quad \Rightarrow \quad \hat{\varphi} = \frac{h\omega_0 \eta_C}{\Omega} \hat{\delta} \Rightarrow \Delta s = \frac{c\eta_C}{\Omega} \frac{\Delta p}{p_0}$$

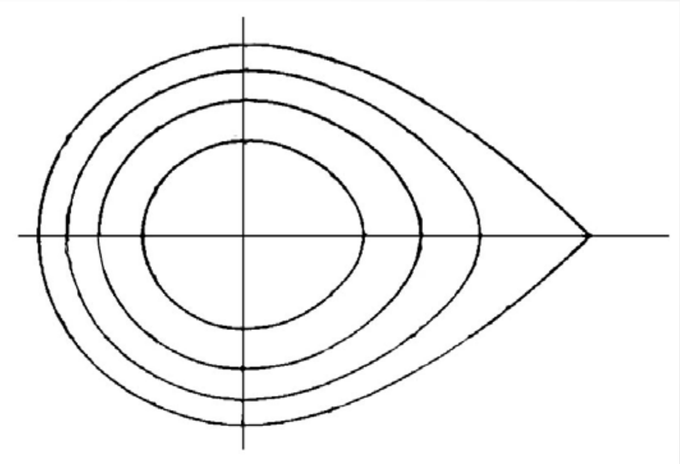
- For σ_p/p_0 = *rms relative momentum spread*, the *rms bunch length* is

$$\sigma_{\Delta S} = \frac{c\eta_C}{\Omega} \frac{\sigma_p}{p_0} = \sqrt{\frac{c^3}{2\pi q} \frac{p_0 \beta_0 \eta_C}{h f_0^2 \hat{V} \cos(\varphi_S)}} \frac{\sigma_p}{p_0}$$

How can particles be lost

- Scattering out of the rf-bucket
  - Particles scatter off the collective field of the beam
  - Large angle particle-particle scattering
- RF-voltage too low for radiation losses

$$\Delta E_{Total} = qV + U(E)$$



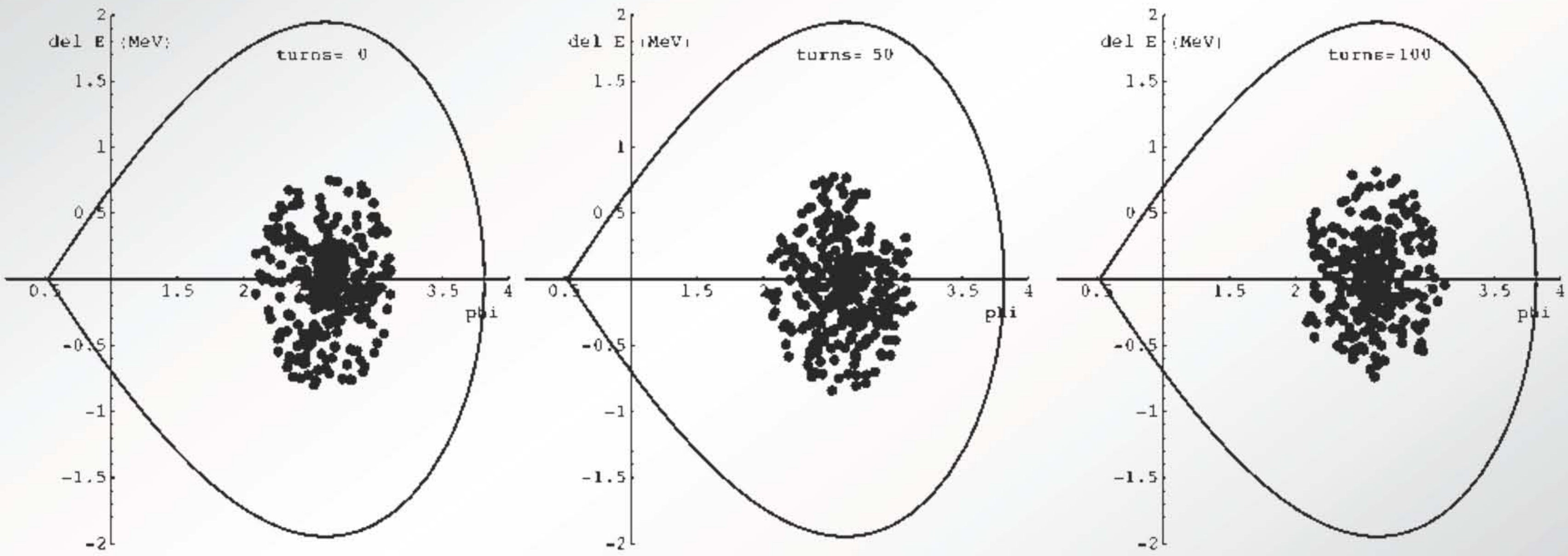
Matching the beam on injection

- Beam injection from another rf-accelerator is typically "bucket-to-bucket"
  - rf systems of machines are phase-locked
  - bunches are transferred directly from the buckets of one machine into the buckets of the other
- This process is efficient for matched beams
  - Injected beam hits the middle of the receiving rf-bucket
  - Two machines are longitudinally matched.
    - They have the same aspect ratio of the longitudinal phase ellipse

Adapted from Dugan: USPAS lectures - Lecture 11

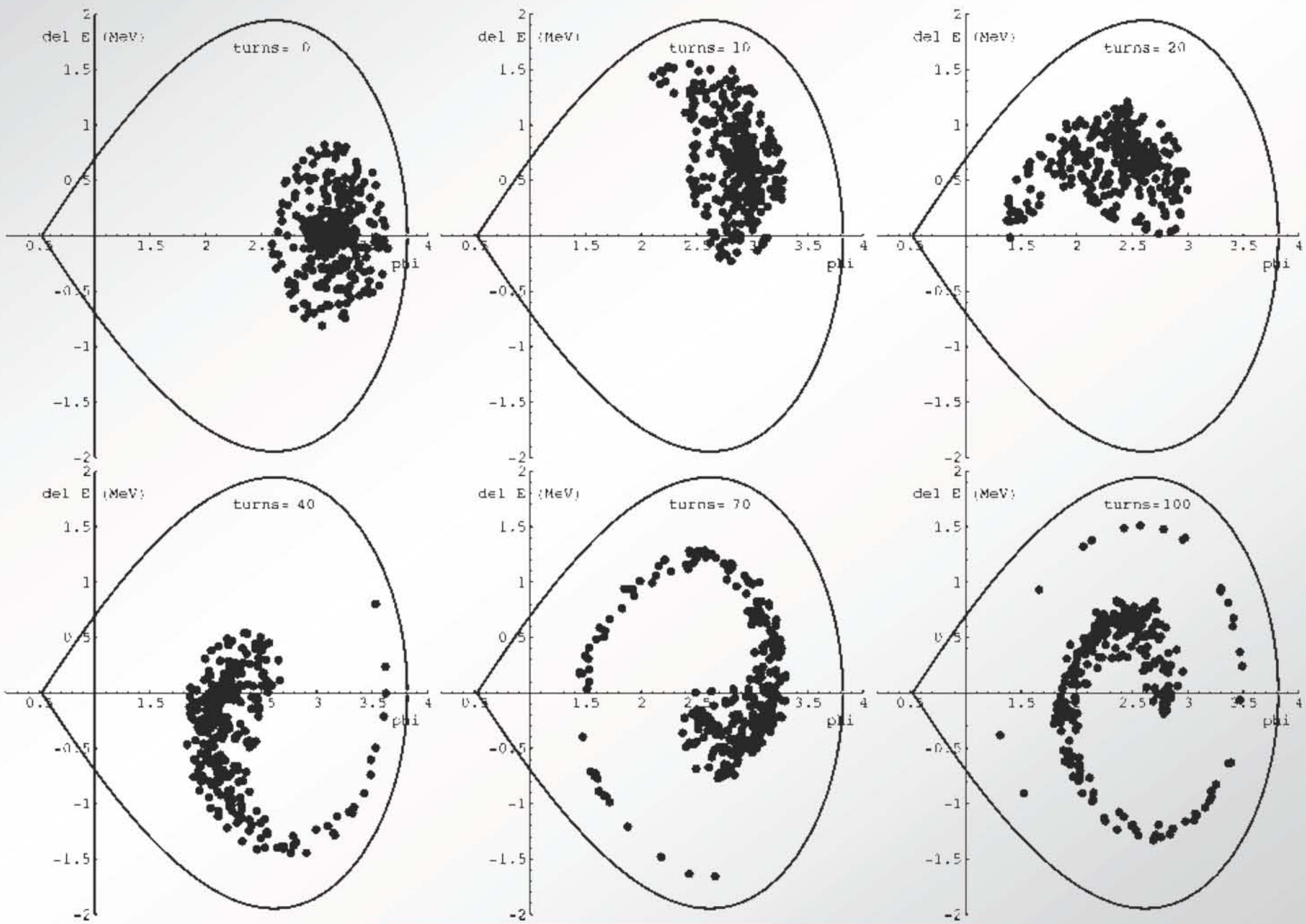
Dugan simulations of CESR injection

Example: a matched transfer, first hundred turns



From Dugan: USPAS lectures - Lecture 11

Example of mismatched CESR transfer:
phase error 60°



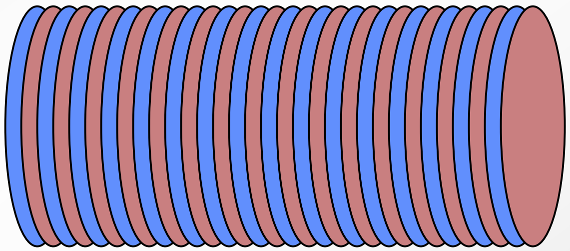
From Dugan: USPAS lectures - Lecture 11

General Envelope Equation for Cylindrically Symmetric Beams

Can be generalized for sheet beams and beams with quadrupole focusing

Assumptions for the derivation

Divide beam into disks



- Rays are paraxial ($v_\perp/c << 1$)
- Axisymmetry
- No mass spread with a disk
- Small angle scattering
- Uniform B_z
- Disks do not overtake disks

Particle equations

$$\dot{\mathbf{p}} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) + \delta\mathbf{F}_{scat}$$

$$\mathbf{p} = \gamma m \mathbf{v}$$

$$So, \quad \frac{\mathrm{d}}{\mathrm{dt}}(\gamma m \mathbf{v}) - q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = \delta\mathbf{F}_{scat} \qquad \text{(EoM)}$$

$$Define \quad \mathrm{w} = \gamma m c^2$$

- Paraxial implies

$$\mathrm{v}_{\perp}/\mathrm{c} << 1$$

and

$$I_{beam} << I_{Alfven} = \gamma\beta \frac{ec}{r_e} = 17{,}000\ \gamma\beta \text{ Amps}$$

Next write the particle equation of motion

- Define the cyclotron frequency & the betatron frequency

$$\omega_c = \frac{qB_z}{\gamma m} \quad \text{and} \quad \omega_\beta = \frac{\beta c B_\vartheta - E_r}{r}$$

- By Maxwell's equations

$$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}$$

$$E_\vartheta = -\frac{r}{2}\frac{\partial B_z}{\partial t}$$

$$\frac{dB_z}{dt} \equiv \dot{B} = \frac{\partial B_z}{\partial t} + \beta c \frac{\partial B_z}{\partial z}$$

- The EoM for a beam particle is

$$\frac{\dot{\gamma}}{\gamma}\mathbf{v} + \dot{\mathbf{v}} + \omega_\beta^2 \mathbf{r} + \omega_\mathbf{c} \hat{\mathbf{z}} \times \mathbf{v} + \frac{1}{2\gamma}\frac{d}{dt}(\gamma \omega_c)\hat{\mathbf{z}} \times \mathbf{r} = \frac{1}{\gamma m}\delta \mathbf{F}_{scat}$$

Take moments of the EoM

- Three moment equations:
  1. **v** • EoM = Energy equation
  2. **r** • EoM = Virial equation
  3. **r** × EoM = Angular momentum equation

- Next take rms averages of the moment equations
  - Yields equations in R, V, L and their derivatives

- Ansatz: The radial motions of the beam are self similar
  - The functional shape of J(r) stays fixes as R changes

Last steps

- Angular momentum conservation implies

$$P_\vartheta = \gamma L + \gamma \omega_c \frac{R^2}{c} = \text{ constant}$$

- The energy & virial equations combine to yield

$$\ddot{R} + \frac{\dot{\gamma}}{\gamma}\dot{R} + \frac{U}{R} + \frac{\omega_c^2 R}{4} - \frac{\mathcal{E}^2}{\gamma^2 R^3} = \frac{1}{\gamma^2 R^3}\int_{t_o}^{t} dt' \left(\frac{2\gamma R^2}{m} \varepsilon' \right)$$

where

$$U = \left\langle \omega_{\beta,self}^2 r^2 \right\rangle = I \big/ I_{Alfven}$$

What is I_{alfven}?

and

$$\mathcal{E}^2 = \gamma^2 R^2 \left(V^2 - (\dot{R})^2 \right) + P_\vartheta^2$$

Without scattering & in equilibrium

$$\underbrace{\ddot{R}}_{=0} + \underbrace{\frac{\dot{\gamma}}{\gamma}\dot{R}}_{=0} + \frac{U}{R} + \frac{\omega_c^2 R}{4} - \frac{\mathcal{E}^2}{\gamma^2 R^3} = \frac{1}{\gamma^2 R^3}\int_{t_o}^{t} dt' \underbrace{\left(\frac{2\gamma R^2}{m}\varepsilon'\right)}_{=0}$$

$$\therefore \quad \frac{U}{R} + \frac{\frac{1}{4}\omega_c^2 R^2}{R} - \frac{\mathcal{E}^2}{\gamma^2 R^3} = 0$$

Self-forces Focusing Emittance

$$\text{More generally,} \quad \frac{U}{R} + \frac{\langle \omega_\beta^2 R^2 \rangle}{R} - \frac{\mathcal{E}^2}{\gamma^2 R^3} = 0$$