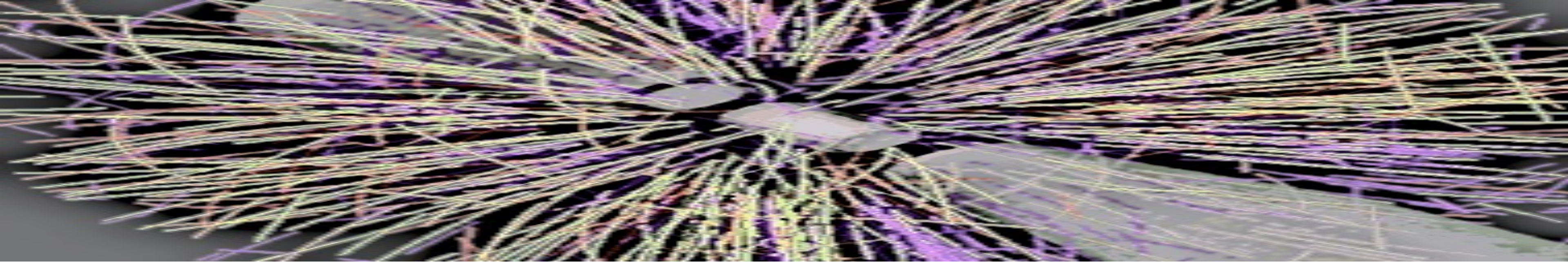
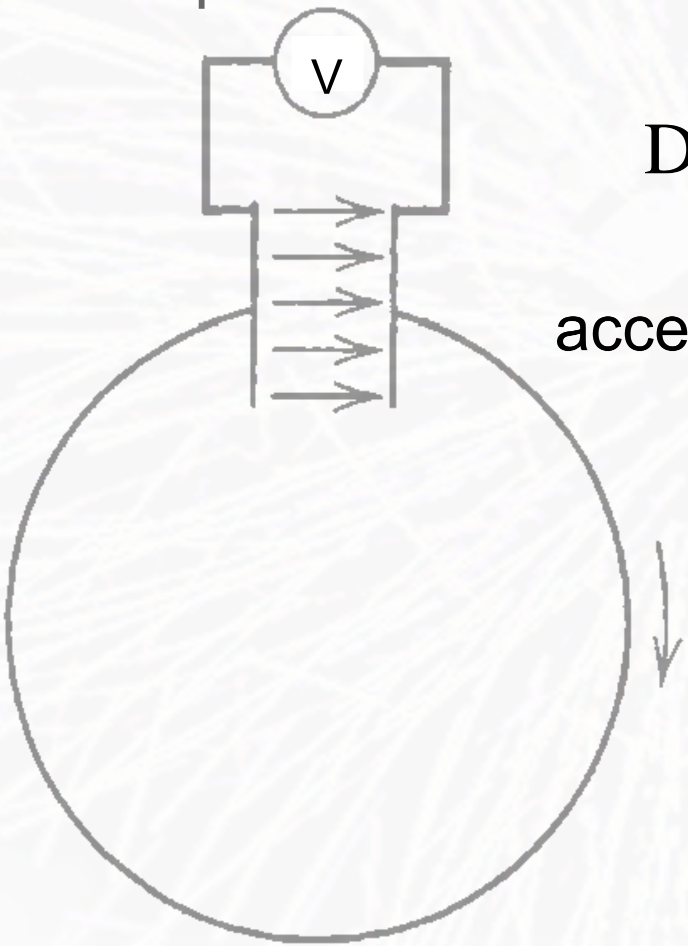


Accelerator Fundermentals
Introduction to RF Cavity

- RF cavity
- circuit model
- impedance
- Pill Box

Acceleration Structure

- Can one accelerate beams with DC electric field?
  - Not possible with circular machine



DC field means $\oint E \bullet dl = 0$

accelerating beam means

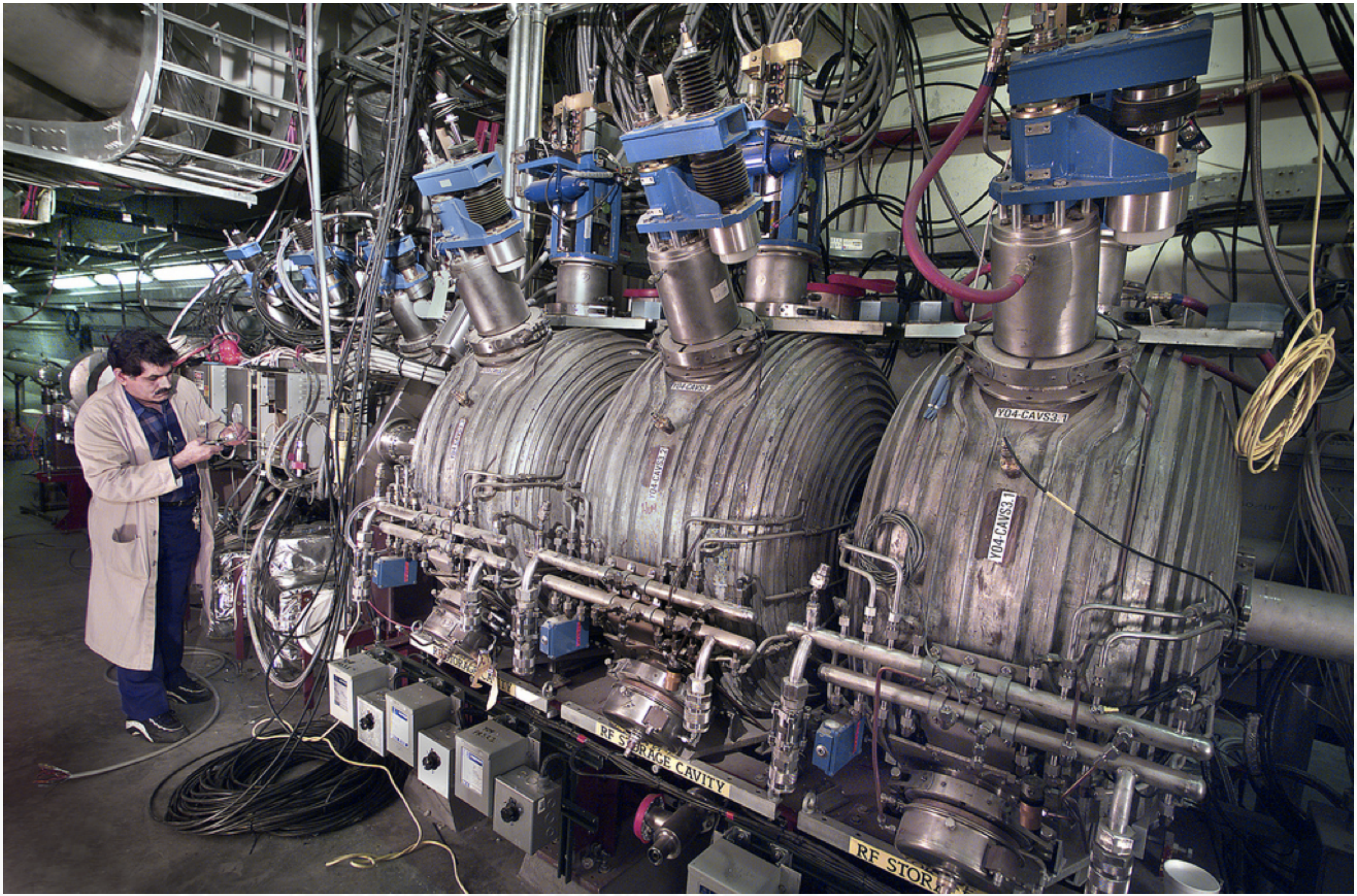
$$\frac{\partial \phi_B}{\partial t} \neq 0$$

This is contradictory to the above condition

RF Cavity

- A metallic chamber that contains electro-magnetic field in the radio frequency region of the spectrum, which can be tuned in a way to boost particle along its velocity, i.e. acceleration
- In a storage ring, an RF cavity is also used to provide longitudinal focusing mechanism to keep the beam stay bunched

RHIC RF Cavities



Electromagnetic wave propagation

- propagation of an electromagnetic wave in vacuum

$$\begin{cases}(\nabla^2 - \mu\varepsilon \dfrac{\partial^2}{\partial t^2})\vec{E} = 0 \\ (\nabla^2 - \mu\varepsilon \dfrac{\partial^2}{\partial t^2})\vec{B} = 0\end{cases} \Rightarrow \begin{cases}\vec{E}(\vec{r}, z, t) = \vec{E}(\vec{r})e^{i\omega t - ik_z z} \\ \vec{B}(\vec{r}, z, t) = \vec{B}(\vec{r})e^{i\omega t - ik_z z}\end{cases}$$

ω : angular frequency

$k_z = \left|\vec{k}_z\right| = \dfrac{2\pi}{\lambda}$: wave number, λ is the wavelength

$$\begin{cases}\nabla_t^2 \vec{E} + (\omega^2 \mu\varepsilon - k_z^2)\vec{E} = 0 \\ \nabla_t^2 \vec{B} + (\omega^2 \mu\varepsilon - k_z^2)\vec{B} = 0\end{cases}$$

$$\nabla_t^2 = \nabla^2 - \partial_z^2$$

In a waveguide

Let $\vec{\mathrm{E}} = \vec{\mathrm{E}}_z + \vec{\mathrm{E}}_t$ and $\vec{\mathrm{B}} = \vec{\mathrm{B}}_z + \vec{\mathrm{B}}_t$

$$\vec{\mathrm{E}}_z = \mathrm{E}_z \hat{e}_z \qquad \vec{B}_z = \vec{B}_z \hat{e}_z$$

$$\vec{\mathrm{E}}_t = (\hat{e}_z \times \vec{E}) \times \hat{e}_z \qquad \vec{B}_t = (\hat{e}_z \times \vec{B}) \times \hat{e}_z$$

$$\nabla \bullet \vec{\mathrm{E}} = 0 \text{ yields } \nabla_t \bullet \vec{\mathrm{E}}_t = -\frac{\partial E_z}{\partial z}$$

$$\nabla \bullet \vec{\mathrm{B}} = 0 \text{ yields } \nabla_t \bullet \vec{\mathrm{B}}_t = -\frac{\partial B_z}{\partial z}$$

The transverse component of the E and B field is known as long as the longitudinal component is defined

Cylindrical Waveguide

- Boundary condition at r = R

$$\vec{\mathrm{n}} \times \vec{\mathrm{E}} = 0, \vec{\mathrm{n}} \bullet \vec{B} = 0$$

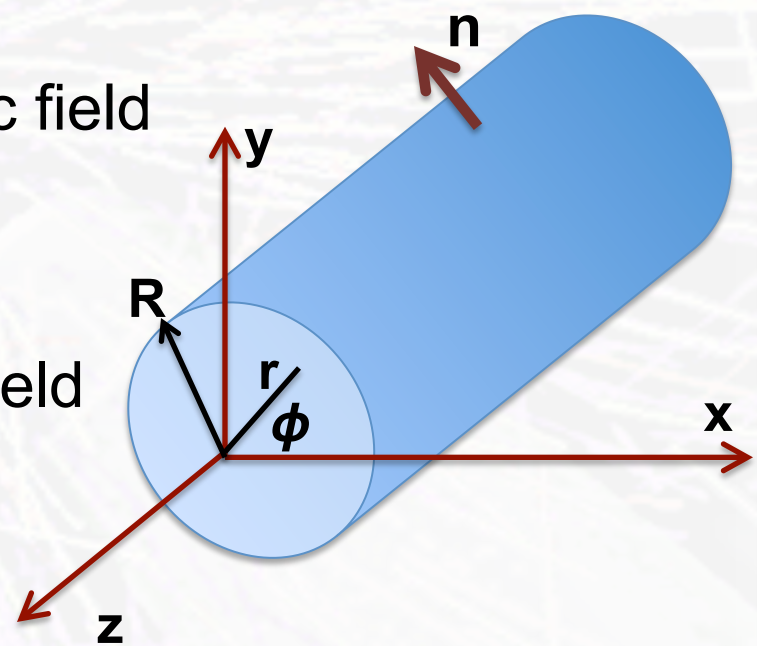
  - At the surface $\mathrm{E}_z = 0, \dfrac{\partial B_z}{\partial n} = 0$

  – TM mode: Transverse magnetic field

$$B_z = 0$$

  – TE mode: Transverse electric field

$$E_z = 0$$



Cylindrical Waveguide: TM modes

- Transverse magnetic field mode(TM modes)

$$\nabla_t^2 E_z(r,\phi) + (\omega^2 \mu\varepsilon - k_z^2) E_z(r,\phi) = 0$$

  – where, $\mathrm{E}_z = \mathrm{E}_z(r,\phi) e^{i\omega t - ik_z z}$

  – And $\nabla_t^2 = \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\phi^2$

  – Take $E_z\left(r,\phi\right) = E_{z0} f(r) g(\phi)$ back to the differential equation

$$\frac{\partial^2 f(r)}{\partial r^2} g(\phi) + \frac{1}{r}\frac{\partial f(r)}{\partial r} g(\phi) + \frac{1}{r^2}\frac{\partial^2 g(\phi)}{\partial \phi^2} f(r) + k^2 f(r) g(\phi) = 0$$

and $k^2 = \omega^2 \mu\varepsilon - k_z^2$

Cylindrical Waveguide: TM modes

$$\frac{\partial^2 f(r)}{\partial r^2} g(\phi) + \frac{1}{r}\frac{\partial f(r)}{\partial r} g(\phi) + \frac{1}{r^2}\frac{\partial^2 g(\phi)}{\partial \phi^2} f(r) + k^2 f(r) g(\phi) = 0$$

Let $\mathrm{g}(\phi) = \cos(n\phi)$

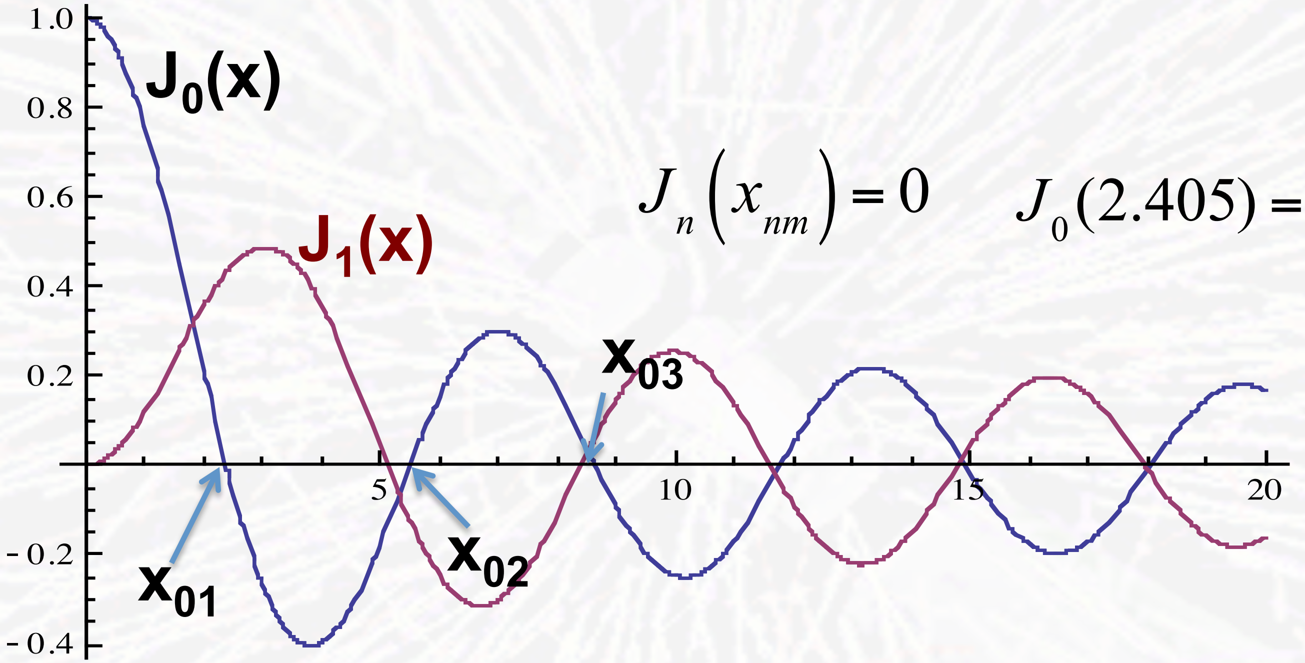
$$r^2 \frac{\partial^2 f(r)}{\partial r^2} + r\frac{\partial f(r)}{\partial r} + [(kr)^2 - n^2] f(r) = 0$$

Hence $f(r) = J_n(kr)$ and $E_z(r,\phi) = E_{z0} J_n(kr)\cos(n\phi)$

$kR = x_{nm}$ $J_n(x_{nm}) = 0$ **n^th order Bessel function**

Cylindrical Waveguide: TM modes

- Bessel functions



$$J_n\left(x_{nm}\right)=0 \qquad J_0(2.405)=0$$

$$E_z\left(r,\phi\right)=E_{z0}J_n(kr)\cos(n\phi) \Rightarrow kR=x_{nm}$$

Cylindrical Waveguide: TM_{nm} mode

$$E_z(r,\phi,z) = E_{z0}\cos(n\phi)J_n\left(x_{nm}\frac{r}{R}\right)e^{\mp jk_z z} \text{ with } k_z^2 = \omega^2\mu\varepsilon - \left(\frac{x_{nm}}{R}\right)^2$$

$$n = 0,1,2\ldots.; \quad m = 1,2,3$$

- Cut-off frequency for TM_{01} mode

$$\omega_c^2\mu\varepsilon = \left(\frac{2.405}{R}\right)^2 \qquad f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{2.405}{R}$$

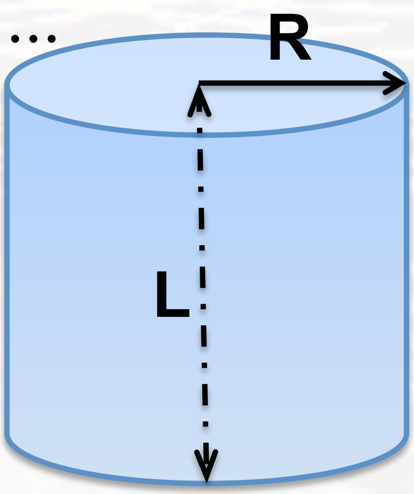
- Phase velocity of the propagating wave
  - Faster than the speed of the light, can't be used to accelerate!

$$v_p = \frac{\omega}{k_z} = \frac{1}{\sqrt{\mu\varepsilon}\sqrt{1-(\frac{\omega_c}{\omega})^2}} > c$$

Pill-Box

- A cylindrical cavity
  - Boundary condition: E field also has to be zero at two end plates. This means that
    - For TM fields, transverse component of E fields should disappear, i.e.

$$E_z = \psi(r,\phi)\cos(\frac{p\pi z}{L}), \;\; p = 0,1,2, \ldots$$



    - For TE fields, longitudinal B field should disappear at the two z=0, and z=d

$$H_z = \psi(r,\phi)\sin(\frac{p\pi z}{L}), \;\; p = 1,2,3, \ldots$$

  - A resonant structure, no wave propagation

Pill-Box Response

- TM modes

$$f_{nmp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{x_{nm}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}\,; \quad J_n\left(x_{nm}\right) = 0$$

  - Lowest TM mode: TM_{010} mode $f_{010} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{2.405}{R}$

- TE modes

$$f_{nmp} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{x'_{nm}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}\,; \quad J'_n\left(x'_{nm}\right) = 0$$

  - Lowest TE mode: TE_{111} mode $f_{111} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{3.832}{R}\right)^2 + \left(\frac{1}{L}\right)^2}$

Pill-Box for Acceleration

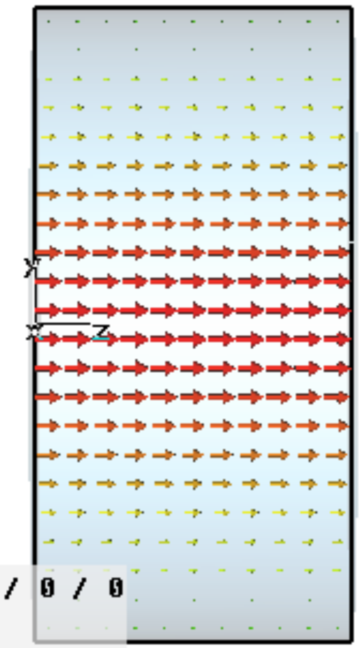
- TM_{010} mode
  - Longitudinal electric field in the center of the cavity, which can be used for acceleration
  - B field has no angular dependence. But it is the magnetic loop causes ohmic heating
  - Frequency depends only on radius, and independent on cavity length

$$f_{010} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \frac{2.405}{a}$$

  - Almost all RF cavities operate with this mode for acceleration

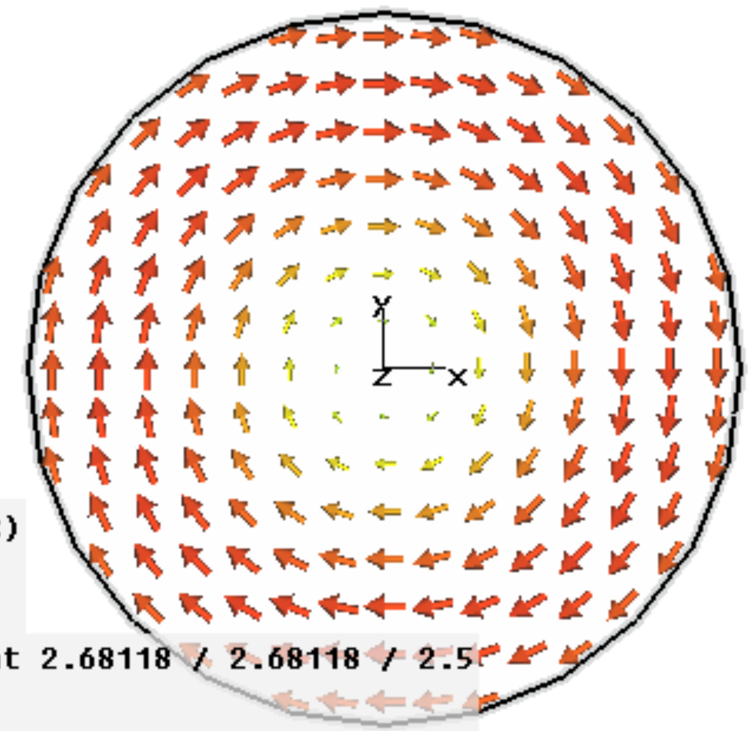
TM_{010} Accelerating Mode

Electric Fields



Type	E-Field (peak)
Monitor	Mode 1
Plane at x	0
Maximum-2d	4.61371e+007 V/m at 0 / 0 / 0
Frequency	2.29257
Phase	0 degrees

Magnetic Fields



Type	H-Field (peak)
Monitor	Mode 1
Plane at z	2.5
Maximum-2d	77383.3 A/m at 2.68118 / 2.68118 / 2.5
Frequency	2.29257
Phase	90 degrees

From G. Burt, Introduction to RF Cavities for Accelerators

Accelerating voltage

- The energy gain that a particle sees when it travels through the cavity's time varying field is

$$\Delta E = eV_c = e\left|\int_{-L/2}^{+L/2} E_z(r,\phi,z)e^{i\omega z/v}\,dz\right|;$$

- For a pill-box TM_{010} mode

$$\Delta E = eV_c = e\left|\int_{-L/2}^{+L/2} E_{z0}(r)e^{i\omega z/v}\,dz\right|; \quad \text{and} \quad V_c = E_{z0}LT$$

$$V_0 = E_{z0}L$$

Effective voltage for the same energy gain

- For a relativistic particle, to receive the maximum kick the particle should traverse the cavity in a half RF period

$$L = c/2f$$

Transit Time Factor

- Accelerating voltage can also be seen as the linear integral of electric field particle sees. Here, T is the the transit time factor and E_{z0} is the peak axial electric field

$$T = \frac{\int_{-L/2}^{+L/2} E_z(r,\phi) e^{i\omega z/v}\, dz}{E_{z0} L} = \frac{\sin(\omega L/2v)}{\omega L/2v}$$

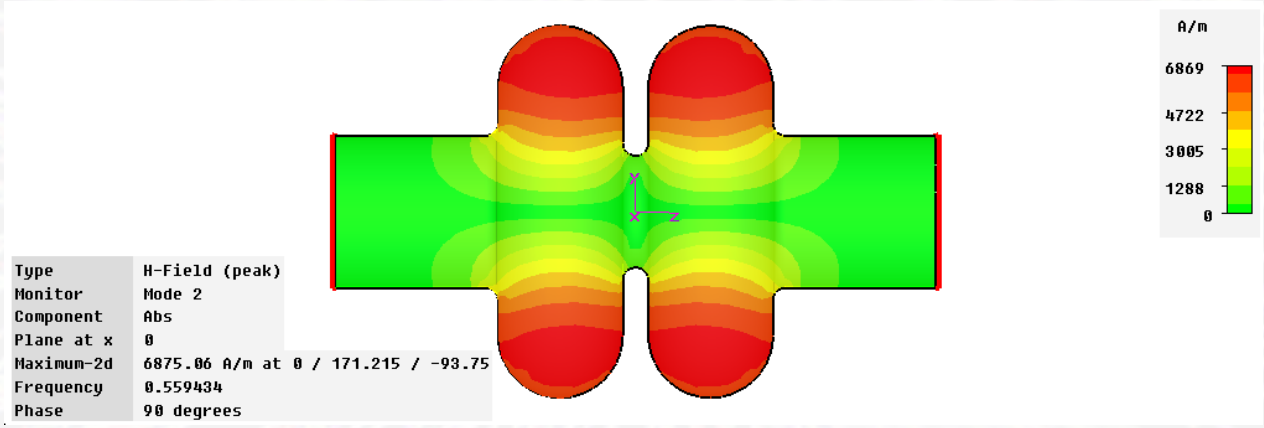
- For a given T of a fixed frequency, the length of the cavity is then definded

Power Dissipation

- The power lost in the cavity walls due to ohmic heating is

$$P_c = \frac{1}{2} R_{surface} \int |H|^2 dS$$

$R_{surface}$ is the surface resistance



- A significant amount of power is dissipated in cavity walls and hence the cavities are heated, this must be water cooled in warm cavities and cooled by liquid helium in superconducting cavities. all power lost in the cavity must be replaced by an rf source

From G. Burt, Introduction to RF Cavities for Accelerators

Power Dissipation For a Pill Box

- Surface current density

Ampere's Law: $J = \frac{1}{\mu_0} B_\theta \qquad B_\theta = \frac{E_{z0}}{c} J_1(\frac{\omega}{c} r) e^{i\omega t}$

$P_c = \frac{1}{2}\rho_s J^2; \quad \rho_s$ is the surface resistivity

$$P_c = \frac{1}{2}\rho_s \left(\frac{E_{z0}}{\mu_0 c}\right)^2 [2\times 2\pi \int_0^a J_1^2(\frac{\omega}{c} r) r\,dr + 2\pi a L J_1^2(\frac{\omega}{c} a)]$$

$$= \frac{1}{2}\rho_s \left(\frac{E_{z0}}{\mu_0 c}\right)^2 2\pi a L(1+\frac{a}{L}) J_1^2(x_{01} = 2.405)$$

Quality Factor (Q factor)

- Defined as the ratio of stored energy in one rf cycle to the dissipation power

$$Q_0 = \frac{\omega U}{P_c}$$

  - Where $U = \frac{1}{2}\mu_0 \int |H|^2 dV = \frac{1}{2}\varepsilon_0 \int |E|^2 dV$

  - And P_c is the power dissipation

- The Q factor determines the maximum energy the cavity can fill to with a given input power

Shunt Impedance

- Defined as the ratio of energy gain per unit power loss

$$R = \frac{(\text{energy gain per unit charge})^2}{\mathrm{p_c}} = \frac{|V_c|^2}{2\mathrm{p_c}}$$

  - V_c is the cavity voltage, $V_c = V_0 T$
  - also important as it is related to the power induced in the mode by the beam (important for unwanted cavity modes)

- Geometric shunt impedance

$$\frac{R}{Q} = \frac{|V_c|^2}{2\omega U}$$

  - independent of both frequency as well as cavity material

For Pill Box

- Stored energy in one rf cycle to the dissipation power

$$U = \frac{1}{2}\mu_0 \int |H|^2 dV = \frac{1}{2}\varepsilon_0 \int |E|^2 dV = \frac{1}{2}\varepsilon_0 E_{z0}^2 J_1^2 (x_{01} = 2.405)$$

- Q-factor for a pill box:

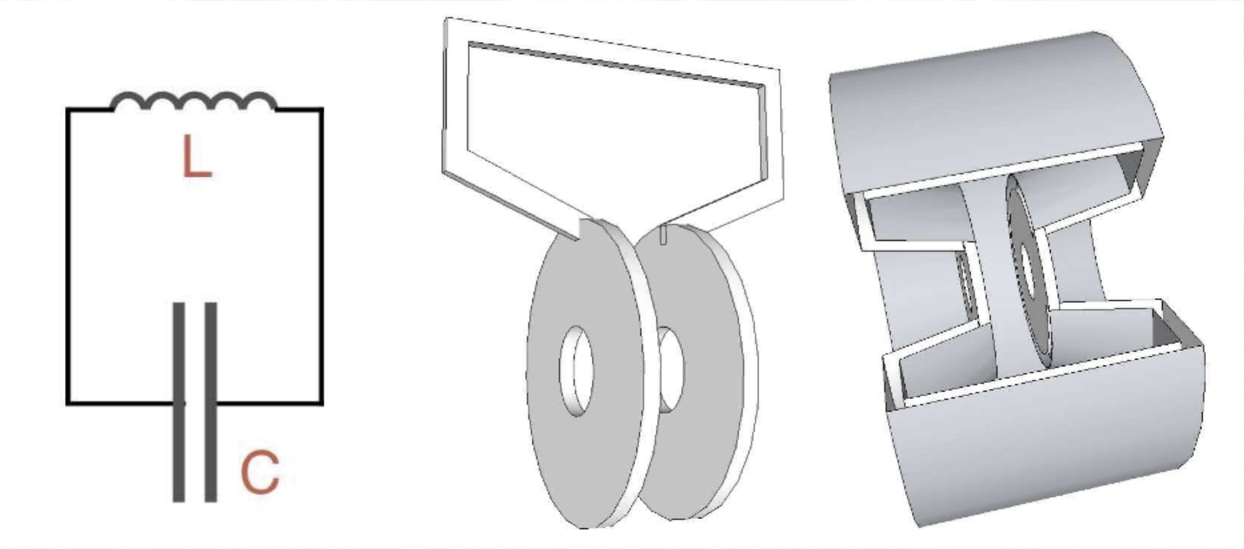
$$Q_0 = \frac{\omega U}{P_c} = \frac{2.405\mu_0 c}{2\rho_s (1 + \frac{a}{L})}$$

- Shunt impedance:

$$R = \frac{\mu_0 c}{\pi \rho_s} \frac{L}{a} \frac{T^2}{(1 + \frac{a}{L}) J_1^2 (2.405)}$$

LC Circuit towards Pill-box

- The resonance frequency of an LC circuit is $f = 1/2\pi\sqrt{LC}$. to Increase resonance frequency
  - Lower inductance, solid wall
  - Lower capacitance, cylindrical shape
  - Beam tubes in the middle for beam to pass



Pill-box ~ LC circuit

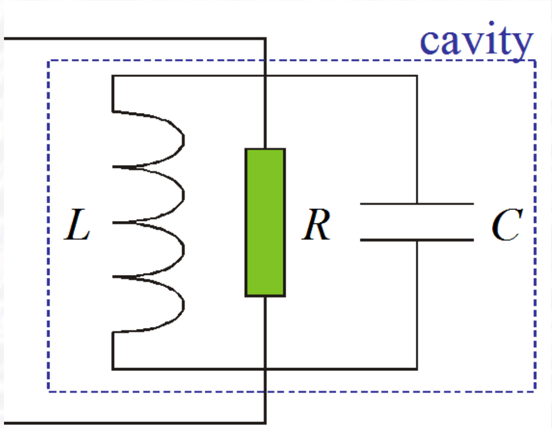
- Inductance of a pill box

$$L = \frac{\psi}{I} = \frac{\int \vec{B}\cdot d\vec{S}}{\oint \vec{H}\cdot d\vec{l}}$$

- Capacitance of a pill box

$$C = \frac{Q}{V} = \frac{\int \vec{D}\cdot d\vec{S}}{\oint \vec{E}\cdot d\vec{l}}$$

- LC circuit



$$P_c = \frac{V_c^2}{2R} \qquad U = \frac{CV_c^2}{2}$$

- The voltage of the circuit is equivalent to a cavity voltage through a transit factor, $V_c = V_0 T$

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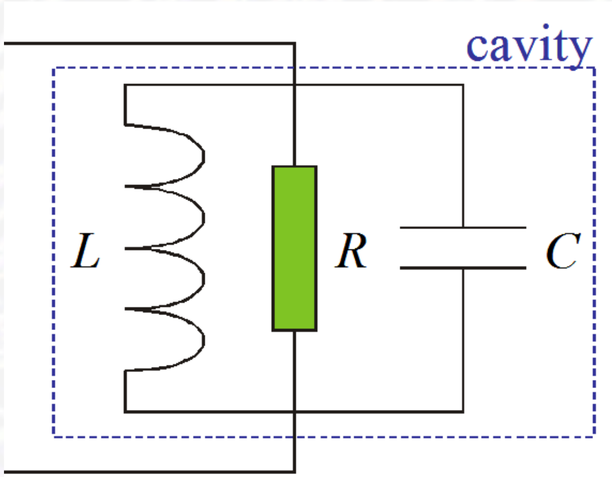
Pill-box ~ LC circuit

- LC circuit

$$Q_0 = \frac{\omega U}{P_c} = \sqrt{\frac{C}{L}} R$$

$$\frac{R}{Q_0} = \frac{V^2}{2\omega U} = \frac{1}{\omega C} = \sqrt{\frac{L}{C}}$$



- equivalent circuits have been proven to accurately model couplers, cavity coupling, micro-phonics, beam loading and field amplitudes including multi-cell cavities

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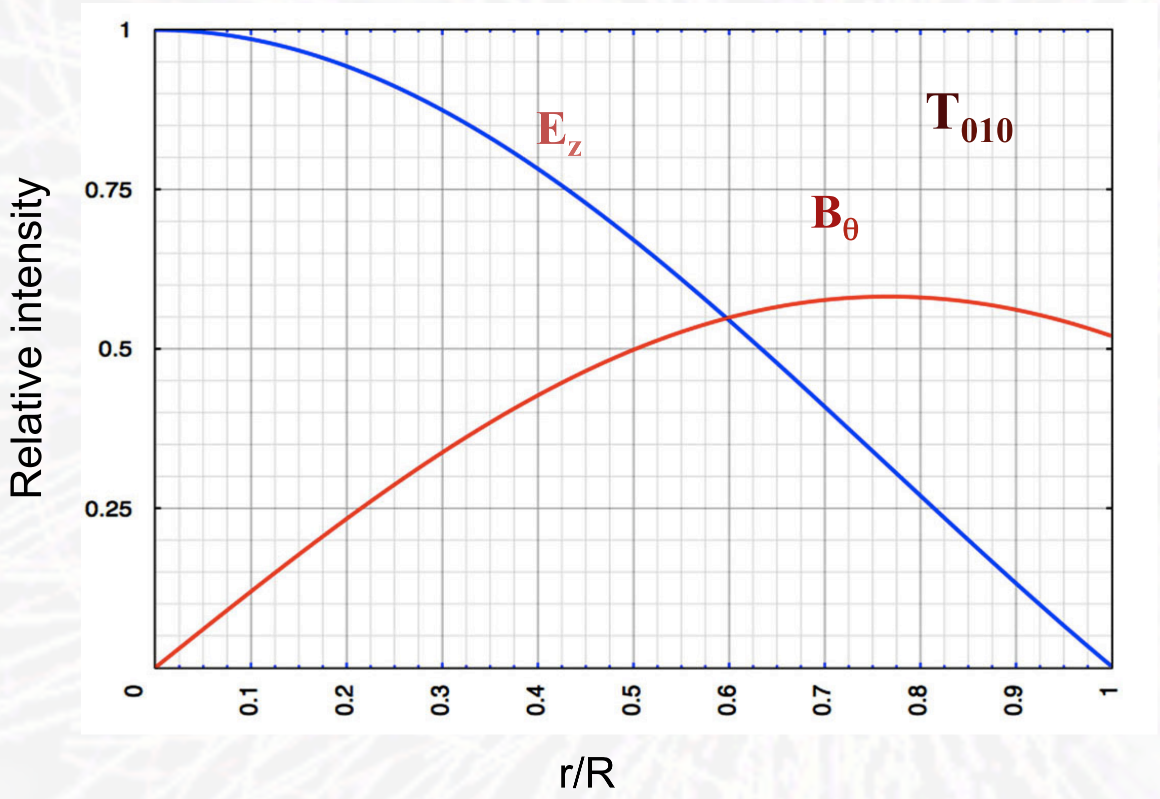
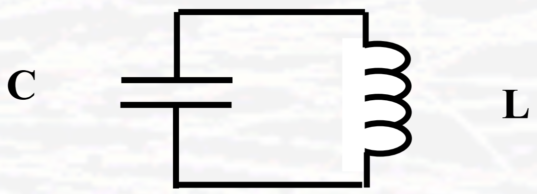
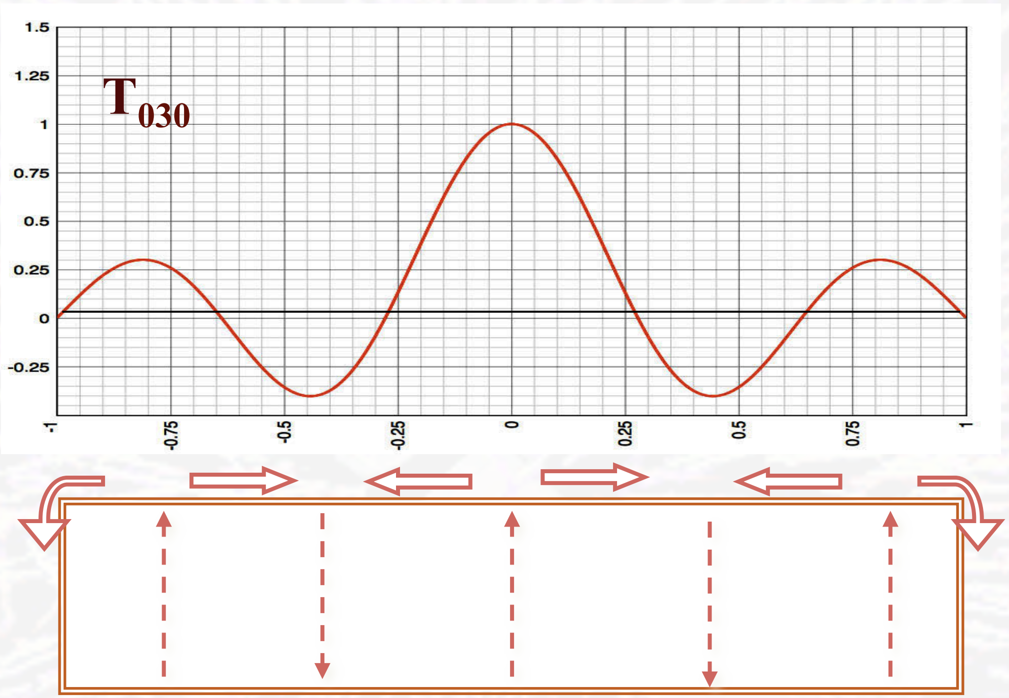
E-fields & equivalent circuit: T_{010} mode

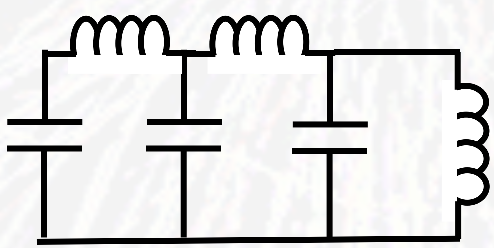




From W. Barletta USPAS Lectures for Accelerator Fundamentals

E-fields & equivalent circuits for T_{030} modes





T_{0n0} has
n coupled, resonant
circuits; each L & C
reduced by 1/n

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Beam Loading

- In addition to ohmic and external losses, the power is also extracted from the cavity by the beam!
  - **"If you can kick a beam, a beam can kick you!"**
- The beam draws a power $P_b = V_c \, I_{beam}$ from the cavity, where $I_{beam} = q\,f$, where q is the bunch charge and f is the repetition rate
- This additional loss can be lumped in with the ohmic heating as an external circuit cannot differentiate between different passive losses.
- This means that the cavity requires different powers without beam or with lower/higher beam currents.

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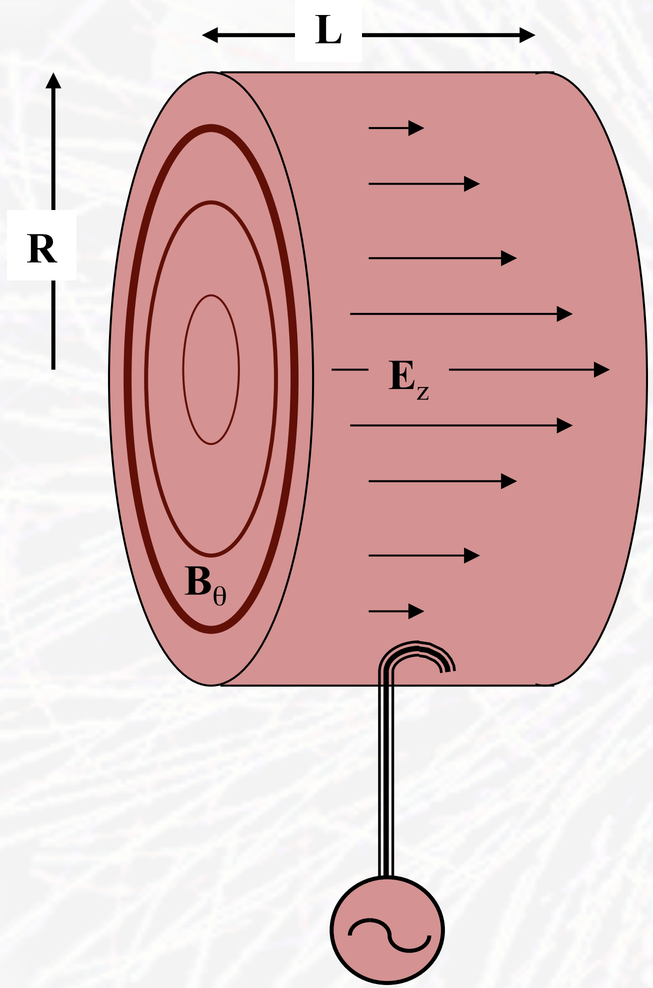
Simple consequences of pillbox model



- Increasing R lowers frequency
  ==> Stored Energy, $\mathcal{E} \sim \omega^{-2}$
- $\mathcal{E} \sim E_z^2$
- Beam loading lowers E_z for the next bunch
- Lowering ω lowers the fractional beam loading
- Raising ω lowers $Q \sim \omega^{-1/2}$
- If time between beam pulses,
  $T_s \sim Q/\omega$
  almost all $\mathcal{E}$ is lost in the walls

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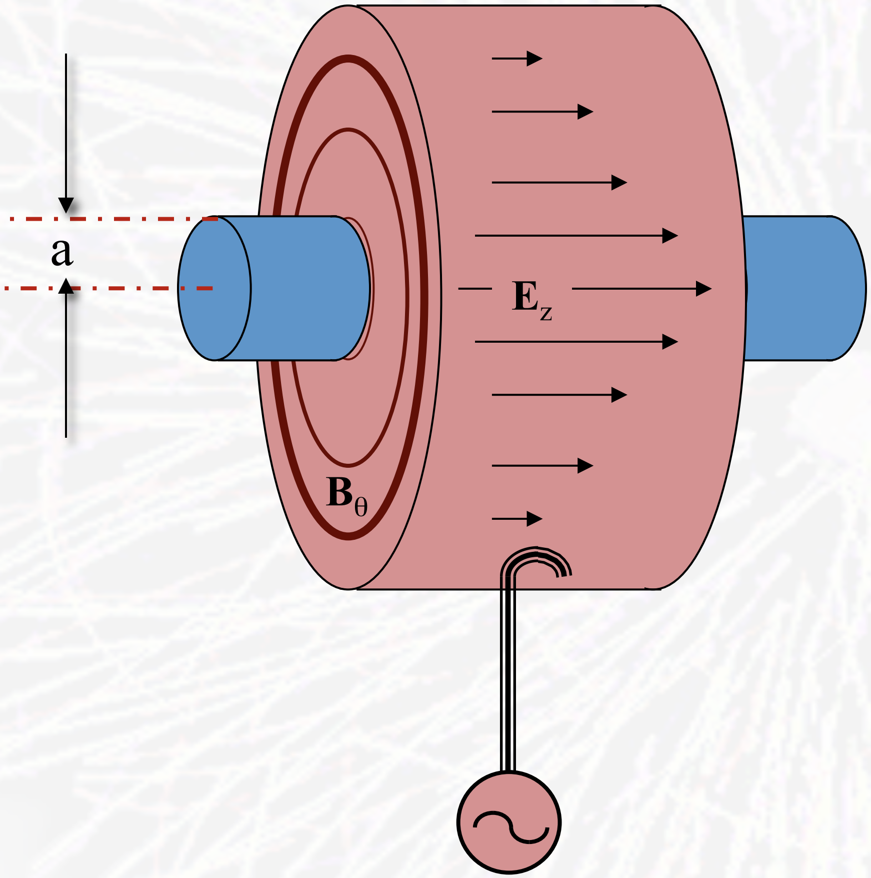
The beam tube complicates the field modes
(& cell design)



- Peak E no longer on axis
  - $E_{pk} \sim 2 - 3 \times E_{acc}$
  - $FOM = E_{pk}/E_{acc}$
- ω_o more sensitive to cavity dimensions
  - Mechanical tuning & detuning
- Beam tubes add length & €' s w/o acceleration
- Beam induced voltages ~ a^{-3}
  - Instabilities

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