Lecture 1:
Introduction to Magnets and Theoretical Fundamentals

Mauricio Lopes – FNAL

Cartoon by **Bruno Touschek**
(3 February 1921–25 May 1978)

A little bit of theory…

Lorentz Force:

$$\boldsymbol{F} = q\,(\cancel{\boldsymbol{E}} + \boldsymbol{v} \times \boldsymbol{B})$$

$$\boldsymbol{F} = q\,\boldsymbol{v} \times \boldsymbol{B}$$

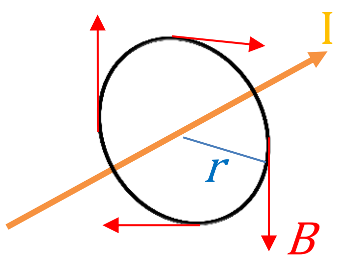
$\boldsymbol{F}$: force q : charge $\boldsymbol{B}$: magnetic field

Magnetic rigidity:

$$Br = \frac{\sqrt{K^2 + 2KE_o}}{qc}$$

T: Beam energy c: speed of light E_o : Particle rest mass

… a little bit more



Biot-Savart law

$$\oint \boldsymbol{H}.\boldsymbol{dl} = I$$

$$\frac{B}{\mu_o} 2\pi r = I$$

$$B = \frac{I\mu_o}{2\pi r}$$

r: radius $\boldsymbol{B}$: magnetic field μ_o: vacuum magnetic permeability

Units

SI units

Variable	Unit
$\boldsymbol{F}$	Newtons (N)
q	Coulombs (C)
$\boldsymbol{B}$	Teslas (T)
I	Amperes (A)
E	Joules (J)

→ (eV) for beams

1 T = 10,000 G

$$\mu_o = 4\pi \times 10^{-7} \frac{T.m}{A}$$

Charge of 1 electron ~ 1.6_x10^{-19} C → 1 eV = 1.6_x10^{-19} J

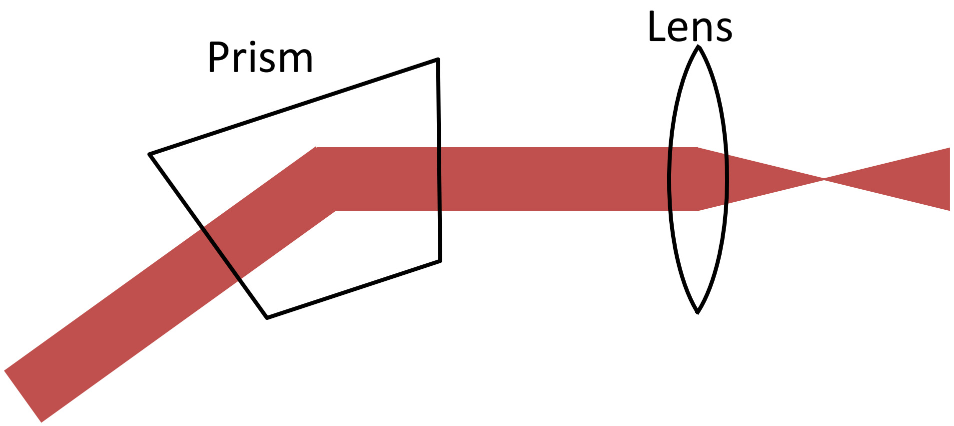
Magnitude of Magnetic Fields

Value	Item
0.1 - 1.0 pT	human brain magnetic field
24 µT	strength of magnetic tape near tape head
31-58 µT	strength of Earth's magnetic field at 0° latitude (on the equator)
0.5 mT	the suggested exposure limit for cardiac pacemakers by American Conference of Governmental Industrial Hygienists (ACGIH)
5 mT	the strength of a typical refrigerator magnet
0.15 T	the magnetic field strength of a sunspot
1 T to 2.4 T	coil gap of a typical loudspeaker magnet
1.25 T	strength of a modern neodymium-iron-boron (Nd2Fe14B) rare earth magnet.
1.5 T to 3 T	strength of medical magnetic resonance imaging systems in practice, experimentally up to 8 T
9.4 T	modern high resolution research magnetic resonance imaging system
11.7 T	field strength of a 500 MHz NMR spectrometer
16 T	strength used to levitate a frog
36.2 T	strongest continuous magnetic field produced by non-superconductive resistive magnet
45 T	strongest continuous magnetic field yet produced in a laboratory (Florida State University's National High Magnetic Field Laboratory in Tallahassee, USA)
100.75 T	strongest (pulsed) magnetic field yet obtained non-destructively in a laboratory (National High Magnetic Field Laboratory, Los Alamos National Laboratory, USA)[10]
730 T	strongest pulsed magnetic field yet obtained in a laboratory, destroying the used equipment, but not the laboratory itself (Institute for Solid State Physics, Tokyo)
2.8 kT	strongest (pulsed) magnetic field ever obtained (with explosives) in a laboratory (VNIIEF in Sarov, Russia, 1998)
1 to 100 MT	strength of a neutron star
0.1 to 100 GT	strength of a magnetar

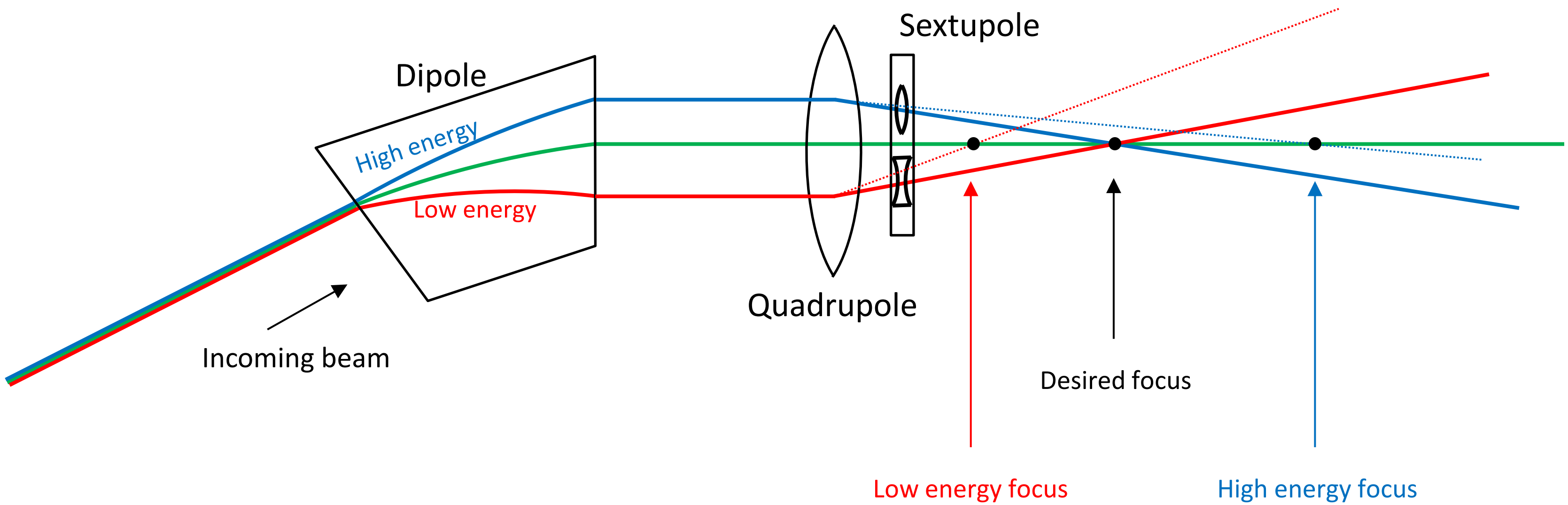
Types of magnets

- Dipoles
- Quadrupoles
- Sextupoles
- Correctors
- Septa
- Kickers

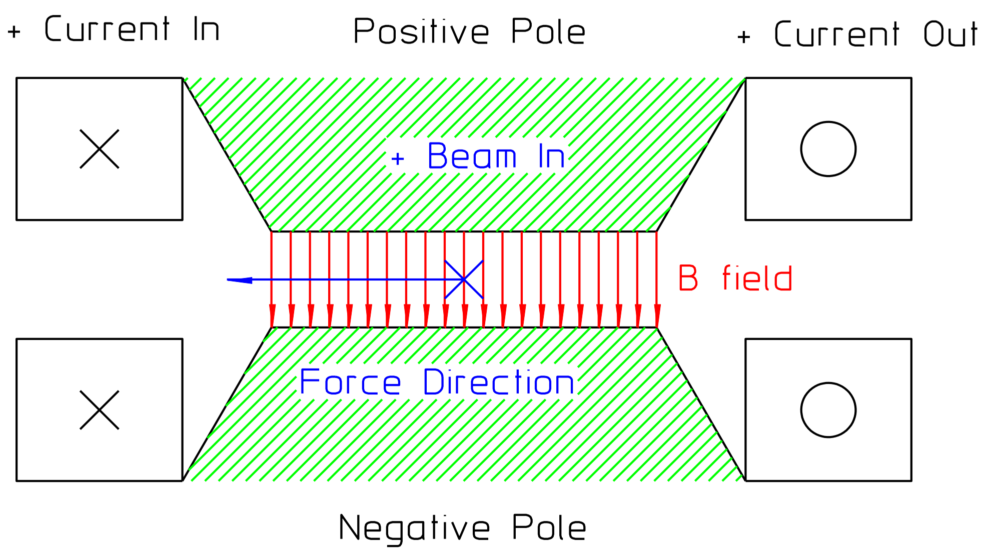
Optics analogy 1



Optics analogy 2

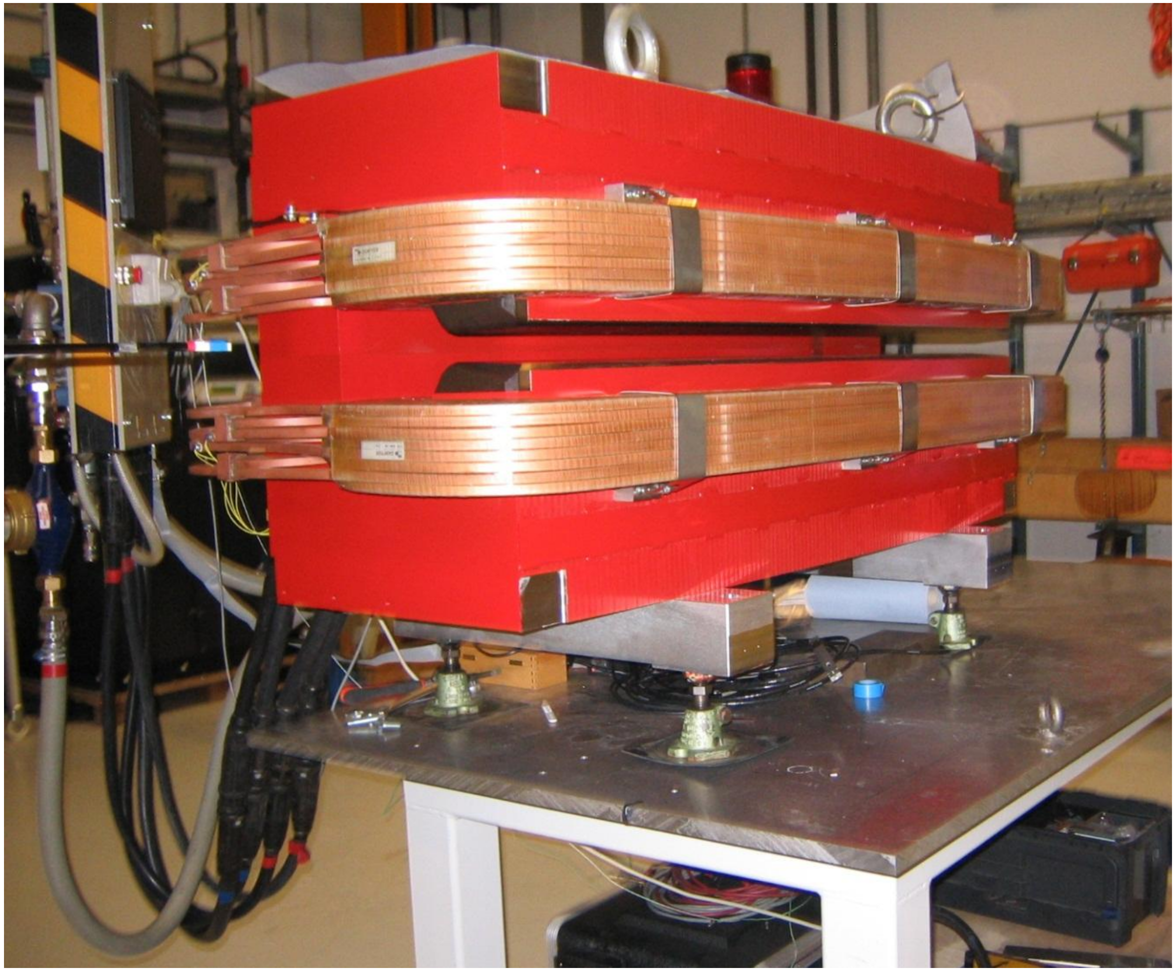


Dipole



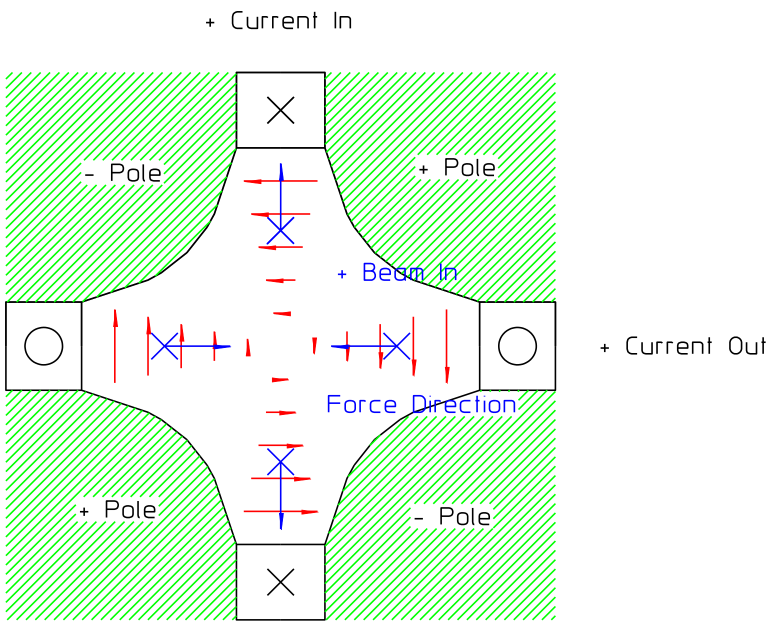
The dipole magnet has two poles, a constant field and steers a particle beam. USing the right hand rule, the positive dipole steer the rotating beam toward the left.

Dipoles



ALBA SR Combined Function Dipole

Quadrupole

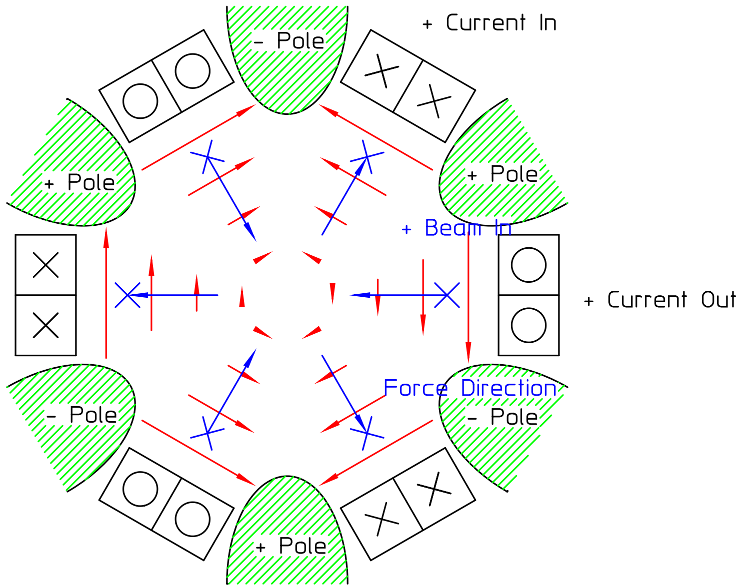


The Quadrupole Magnet has four poles. The field varies *linearly* with the distance from the magnet center. It focuses the beam along one plane while defocusing the beam along the orthogonal plane. An *F* or focusing quadrupole focuses the particle beam along the *horizontal* plane.

Quadrupole

ALBA SR Quadrupole

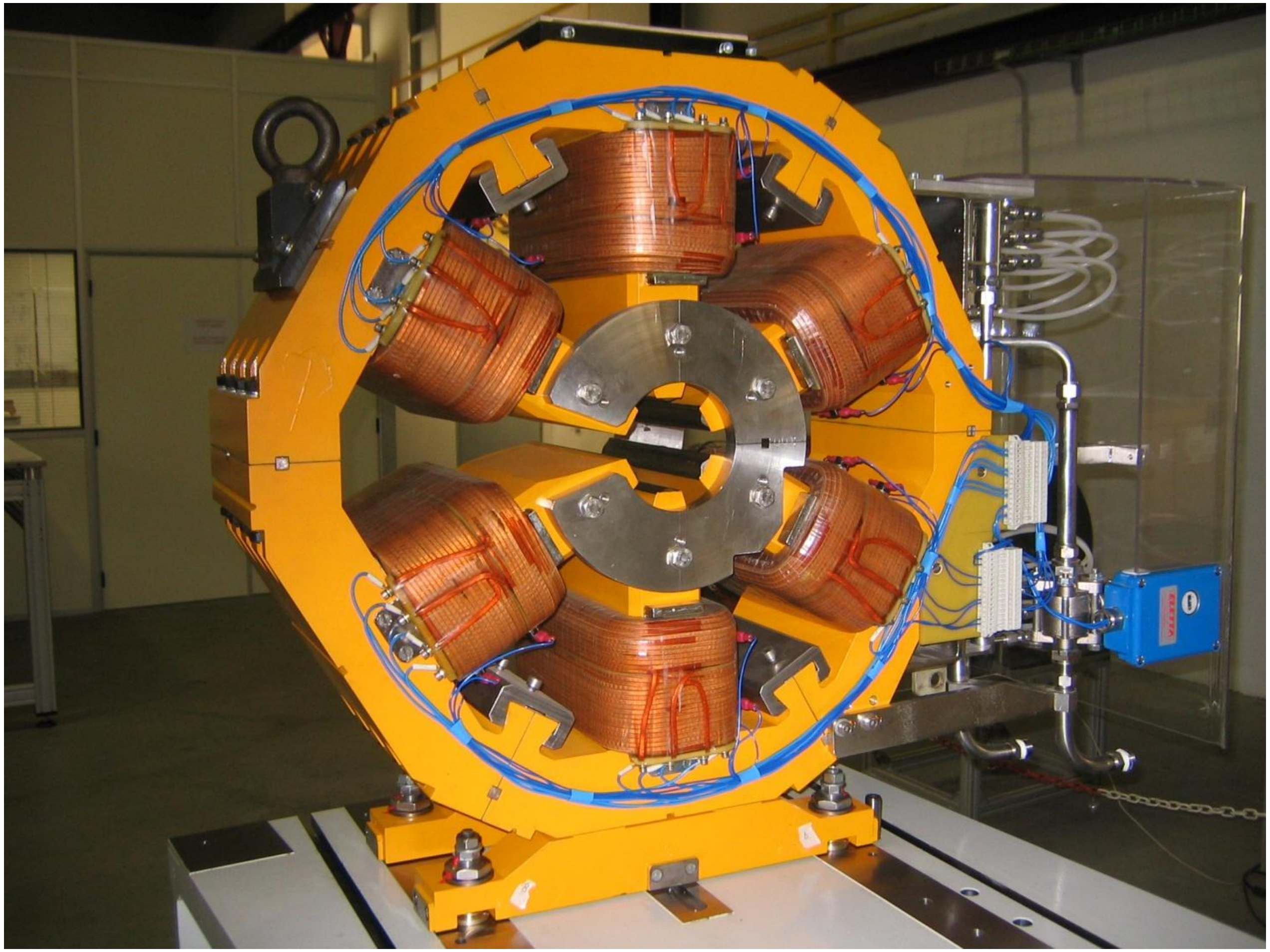
Sextupole



The Sextupole Magnet has six poles. The field varies *quadratically* with the distance from the magnet center. It's purpose is to affect the beam at the edges, much like an optical lens which corrects chromatic aberration. An *F* sextupole will steer the particle beam toward the center of the ring.

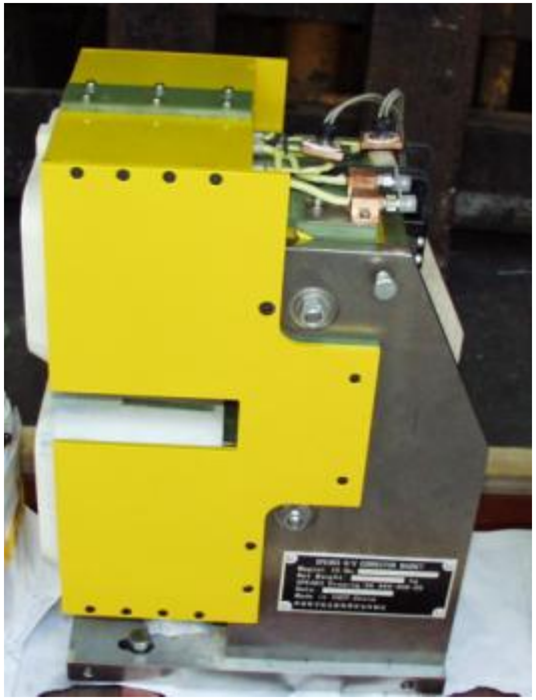
Note that the sextupole also steers along the 60 and 120 degree lines.

Sextupole



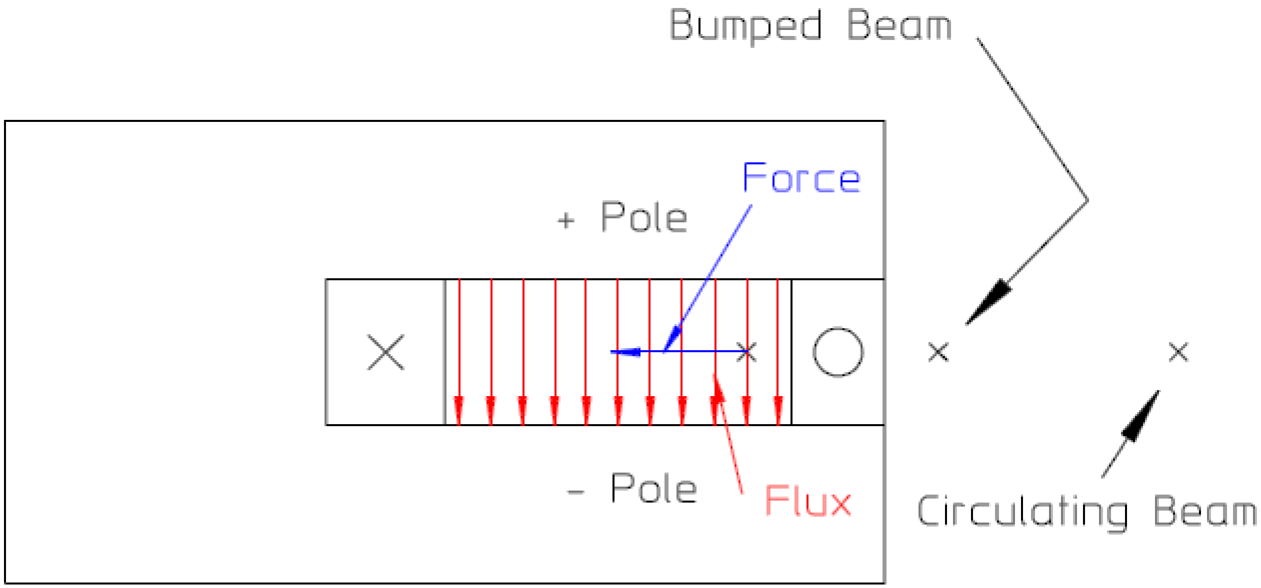
ALBA SR Sextupole

Correctors

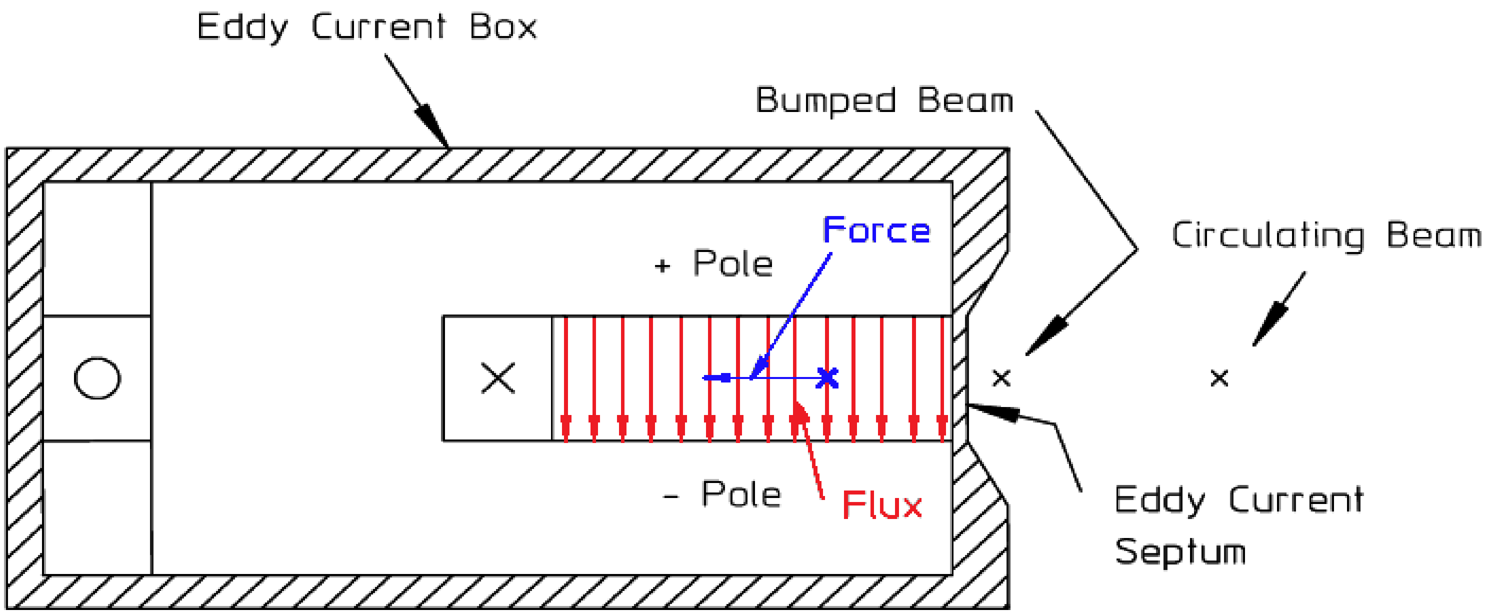


SPEAR3 Corrector

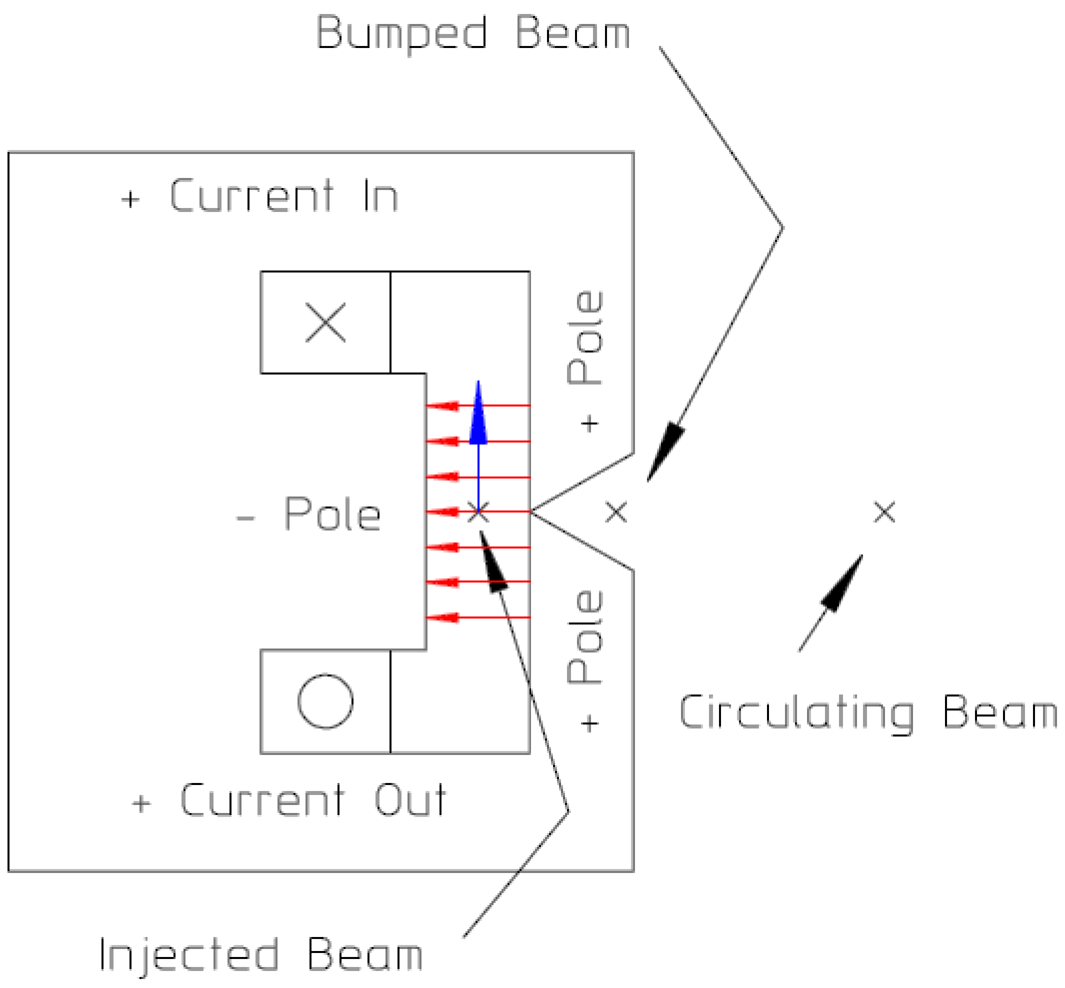
Current Carrying Septum



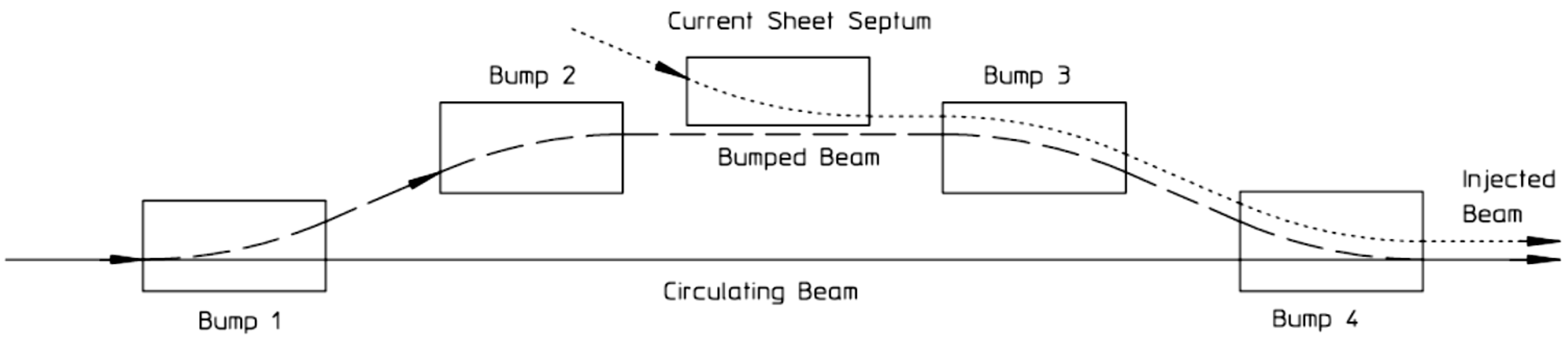
Eddy-Current Septum



Lambertson Septum



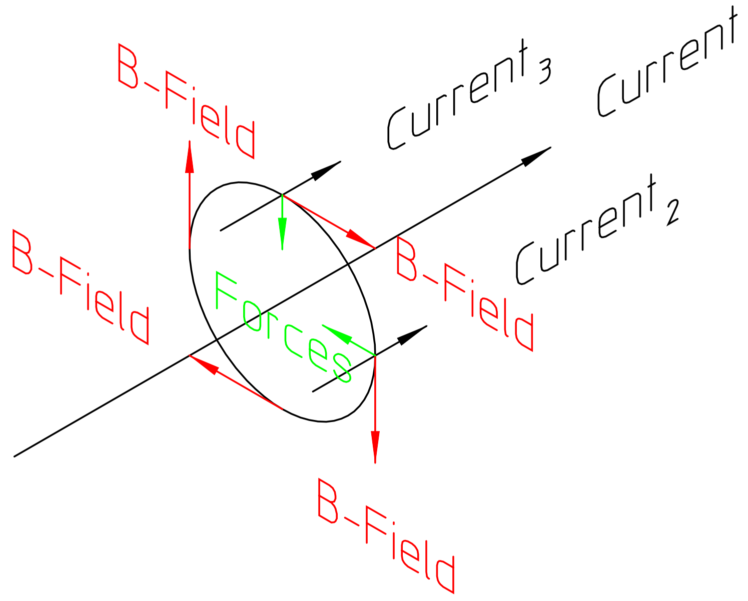
Kicker magnets



Magnetostriction

The forces on parallel currents is illustrated in the following figure. The force on a charge moving with a given velocity through a magnetic field is expressed with the lorentz force:

$$\boldsymbol{F} = q\,\boldsymbol{v} \times \boldsymbol{B}$$



Magnetostriction

- Currents with the same charge travelling in the same direction *attract*.
- Currents with opposite charge travelling in the same direction *repel*.
- Currents with the same charge travelling in the opposite direction *repel*.
- Currents with the opposite charge travelling in the opposite direction *attract*.

Introduction to the Mathematical Formulation

- An understanding of magnets is not possible without understanding some of the mathematics underpinning the theory of magnetic fields. The development starts from Maxwell's equation for the three-dimensional magnetic fields in the presence of steady currents both in vacuum and in permeable material.

- For vacuum and in the absence of current sources, the magnetic fields satisfy Laplace's equation.

- In the presence of current sources (in vacuum and with permeable material) the magnetic fields satisfy Poisson's equation. Although three dimensional fields are introduced, most of the discussion is limited to two dimensional fields.

  – This restriction is not as limiting as one might imagine since it can be shown that the line integral of the three dimensional magnetic fields, when the domain of integration includes all regions where the fields are non-zero, satisfy the two dimensional differential equations.

Maxwell's Equations

(in vacuum)

Gauss's law	$\nabla . \boldsymbol{E} = \frac{\rho}{\varepsilon_o}$	$\oiint \boldsymbol{E} . d\boldsymbol{A} = \frac{Q}{\varepsilon_o}$
	$\nabla . \boldsymbol{B} = 0$	$\oiint \boldsymbol{B} . d\boldsymbol{A} = 0$
Faraday's law	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	$\oint \boldsymbol{E} . d\boldsymbol{l} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} . d\boldsymbol{A}$
Ampere's law	$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} + \mu_o \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$	$\oint \boldsymbol{B} . d\boldsymbol{l} = \mu_o \boldsymbol{I} + \mu_o \varepsilon_o \iint \frac{\partial \boldsymbol{E}}{\partial t} . d\boldsymbol{A}$

Maxwell's Equations

(in media)

Gauss's law

$$\nabla . \boldsymbol{D} = \rho_f \qquad \oint\!\!\oint \boldsymbol{D} . d\boldsymbol{A} = Q_f$$

$$\nabla . \boldsymbol{B} = 0 \qquad \oint\!\!\oint \boldsymbol{B} . d\boldsymbol{A} = 0$$

Faraday's law

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \oint \boldsymbol{E} . d\boldsymbol{l} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} . d\boldsymbol{A}$$

Ampere's law

$$\nabla \times \boldsymbol{H} = \boldsymbol{J_f} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \oint \boldsymbol{H} . d\boldsymbol{l} = \boldsymbol{I_f} + \iint \frac{\partial \boldsymbol{D}}{\partial t} . d\boldsymbol{A}$$

Maxwell's Steady State Magnet Equations

$$\nabla . \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$$

↓ in the absence of sources

$$\nabla \times \boldsymbol{B} = 0$$

The function of a complex variable

$$F = A + iV$$

$\boldsymbol{A}$: Vector potential
V: Scalar potential

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad \boldsymbol{B} = -\boldsymbol{\nabla} V = -\left(\boldsymbol{i}\frac{\partial V}{\partial x} + \boldsymbol{j}\frac{\partial V}{\partial y} + \boldsymbol{k}\frac{\partial V}{\partial z}\right)$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} = 0 \longrightarrow \boldsymbol{\nabla}^2 \boldsymbol{A} = 0$$

0 (Coulomb gauge)

$\boldsymbol{A}$ satisfies the Laplace equation!

$$\boldsymbol{\nabla}.\boldsymbol{B} = \boldsymbol{\nabla}.(-\boldsymbol{\nabla} V) = -\boldsymbol{\nabla}^2 V = 0 \longrightarrow \boldsymbol{\nabla}^2 V = \boldsymbol{0}$$

V also satisfies the Laplace equation!

The complex function $\boldsymbol{F} = \boldsymbol{A} + iV$ must also satisfy the Laplace equation $\boldsymbol{\nabla}^2 \boldsymbol{F} = 0$

The Two-Dimensional Fields

$$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J} \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \boldsymbol{i}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \mu_o J_z$$

Fields from the two-dimensional Function of a complex variable

$$\mathbb{z} = x + iy$$

$$\boldsymbol{F}(\mathbb{z}) = \boldsymbol{A} + iV$$

$$\boldsymbol{B}^{*} = B_x - iB_y = i\boldsymbol{F}'(\mathbb{z})$$

$$Cauchy - Riemann:$$

$$\frac{\partial \boldsymbol{A}}{\partial y} = -\frac{\partial V}{\partial x}$$

$$\frac{\partial \boldsymbol{A}}{\partial x} = \frac{\partial V}{\partial y}$$

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\partial F(\mathbb{z})}{\partial \mathbb{z}} = \frac{\partial \boldsymbol{A} + i\partial V}{\partial x + i\partial x}$$

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\frac{\partial \boldsymbol{A}}{\partial x} + i\frac{\partial V}{\partial x}}{\frac{\partial x}{\partial x} + i\frac{\partial y}{\partial x}}$$

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\partial \boldsymbol{A}}{\partial x} + i\frac{\partial V}{\partial x}$$

$$\boldsymbol{B}^{*} = B_x - iB_y = i\frac{\partial \boldsymbol{A}}{\partial x} - \frac{\partial V}{\partial x}$$

$$\boldsymbol{F}'(\mathbb{z}) = \frac{\frac{\partial \boldsymbol{A}}{\partial y} + i\frac{\partial V}{\partial y}}{\frac{\partial x}{\partial y} + i\frac{\partial y}{\partial y}}$$

$$\boldsymbol{F}'(\mathbb{z}) = -i\frac{\partial \boldsymbol{A}}{\partial y} + \frac{\partial V}{\partial y}$$

$$\boldsymbol{B}^{*} = B_x - iB_y = \frac{\partial \boldsymbol{A}}{\partial y} + i\frac{\partial V}{\partial y}$$

$$B_x = -\frac{\partial V}{\partial x}$$

$$B_x = \frac{\partial \boldsymbol{A}}{\partial y}$$

$$B_y = -\frac{\partial \boldsymbol{A}}{\partial x}$$

$$B_y = -\frac{\partial V}{\partial y}$$

Solution to Laplace's equation
(2D)

$$\boldsymbol{\nabla^2 F} = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$$

$$\frac{\partial F}{\partial x} = \frac{dF}{d\mathbb{z}}\frac{\partial \mathbb{z}}{\partial x} = \frac{dF}{d\mathbb{z}}$$

$$\frac{\partial F}{\partial y} = \frac{dF}{d\mathbb{z}}\frac{\partial \mathbb{z}}{\partial y} = \frac{dF}{d\mathbb{z}}i$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\frac{dF}{d\mathbb{z}} = \frac{d^2 F}{d\mathbb{z}^2}\frac{\partial \mathbb{z}}{\partial x} = \frac{d^2 F}{d\mathbb{z}^2}$$

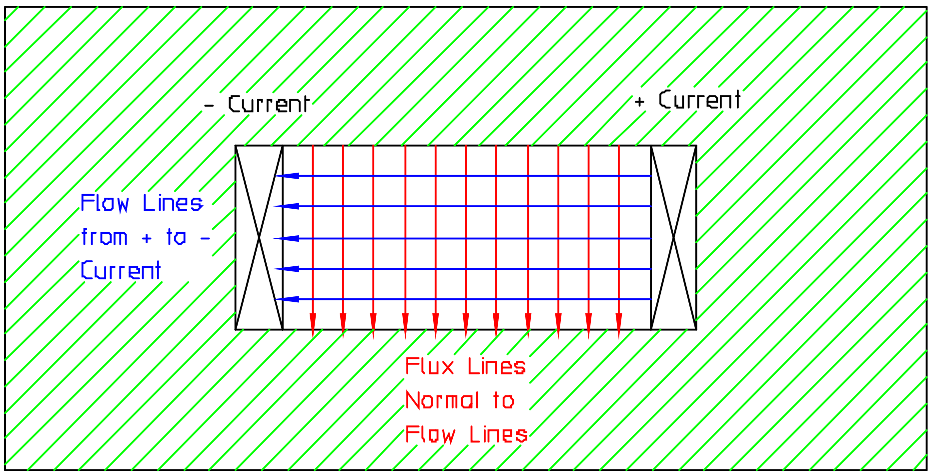
$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\frac{dF}{d\mathbb{z}}i = \frac{d^2 F}{d\mathbb{z}^2}i\frac{\partial \mathbb{z}}{\partial y} = -\frac{d^2 F}{d\mathbb{z}^2}$$

Orthogonal Analog Model

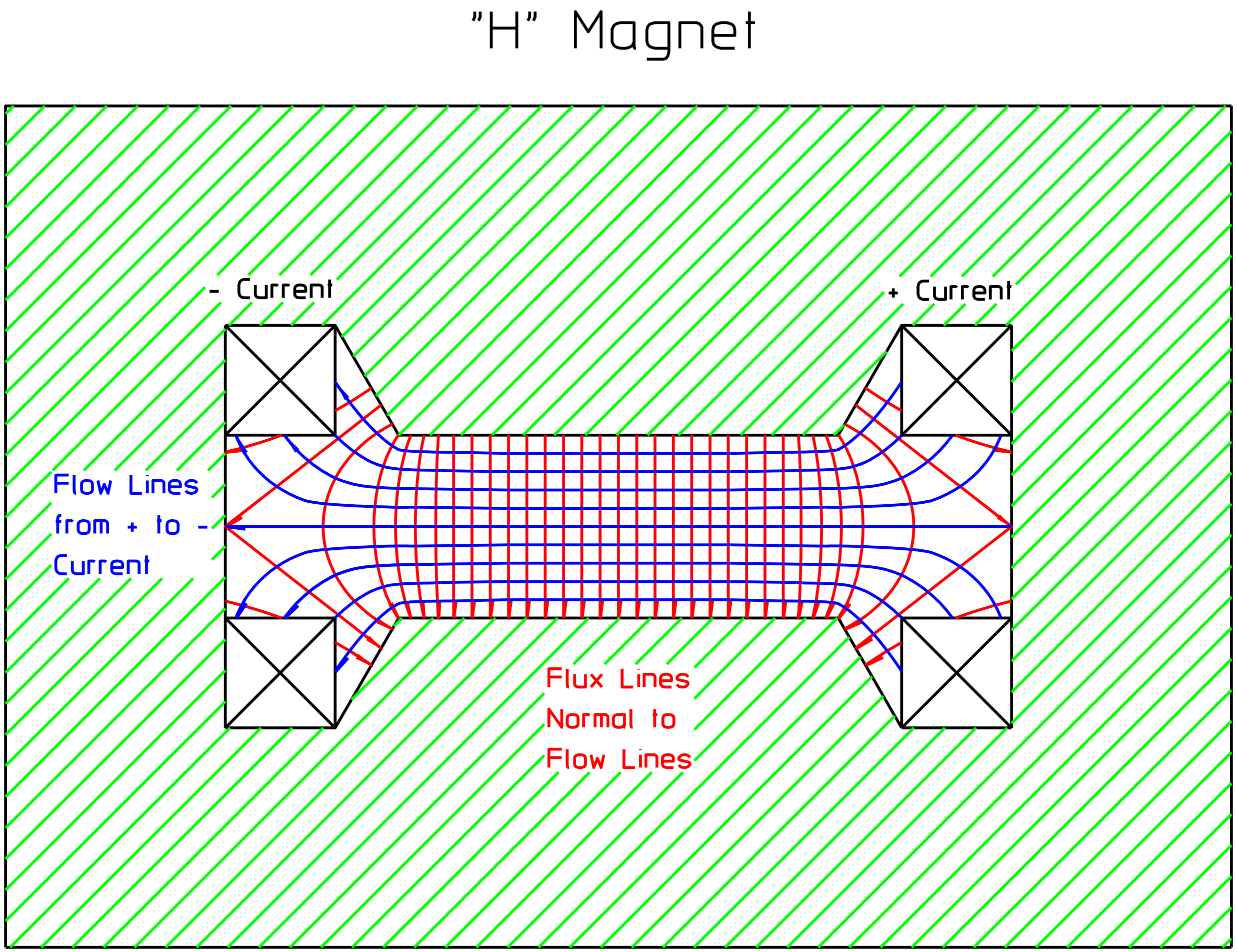
The name of the method for picturing the field in a magnet is called the Orthogonal Analog Model by Klaus Halbach. This concept is presented early in the lecture in order to facilitate visualization of the magnetic field and to aid in the visualization of the *vector* and *scalar* potentials.

- Flow Lines go from the + to - Coils.
- Flux Lines are ortho-normal to the Flow Lines.
- Iron Surfaces are impervious to Flow Lines.

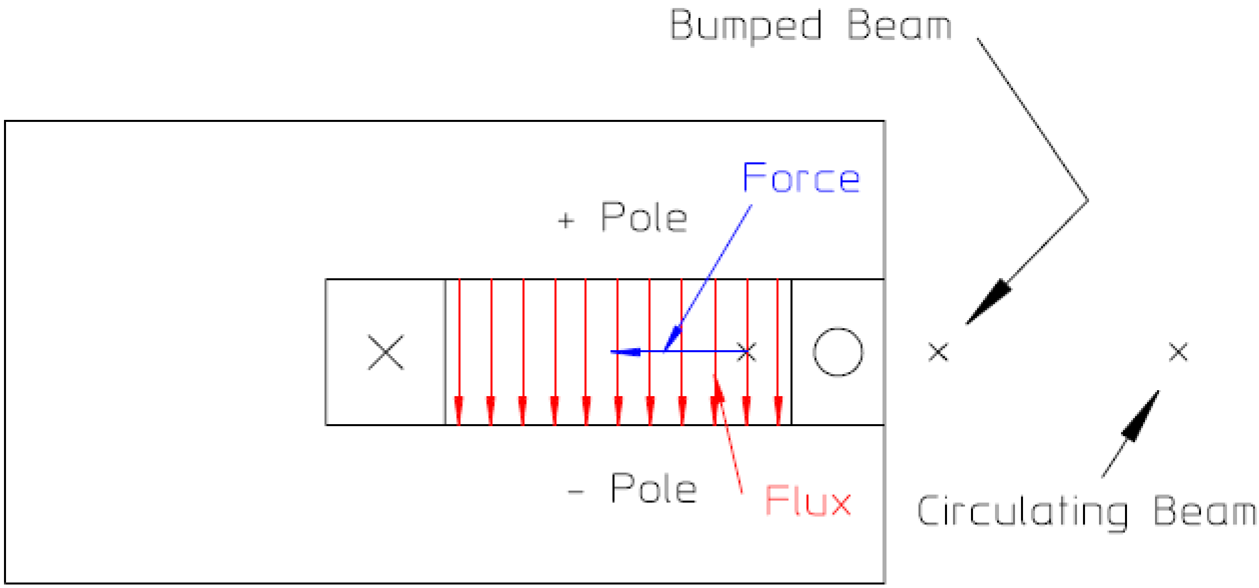
"Window Frame" Magnet



Orthogonal Analog Model



Current Carrying Septum



Multipoles Expansion

$$\mathbb{z} = x + iy$$

$$F(\mathbb{z}) = A + iV = \sum_{n=1}^{\infty} C_n \mathbb{z}^n$$

where n is the order of the multipole

The ideal pole contour can be computed using the *scalar equipotential*.

The field shape can be computed using the *vector equipotential*.

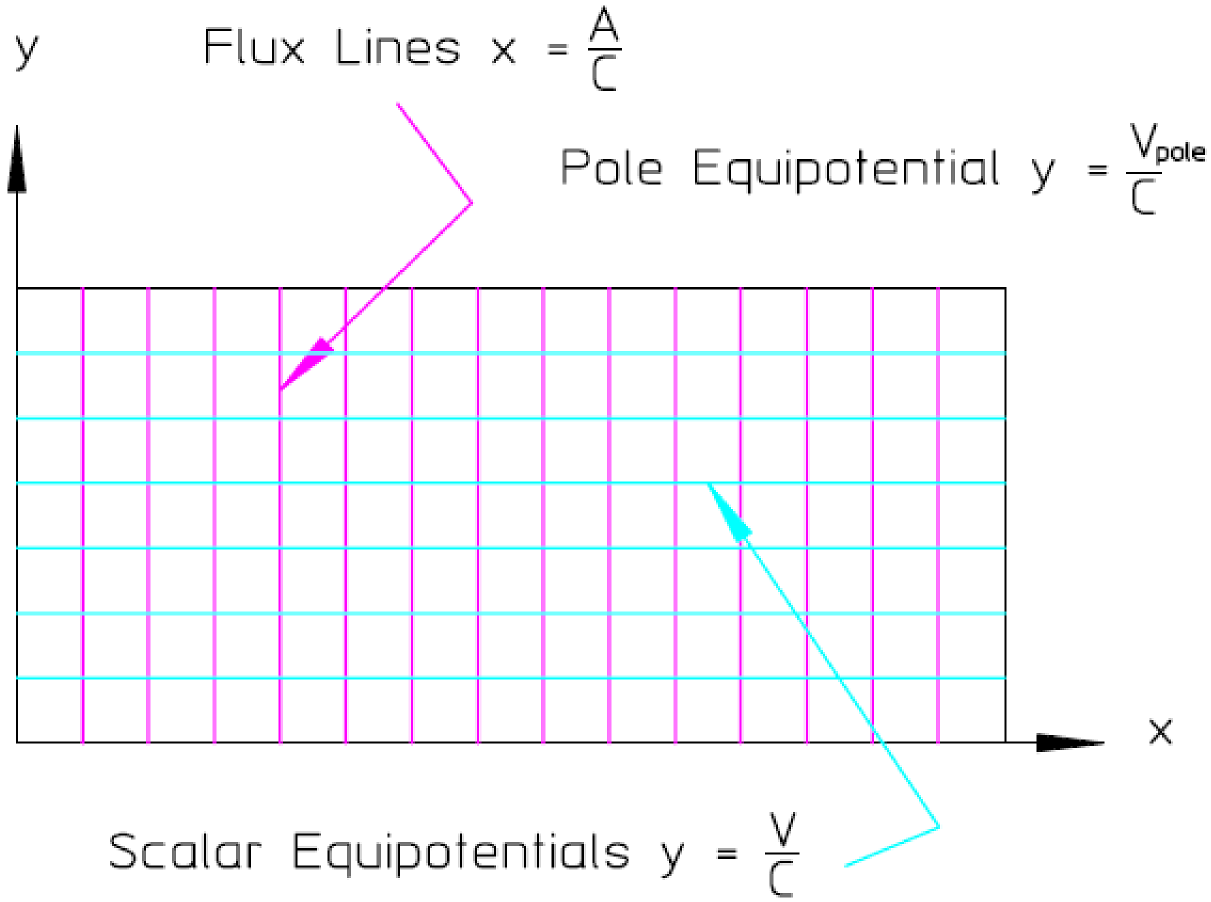
Example 1: Dipole (n=1)

$$A + iV = C_1 \mathbb{z}^1 = C_1(x + iy)$$

$$x = \frac{A}{C_1}$$

$$y = \frac{V}{C_1}$$

Dipole (n=1)



y
Flux Lines $x = \frac{A}{C}$
Pole Equipotential $y = \frac{V_{pole}}{C}$
x
Scalar Equipotentials $y = \frac{V}{C}$

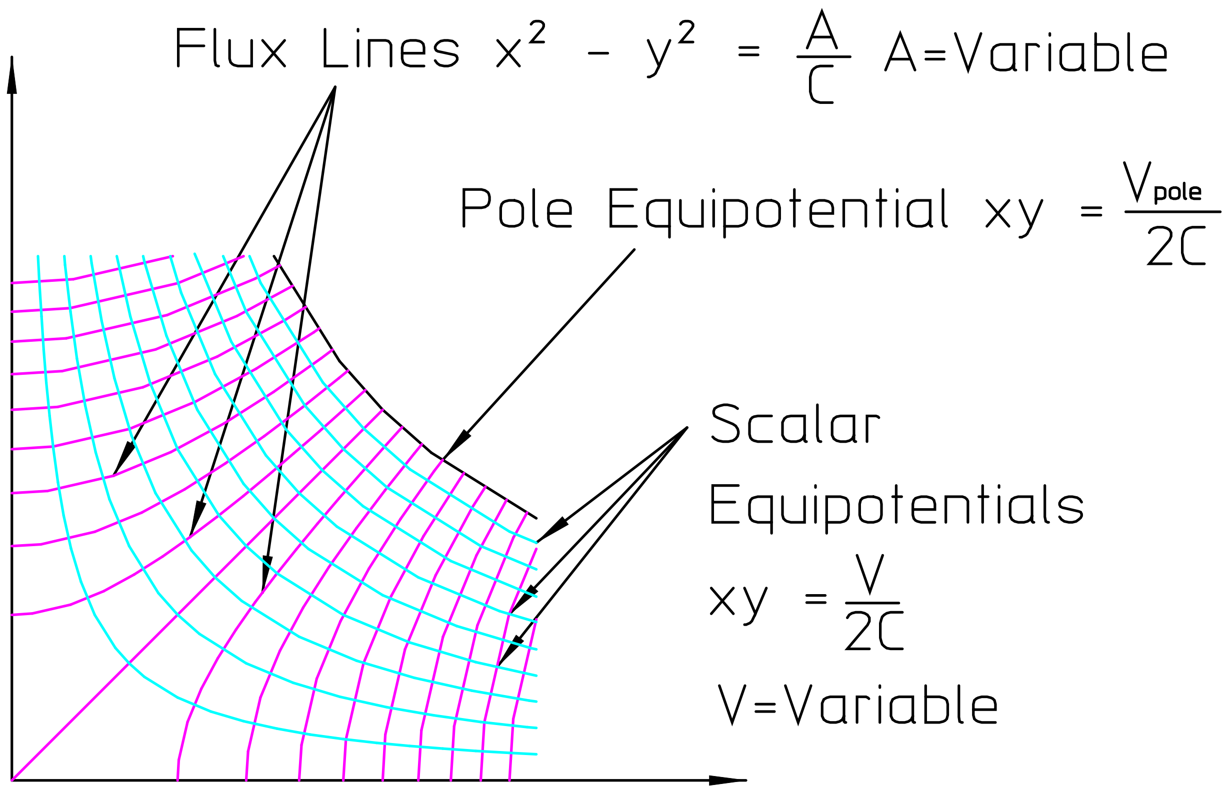
Example 2: Quadrupole (n=2)

$$A + iV = C_2\mathbb{Z}^2 = C_2(x + iy)^2 = C_2(x^2 - y^2 + i2xy)$$

$$x^2 - y^2 = \frac{A}{C_2}$$

$$xy = \frac{V}{2C_2}$$

Quadrupole



Flux Lines $x^2 - y^2 = \frac{A}{C}$ A=Variable
Pole Equipotential $xy = \frac{V_{pole}}{2C}$
Scalar
Equipotentials
$xy = \frac{V}{2C}$
V=Variable

Example 3: Sextupole (n=3)

- For the sextupole case, the function of a complex variable is written in polar form.
  - This case is presented to illustrate that both polar and Cartesian coordinates can be used in the computation.

$$\mathbb{z} = x + iy = |\mathbb{z}|e^{-i\theta}$$

$$A = C|z|^3 \cos 3\theta$$

$$F = Cz^3 = C|z|^3 e^{i3\theta}$$

$$V = C|z|^3 \sin 3\theta$$

$$= C|z|^3 (\cos 3\theta + i \sin 3\theta)$$

$$|z|_{VectorPotential} = \left(\frac{A}{C\cos 3\theta}\right)^{\frac{1}{3}}$$

Vector Potentials

$$x_{VectorPotential} = |z|\cos\theta = \left(\frac{A}{C\cos 3\theta}\right)^{\frac{1}{3}}\cos\theta$$

$$y_{VectorPotential} = |z|\sin\theta = \left(\frac{A}{C\cos 3\theta}\right)^{\frac{1}{3}}\sin\theta$$

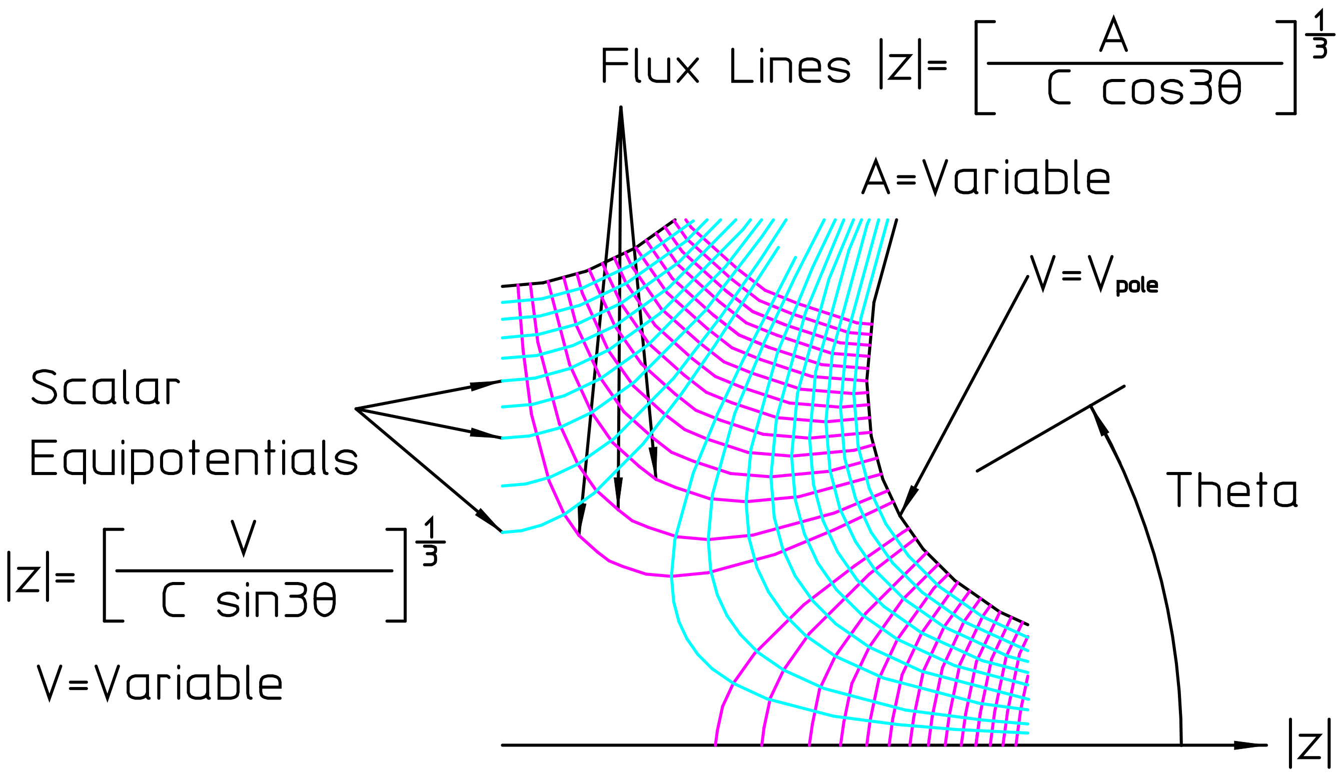
$$|z|_{SclarPotential} = \left(\frac{V}{C\sin 3\theta}\right)^{\frac{1}{3}}$$

Scalar Potentials

$$x_{ScalarPotential} = |z|\cos\theta = \left(\frac{V}{C\sin 3\theta}\right)^{\frac{1}{3}}\cos\theta$$

$$y_{ScalarPotential} = |z|\sin\theta = \left(\frac{V}{C\sin 3\theta}\right)^{\frac{1}{3}}\sin\theta$$

Sextupole Equipotentials



Real and Skew Magnets

- Magnets are described as real when the magnetic fields are vertical along the horizontal centerline :

$$B_x = 0 \text{ and } B_y \neq 0 \text{ for } y=0$$

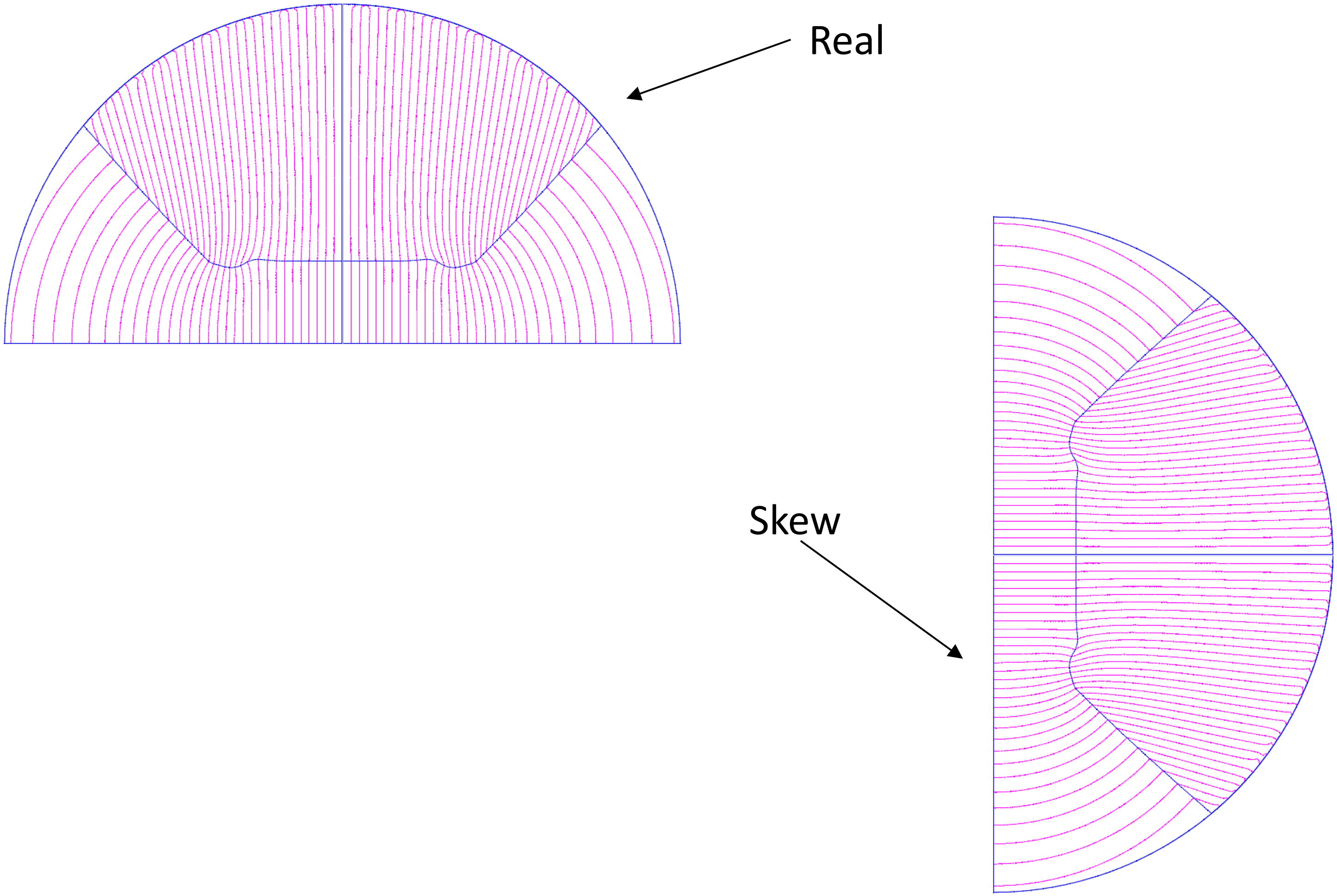
- Real magnets are characterized by C = real.

- Magnets are described as skew when the field are horizontal along the horizontal centerline:

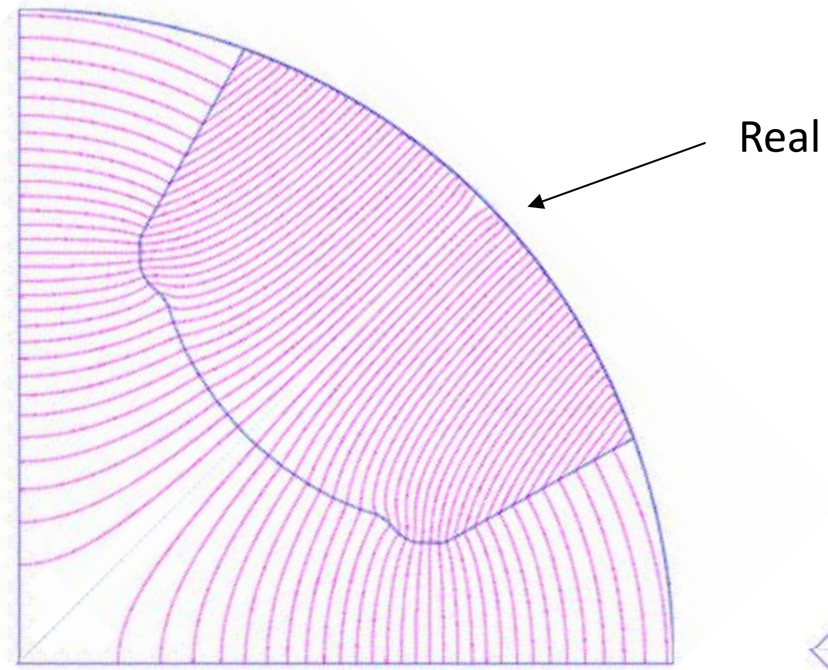
$$B_x \neq 0 \text{ and } B_y = 0 \text{ for } y=0$$

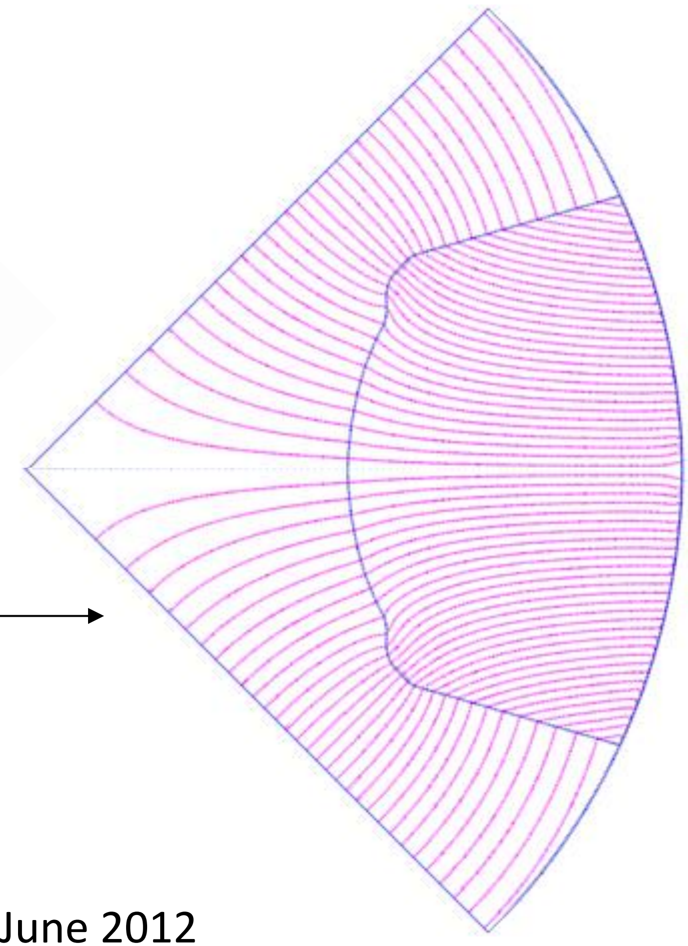
- Skew magnets are characterized by C = imaginary.

Dipole Example



Quadrupole Example





Ideal pole shapes

$$B_y = -\frac{\partial \boldsymbol{A}}{\partial x} \qquad B_y = -\frac{\partial V}{\partial y}$$

Dipole

$$x = \frac{\boldsymbol{A}}{C_1} \qquad V = -\int B_o dy = -B_o y \qquad -B_o h = -B_o y$$

$$B_y = B_o \qquad V(x = 0, y = h) = -B_o h \qquad \boxed{y = h}$$

h = half gap

Ideal pole shapes

$$B_y = -\frac{\partial \boldsymbol{A}}{\partial x} \qquad B_y = -\frac{\partial V}{\partial y}$$

Quadrupole

$$x^2 - y^2 = \frac{\boldsymbol{A}}{C_2}$$

$$B_y = B'.x \ \ @\, y = 0$$

$$V = -\int (B'.x)dy = -B'.xy = -B'.r^2 cos\theta sin\theta$$

$$V(r = h, \theta = \pi/4) = -\frac{B'h^2}{2}$$

$$-\frac{B'h^2}{2} = -B'xy$$

$$\boxed{y = \frac{h^2}{2x}}$$

Hyperbola!

Sextupole

$$|\mathbb{z}| = \sqrt{x^2 + y^2} = \left(\frac{\boldsymbol{A}}{C_3 cos3\theta}\right)^{\frac{1}{3}}$$

$$B_y = B''.x^2 \ \ @\, y = 0$$

$$\boxed{|\mathbb{z}| = \sqrt{x^2 + y^2} = \frac{h}{\sqrt[3]{sin3\theta}}}$$

Complex Extrapolation

- Using the concept of the magnetic potentials, the ideal pole contour can be determined for a desired field.

- Combined Magnet Example
  - The desired gradient magnet field requires a field at a point and a *linear* gradient.
  - Given:
    - A central field and gradient.
    - The magnet half gap, h, at the magnet axis.

  - What is the ideal pole contour?

The desired field is;

$$B_y = B_0 + B' x$$

The *scalar* potential satisfies the relation;

$$B_y = -\frac{\partial V}{\partial y}$$

Therefore;

$$V = -\int \left(B_0 + B' x \right) dy = -B_0 y - B' xy$$

For $\left(x, y \right) = \left(0, h \right)$ on the pole surface,

$$V_{pole} = -B_0 h - B' \left(0 \times h \right) = -B_0 h$$

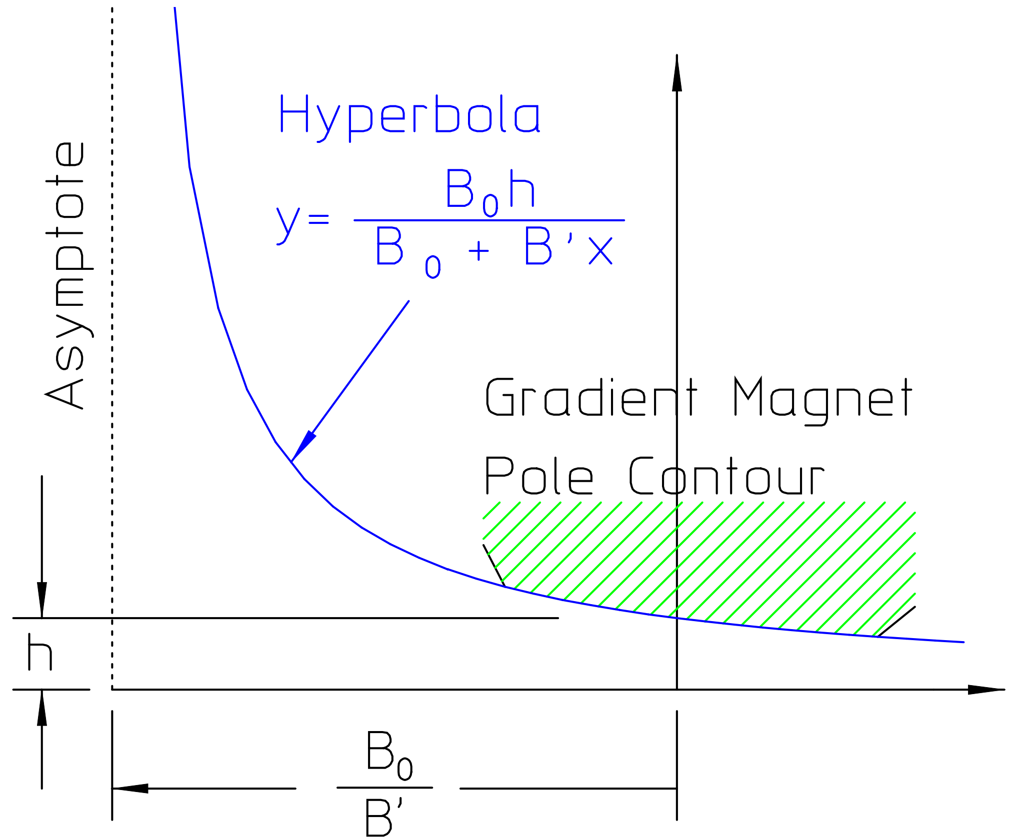
Therefore, the equation for the pole is,

$$-B_0 y - B' xy = V_{pole} = -B_0 h$$

or solving for *y*,

$$y = \frac{B_0 h}{B_0 + B' x}$$

Hyperbola $$y = \frac{B_0 h}{B_0 + B' x}$$ with asymptote at, $$x = -\frac{B_0}{B'}$$



Section Summary

- We learned about the different kinds of magnets and their functions.

$$\boldsymbol{F}(\mathbb{z}) = \boldsymbol{A} + iV = \sum_{n=1}^{\infty} C_n \mathbb{z}^n$$

- The ideal pole contour can be computed using the *scalar equipotential*.

- The field shape can be computed using the *vector equipotential*.

$$B_x = -\frac{\partial V}{\partial x} \qquad B_y = -\frac{\partial \boldsymbol{A}}{\partial x}$$

$$B_x = \frac{\partial \boldsymbol{A}}{\partial y} \qquad B_y = -\frac{\partial V}{\partial y}$$

Next…

- Multipoles
- Pole tip design
- Conformal mapping