



Lecture 4

RF Acceleration in Linacs

Part 2



Outline

- Traveling-wave linear accelerators
- Longitudinal beam dynamics

- Material from
 - *Wiedemann, Chapters 15, 6*
 - *Wangler, Chapters 3, 6*



Guided Electromagnetic Waves in a Cylindrical Waveguide

- Can we accelerate particles by transporting EM waves in a waveguide?
- Consider a cylindrical geometry. The wave equation in cylindrical coordinates for the z field component is

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2} = 0$$

- Assume the EM wave propagates in the Z direction. Let's look for a solution that has a finite electric field in that same direction:

$$E_z = E_z(r, \phi, z, t) = E_0(r, \phi) \cos(k_z z - \omega t)$$

- The azimuthal dependence must be repetitive in ϕ :

$$E_z = R(r) \cos(n\phi) \cos(k_z z - \omega t)$$

- The wave equation yields:

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} + \left(\underbrace{\frac{\omega^2}{c^2} - k_z^2}_{k_c^2} - \frac{n^2}{r^2} \right) R(r) = 0$$

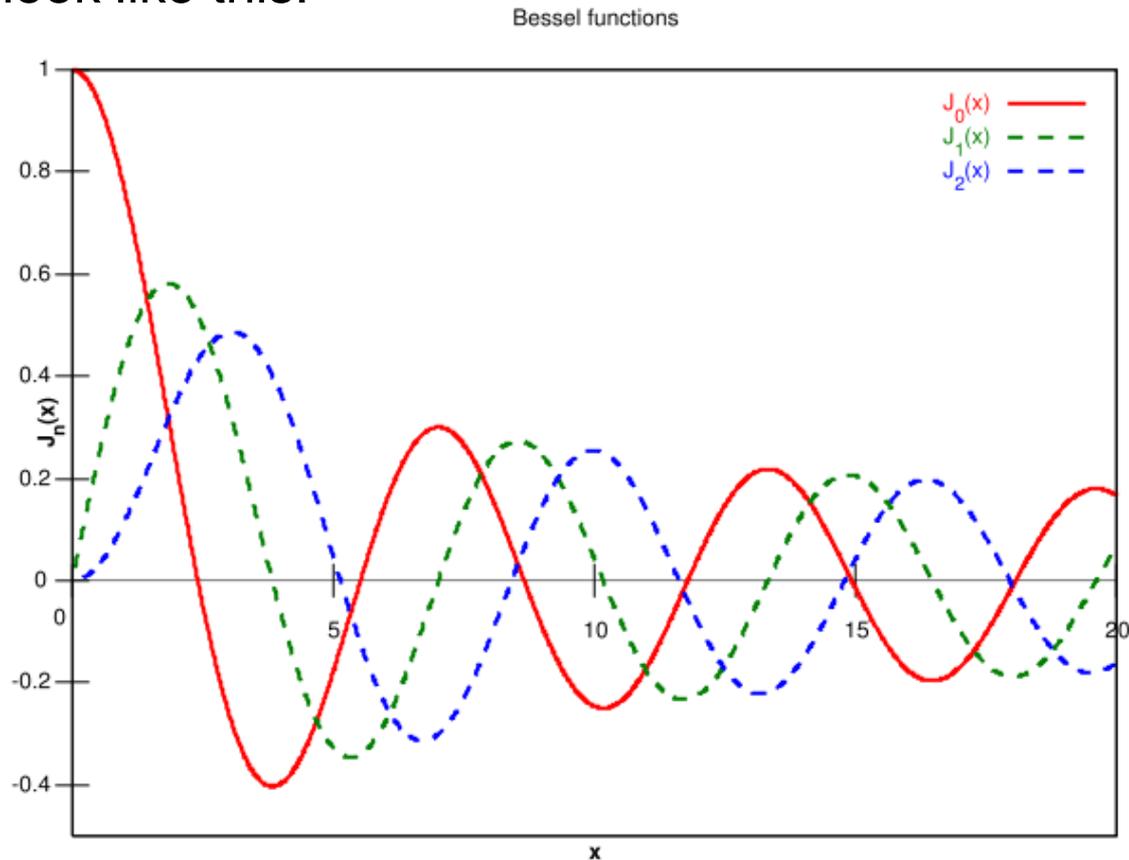


Cylindrical Waveguides

- Which results in the following differential equation for $R(r)$ (with $x=k_c r$)

$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + (1 - n^2 / x^2) R = 0$$

- The solutions to this equation are *Bessel functions of order n* , $J_n(k_c r)$, which look like this:





Cylindrical Waveguides

- The solution is:

$$E_z = J_n(k_c r) \cos(n\phi) \cos(k_z z - \omega t)$$

- The boundary conditions require that

$$E_z(r = a) = 0$$

- Which requires that

$$J_n(k_c a) = 0 \text{ for all } n$$

- Label the n -th zero of J_m : $J_m(x_{mn}) = 0$

- For $m=0$, $x_{01} = 2.405$

$$\frac{\omega^2}{c^2} = k_c^2 + k_z^2 = \left(\frac{2.405}{a}\right)^2 + k_z^2$$



Cutoff Frequency and Dispersion Curve

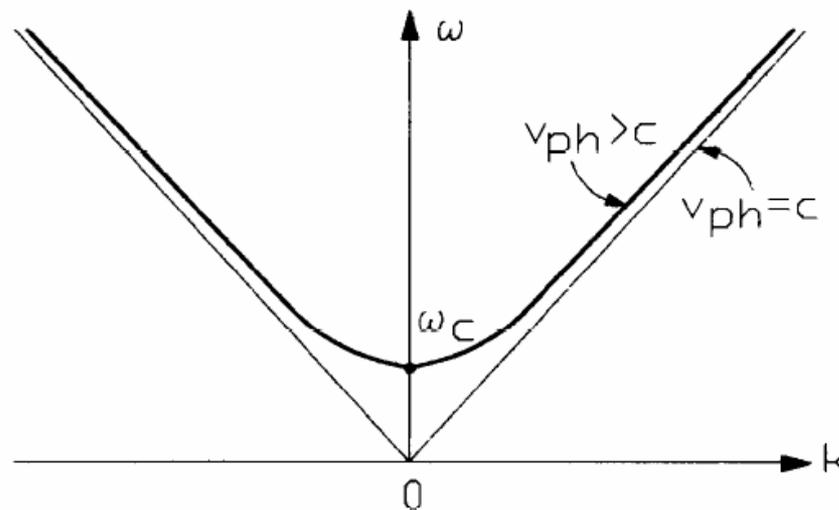
- The cylindrically symmetric waveguide has

$$k_0^2 = k_c^2 + k_z^2 \qquad \omega^2 = \omega_c^2 + (k_z c)^2$$

- A plot of ω vs. k is a hyperbola, called the Dispersion Curve.

Two cases:

- $\omega > \omega_c$: k_z is a real number and the wave propagates.
- $\omega < \omega_c$: k_z is an imaginary number and the wave decays exponentially with distance.
- Only EM waves with frequency above cutoff are transported!





Phase Velocity and Group Velocity

- The propagating wave solution has

$$E_z = E_0(r, z) \cos(\phi) \quad \phi = k_z z - \omega t$$

- A point of constant ϕ propagates with a velocity, called the phase velocity,

$$v_p = \frac{\omega}{k_z}$$

- The electromagnetic wave in cylindrical waveguide has phase velocity that is faster than the speed of light:

$$v_p = \frac{c}{\sqrt{1 - \omega_c^2 / \omega^2}} > c$$

- This won't work to accelerate particles. We need to modify the phase velocity to something smaller than the speed of light to accelerate particles.
- The *group velocity* is the velocity of energy flow:

$$P_{RF} = v_g U$$

- And is given by:

$$v_g = \frac{d\omega}{dk}$$

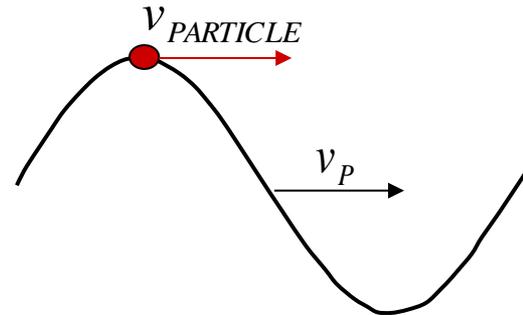


Traveling Wave Structures

- Recall that in the cylindrical waveguide, the electromagnetic wave has phase velocity that is faster than the speed of light:

$$v_p = \frac{c}{\sqrt{1 - \omega_c^2 / \omega^2}} > c$$

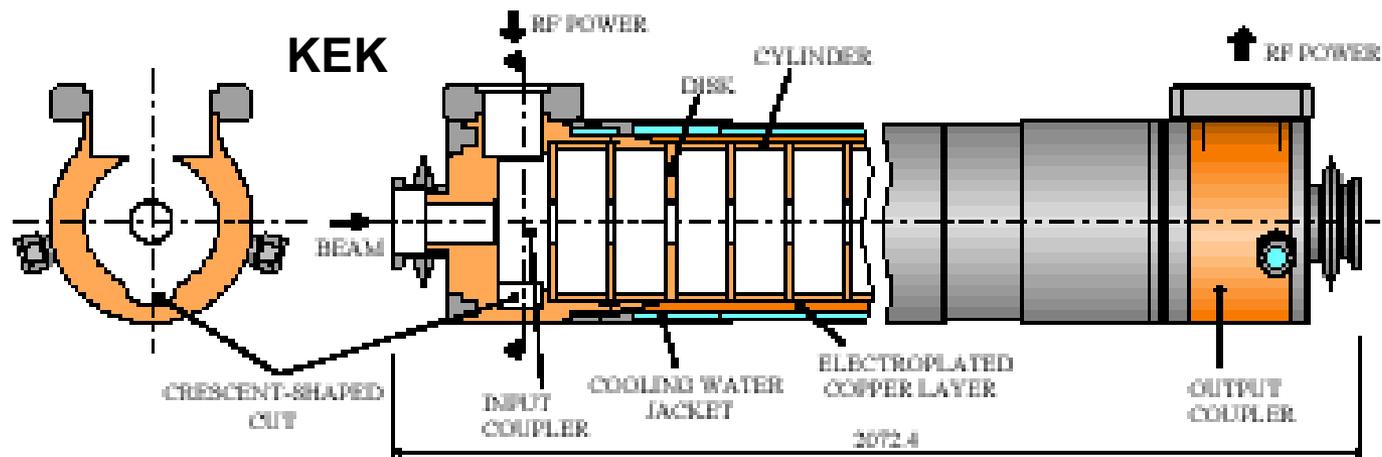
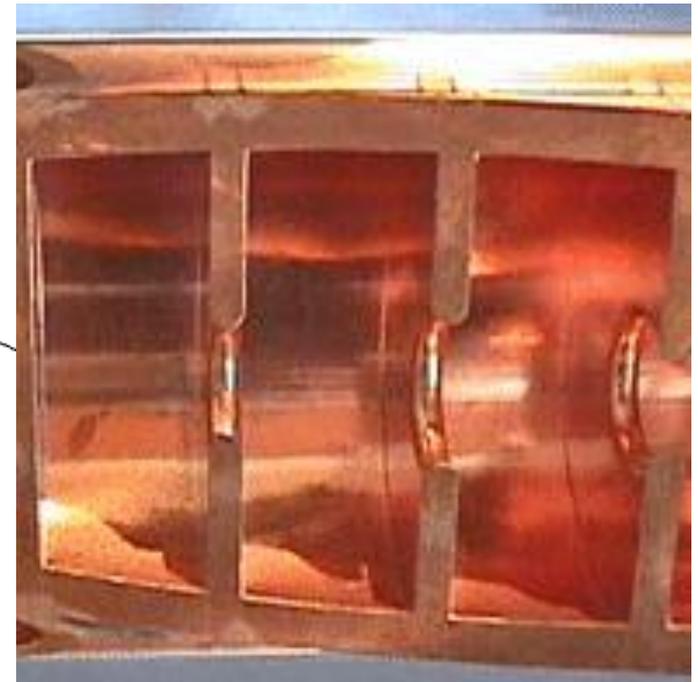
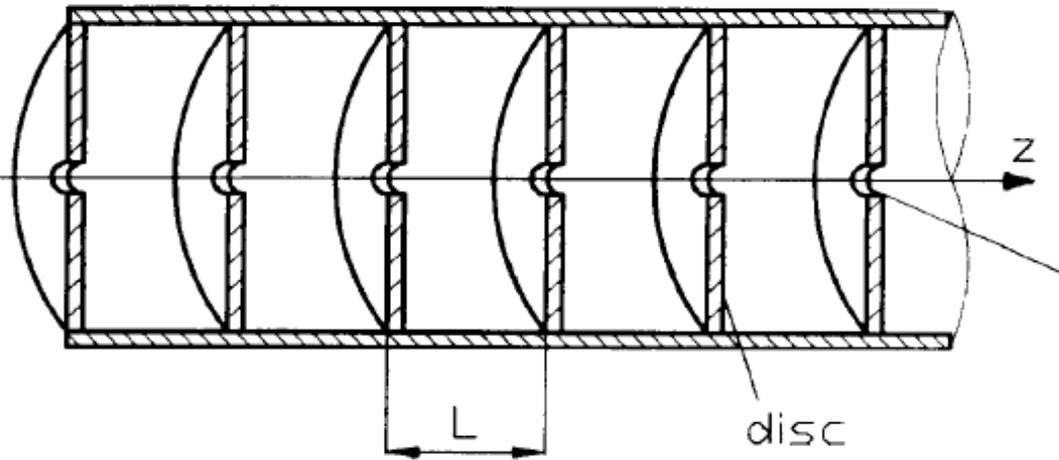
- This won't work to accelerate particles. We need to modify the phase velocity to the speed of light (or slower) to accelerate particles in a traveling wave.
- Imagine a situation where the EM wave phase velocity equals the particle velocity.
- Then the particle “rides the wave”.



- A “disk-loaded waveguide” can be made to have a phase velocity equal to the speed of light. These structures are often used to accelerate electrons.
- The best and largest example of such an accelerator is the SLAC two-mile long linac.



Disk-loaded waveguide structure





Energy Gain in a Disk-Loaded Waveguide

Define

- E_a : longitudinal accelerating field amplitude
- U : stored energy per unit length
- P_w : traveling wave power
- dP_w/dz : power dissipation per unit length
- Shunt impedance per unit length $r_L = E_a^2 / (-dP_w / dz)$

• We have
$$Q = \omega U / (-dP_w / dz)$$

$$P_w = v_g U$$

$$E_a^2 = \omega r_L P_w / Q v_g$$

$$\frac{dP_w}{dz} = -\frac{\omega}{Q v_g} P_w = -2\alpha_0 P_w$$

We have two choices for the accelerating structure, considered now in turn:



Constant Impedance Traveling Wave Structure

- Consider a disk-loaded waveguide with uniform cell geometry along the length, then Q , v_g , r_L , α_0 are independent of z :

$$P_w(z) = e^{-2\alpha_0 z}$$

- Power decays exponentially along the length of the structure.
- The Electric field amplitude is

$$dE_a / dz = -\alpha_0 E_a$$

$$E_a(z) = E_0 e^{-\alpha_0 z}$$

- At the end of a waveguide of length L

$$P_w(L) = P_0 e^{-2\tau_0} \quad E_a(L) = E_0 e^{-\tau_0}$$

$$\tau_0 = \alpha_0 L = \frac{\omega L}{2Qv_g}$$

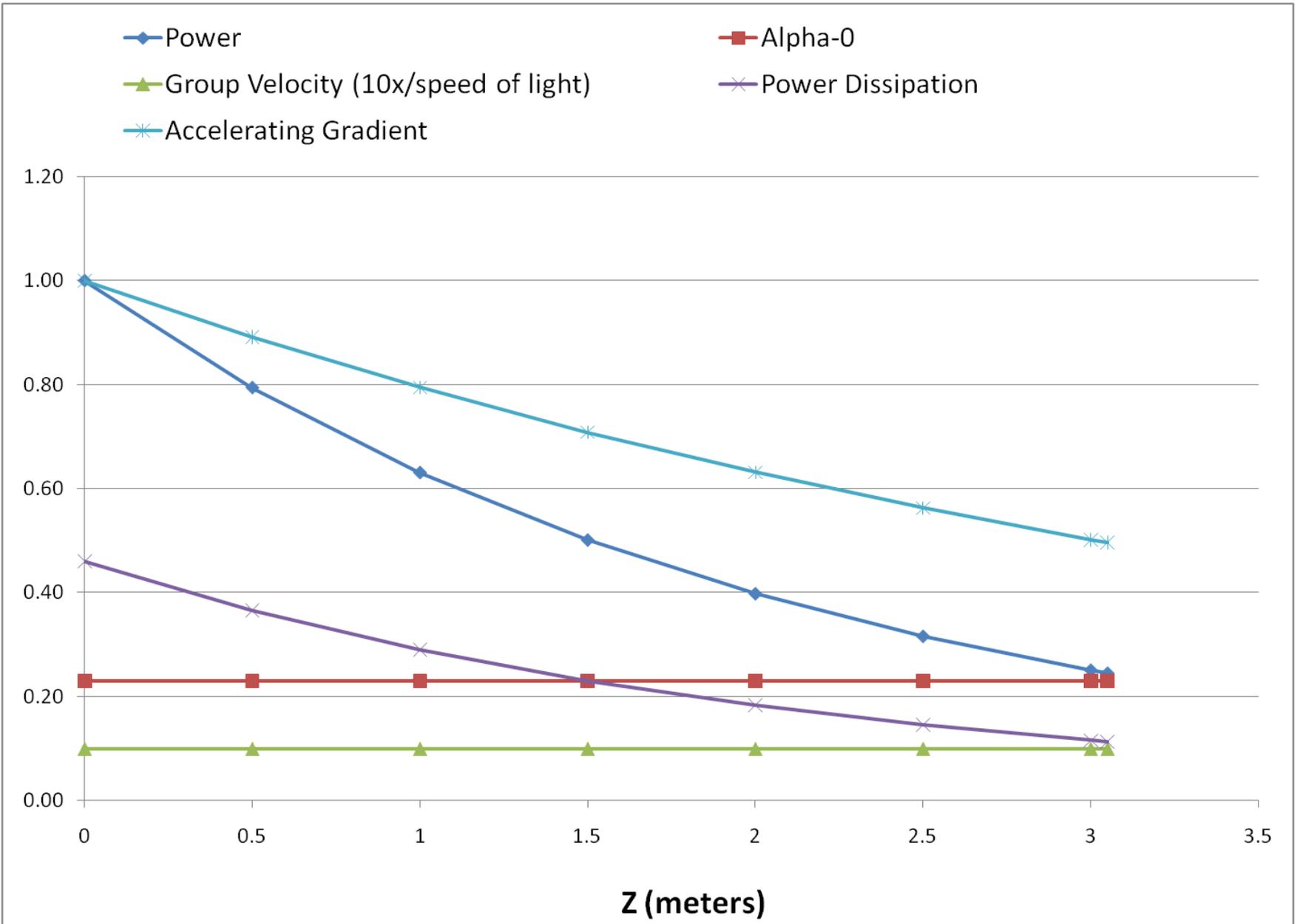
- The energy gain is

$$\Delta W = q \cos \phi \int_0^L E_a(z) dz = q E_0 L \frac{1 - e^{-\tau_0}}{\tau_0} \cos \phi$$

$$\Delta W = q \sqrt{2r_L P_0 L} \frac{1 - e^{-\tau_0}}{\sqrt{\tau_0}} \cos \phi$$



Constant Impedance Structure Parameters





Constant Gradient Traveling Wave Structure

- A more common design keeps the gradient constant over the length, which requires that the attenuation α_0 depend on z

$$\frac{dP_w}{dz} = -2\alpha_0(z)P_w$$

- Which can be integrated to yield $P_w(z) = P_0 \left[1 - \frac{z}{L} (1 - e^{-2\tau_0}) \right]$

- The attenuation factor is

$$\alpha_0(z) = \frac{1}{2L} \frac{1 - e^{-2\tau_0}}{1 - (z/L)(1 - e^{-2\tau_0})}$$

- The energy gain is

$$\Delta W = q \cos \phi \int_0^L E_a(z) dz = qE_0 L \cos \phi$$

$$\Delta W = q \sqrt{r_L P_0 L (1 - e^{-2\tau_0})} \cos \phi$$

- To achieve a constant gradient, the SLAC linac structure tapers from a radius of 4.2 to 4.1 cm, and the iris radii taper from 1.3 to 1.0 cm over 3 meters.

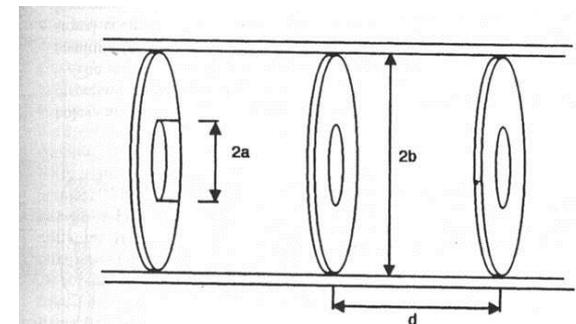


Figure 3.14 Iris (disk)-loaded traveling-wave structure.

⁴ For a comprehensive article on this subject, see G. A. Loew and R. B. Neal, in *Linear Accelerators*, P. M. Lapostolle and A. L. Septier, Wiley, New York, 1970, pp. 39–113.



Constant Gradient Traveling Wave Structure

- The group velocity is

$$v_g(z) = \frac{\omega}{2Q\alpha_0(z)} = \frac{\omega L}{Q} \frac{1 - (z/L)(1 - e^{-2\tau_0})}{1 - e^{-2\tau_0}}$$

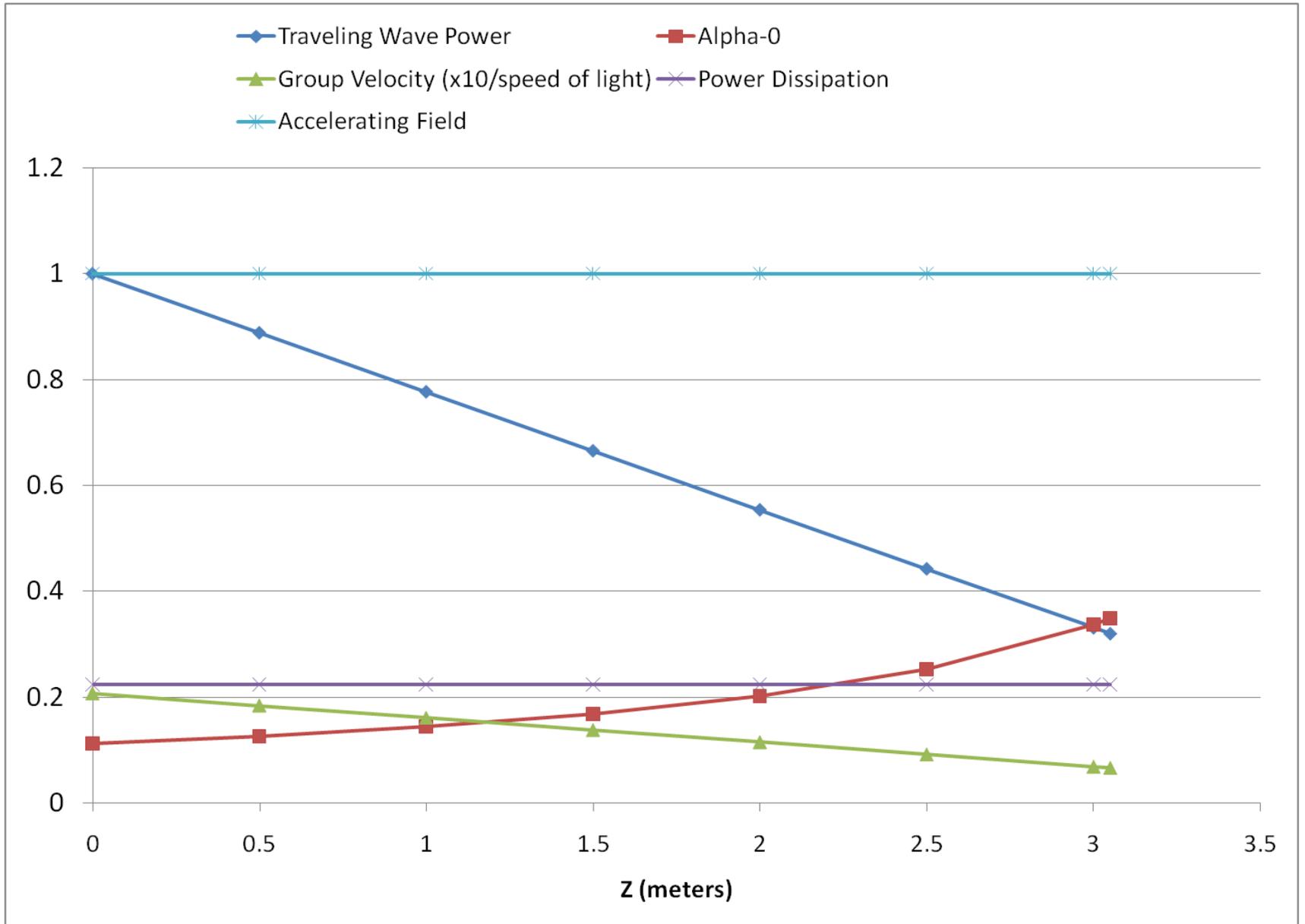
- The filling time is

$$t_F = \int_0^L \frac{dz}{v_g(z)} = \frac{Q}{\omega L} (1 - e^{-2\tau_0}) \int_0^L \frac{dz}{1 - (z/L)(1 - e^{-2\tau_0})} = \tau_0 \frac{2Q}{\omega}$$

- For typical parameters, the filling time is $\sim 1 \mu\text{sec}$, and the beam pulse is $1\text{-}2 \mu\text{sec}$.



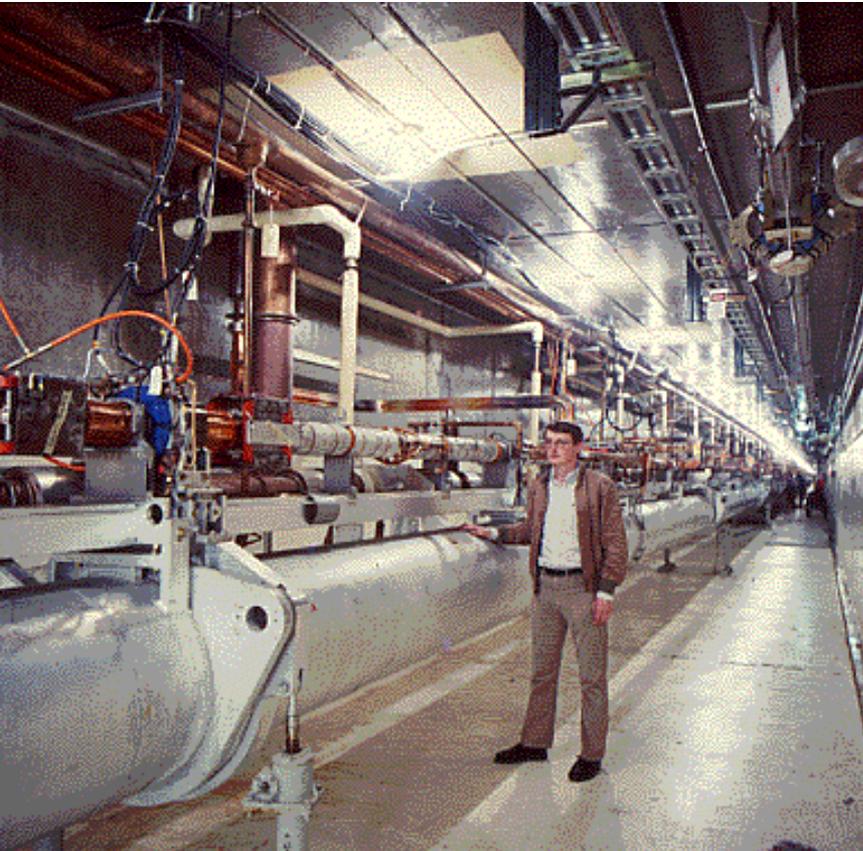
Constant Gradient Structure Parameters





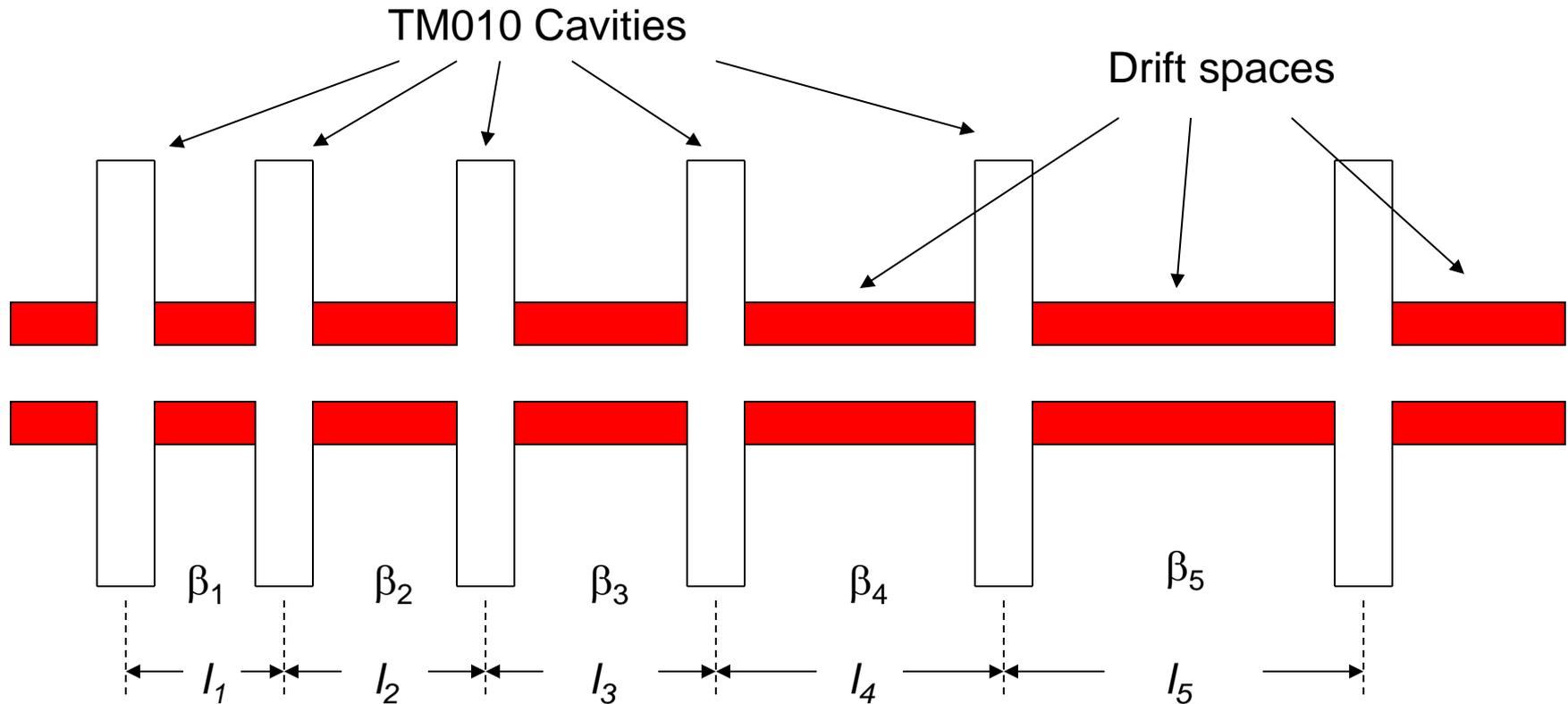
SLAC Linac

- Largest in the world.
Reached energies of 50 GeV.





Synchronicity condition in multicell RF structures



- Suppose we want a particle to arrive at the center of each gap at $\phi=0$. Then we would have to space the cavities so that the RF phase advanced by
 - 2π if the coupled cavity array was driven in zero-mode,
 - Or by π if the coupled cavity array was driven in pi-mode.



Synchronicity Condition

Zero-mode:

$$\phi = \omega t = \frac{2\pi c}{\lambda} t = \frac{2\pi c}{\lambda} \frac{l_n}{c\beta_n} = 2\pi$$

$$l_n = \beta_n \lambda$$

- RF gaps (cells) are spaced by $\beta\lambda$, which increases as the particle velocity increases.

Pi-mode:

$$\phi = \omega t = \frac{2\pi c}{\lambda} t = \frac{2\pi c}{\lambda} \frac{l_n}{c\beta_n} = \pi$$

$$l_n = \beta_n \lambda / 2$$

- RF gaps (cells) are spaced by $\beta\lambda/2$, which increases as the particle velocity increases.



Longitudinal Dynamics

- The drift space length between gaps is calculated for a particular particle with a very specific energy. This is the *reference* particle, or the *synchronous* particle.
- What happens to particles slightly faster or slower than the *synchronous* particle that the linac was designed to accelerate?
- Linacs are operated to provide *longitudinal focusing* to properly accelerate particles over a range in energies or arrival time.
- Slower particles arrive at the next gap later than the synchronous particle.
 - They experience a larger accelerating field.
- Faster particles arrive at the next gap earlier than the synchronous particle.
 - They experience a smaller accelerating field.

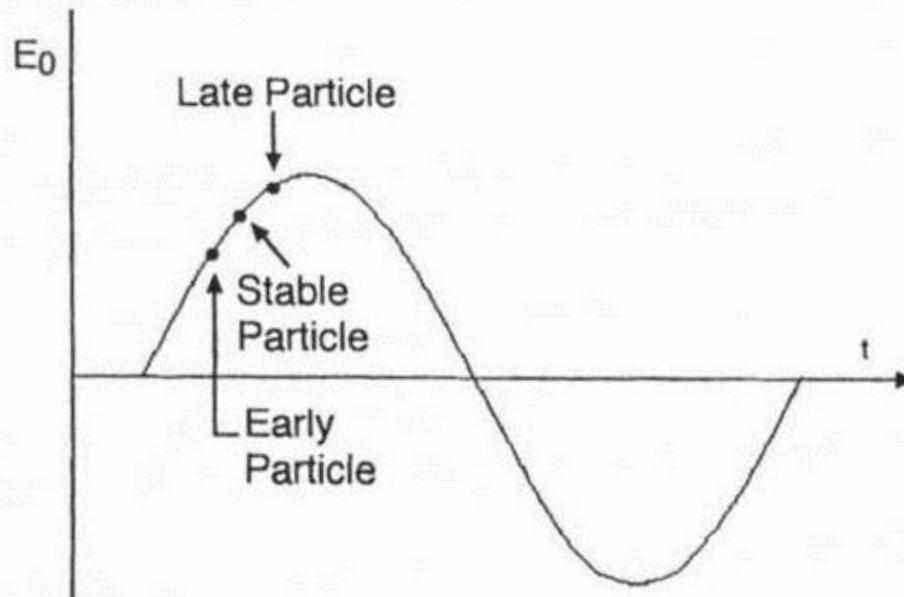


Figure 6.1. Stable phase.



Equations of Motion I

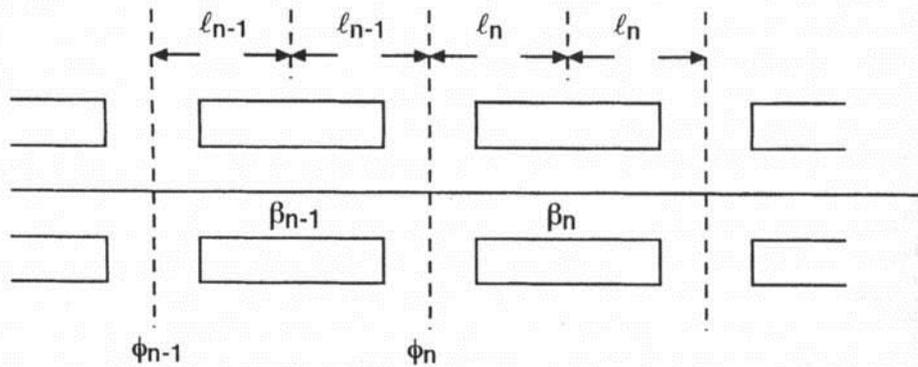


Figure 6.2. Accelerating cells for describing the longitudinal motion.

- Consider an array of accelerating cells with drift tubes and accelerating gaps.
- The array is designed at the n -th cell for a particle with synchronous phase, kinetic energy, and velocity ϕ_{sn} , W_{sn} , β_{sn} . Note that the synchronous phase is not zero!
- We express the phase, energy and velocity for an arbitrary particle in the n -th cell as ϕ_n , W_n , β_n .
- Assume that the particles receive a longitudinal kick at the geometric center of the cell, and drift freely to the center of the next cell.

- The half-cell length is

$$l_{n-1} = \frac{N\beta_{s,n-1}\lambda}{2}$$

where $N=1/2$ for Pi-mode and 1 for zero-mode.

- The cell length (center of one drift tube to center of next) is therefore

$$L_n = N(\beta_{s,n-1} + \beta_{s,n})\lambda / 2$$



Equations of Motion II

- The RF phase changes as the particle advances from one gap to the next according to

$$\phi_n = \phi_{n-1} + \omega \frac{2l_{n-1}}{\beta_{n-1}c} + \begin{cases} \pi & \text{\textbf{\pi mode}} \\ 0 & \text{\textbf{0 mode}} \end{cases}$$

- The phase change during the time an arbitrary particle travels from gap n-1 to gap n, relative to the synchronous particle is

$$\Delta(\phi - \phi_s)_n = \Delta\phi_n - \Delta\phi_{s,n} = 2\pi N \beta_{s,n-1} \left[\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right] \cong -2\pi N \beta_{s,n-1} \frac{\delta\beta_{n-1}}{\beta_{s,n-1}^2}$$

where we have used

$$\frac{1}{\beta} - \frac{1}{\beta_s} = \frac{1}{\beta_s + \delta\beta} - \frac{1}{\beta_s} \cong -\frac{\delta\beta}{\beta_s^2}, \text{ for } \delta\beta \ll 1$$

- Using

$$\delta\beta = \frac{\delta W}{mc^2 \gamma_s^3 \beta_s}$$

- We get

$$\Delta(\phi - \phi_s)_n = -2\pi N \frac{(W_{n-1} - W_{s,n-1})}{mc^2 \gamma_{s,n-1}^3 \beta_{s,n-1}^2}$$



Equations of Motion III

- Next, derive the difference in kinetic energies of the arbitrary particle and the synchronous particle:

$$\Delta(W - W_s)_n = qE_0 T L_n (\cos \phi_n - \cos \phi_{s,n})$$

- To figure out the dynamics, we could track particles through gaps on a computer using these difference equations.
- To get a feeling for the dynamics “on paper”, we can convert these difference equations to differential equations by replacing the discrete action of the fields with a continuous field.
- So we replace

$$\Delta(\phi - \phi_s) \rightarrow \frac{d(\phi - \phi_s)}{dn} \quad \Delta(W - W_s) \rightarrow \frac{d(W - W_s)}{dn} \quad n = \frac{s}{N\beta_s \lambda}$$

- giving

$$\gamma_s^3 \beta_s^3 \frac{d(\phi - \phi_s)}{ds} = -2\pi \frac{W - W_s}{mc^2 \lambda} \quad \frac{d(W - W_s)}{ds} = qE_0 T (\cos \phi - \cos \phi_s)$$



Equations of Motion IV

- Assume acceleration rate is small, and that E_0T , ϕ_s and β_s are constant.
- We arrive at the equations of motion:

$$w' = \frac{dw}{ds} = B(\cos \phi - \cos \phi_s) \quad \text{and} \quad \phi' = \frac{d\phi}{ds} = -Aw$$

$$\text{with } w = \frac{W - W_s}{mc^2} \quad \text{and} \quad A = \frac{2\pi}{\beta_s^3 \gamma_s^3 \lambda} \quad B = \frac{qE_0T}{mc^2}$$

$$\frac{d^2\phi}{ds^2} = -AB(\cos \phi - \cos \phi_s)$$

- Finally

$$\frac{Aw^2}{2} + B(\sin \phi - \phi \cos \phi_s) = H_\phi$$

$$\frac{1}{2}Aw^2 + V_\phi = H_\phi$$

- Where V is the potential energy term, and H (the Hamiltonian) is total energy. Technically, $-f$ is the canonical conjugate of w .



Stable RF Bucket

- There is a potential well when $-\pi < \phi_s < 0$.
- There is acceleration for $-\pi/2 < \phi_s < \pi/2$.
- The stable region for phase motion is $\phi_2 < \phi < -\phi_s$.
- The “separatrix” defines the area within which the trajectories are stable.
- The stable area is called the “bucket”.
- Stable motion means that particles follow a trajectory about the stable phase, with constant amplitude given by H_ϕ .

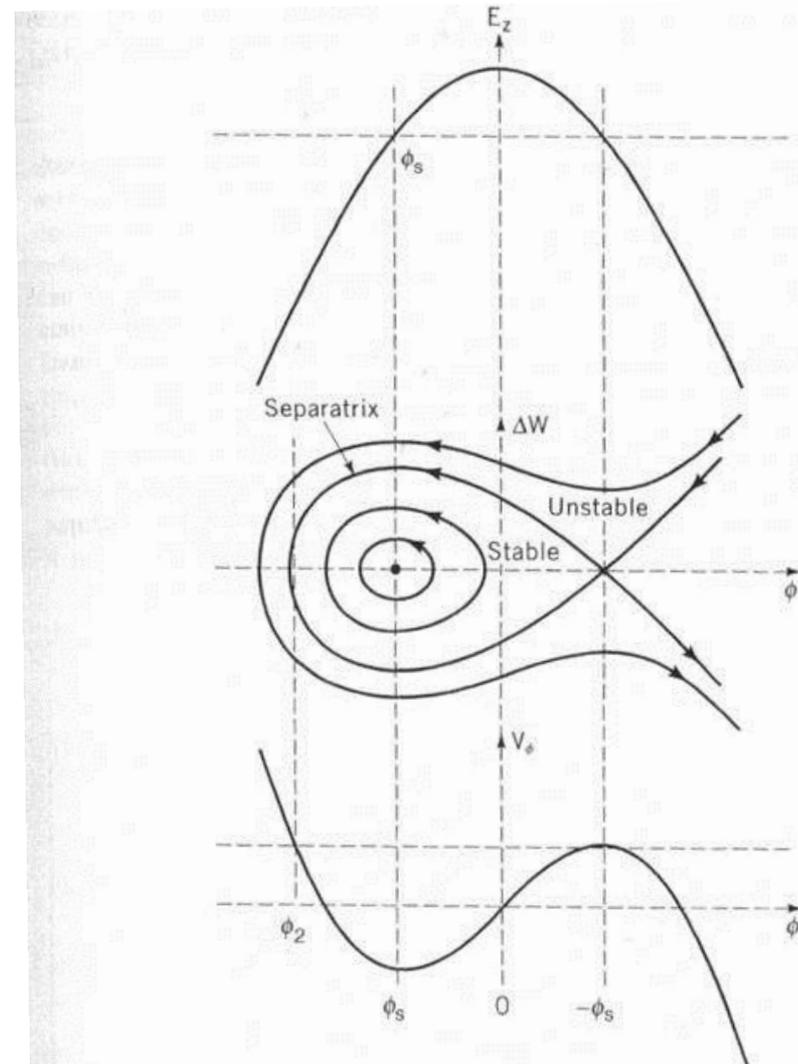


Figure 6.3. At the top, the accelerating field is shown as a cosine function of the phase; the synchronous phase ϕ_s is shown as a negative number, which lies earlier than the crest where the field is rising in time. The middle plot shows some longitudinal phase-space trajectories, including the separatrix, the limiting stable trajectory, which passes through the unstable fixed point at $\Delta W = 0$, and $\phi = -\phi_s$. The stable fixed point lies at $\Delta W = 0$ and $\phi = \phi_s$, where the longitudinal potential well has its minimum, as shown in the bottom plot.



Hamiltonian and Separatrix Parameters

- We can calculate the Hamiltonian to complete the discussion.
- At the potential maximum where, $\phi = -\phi_s$, $\phi' = 0$ and $w=0$

$$H_\phi = B(\sin(-\phi_s) - (-\phi_s \cos \phi_s))$$

- The points on the separatrix must therefore satisfy

$$\frac{Aw^2}{2} + B(\sin \phi - \phi \cos \phi_s) = -B(\sin \phi_s - \phi_s \cos \phi_s)$$

- We can calculate the “size” of the separatrix. We will do the energy width. The maximum energy width corresponds to $\phi = \phi_s$

$$\frac{Aw_{\max}^2}{2} + B(\sin \phi_s - \phi_s \cos \phi_s) = -B(\sin \phi_s - \phi_s \cos \phi_s)$$

- Giving for the energy half-width of the separatrix. The **energy acceptance** is twice this value:

$$w_{\max} = \frac{\Delta W_{\max}}{mc^2} = \sqrt{\frac{2qE_0 T \beta_s^3 \gamma_s^3 \lambda}{\pi mc^2}} (\phi_s \cos \phi_s - \sin \phi_s)$$



Phase Width

- The maximum ***phase width*** is determined from the two solutions for $w=0$. One solution is $\phi_1 = -\phi_s$. The other solution ϕ_2 is given by

$$\sin \phi_2 - \phi_2 \cos \phi_s = \phi_s \cos \phi_s - \sin \phi_s$$

- The total phase width is $\Psi = -\phi_s - \phi_2$
- The phase width is zero at $\phi_s = 0$ and maximum at $\phi_s = -\pi/2$, giving $\psi = 2\pi$ (see Wangler figure 6.4).



Small Amplitude Oscillations

- Look at small amplitude oscillations. Letting $\phi - \phi_s$ be small,

$$(\dot{\phi} - \dot{\phi}_s) + AB \sin(-\phi_s)(\phi - \phi_s) = 0$$

- This is an equation for simple harmonic motion with an angular frequency given by

$$\omega_t^2 = \frac{\omega^2 q E_0 T \lambda \sin(-\phi_s)}{2\pi m c^2 \gamma_s^3 \beta_s}$$

- Note that as the beam becomes relativistic, the frequency goes to zero.
- From the equation of motion we can calculate the trajectory of a particle:

$$\frac{w^2}{w_0^2} + \frac{(\phi - \phi_s)^2}{(\Delta\phi_0)^2} = 1 \quad w_0 = \frac{\Delta W}{m c^2} = \sqrt{q E_0 T \beta_s^3 \gamma_s^3 \lambda \sin(-\phi_s) \Delta\phi_0^2 / 2\pi m c^2}$$

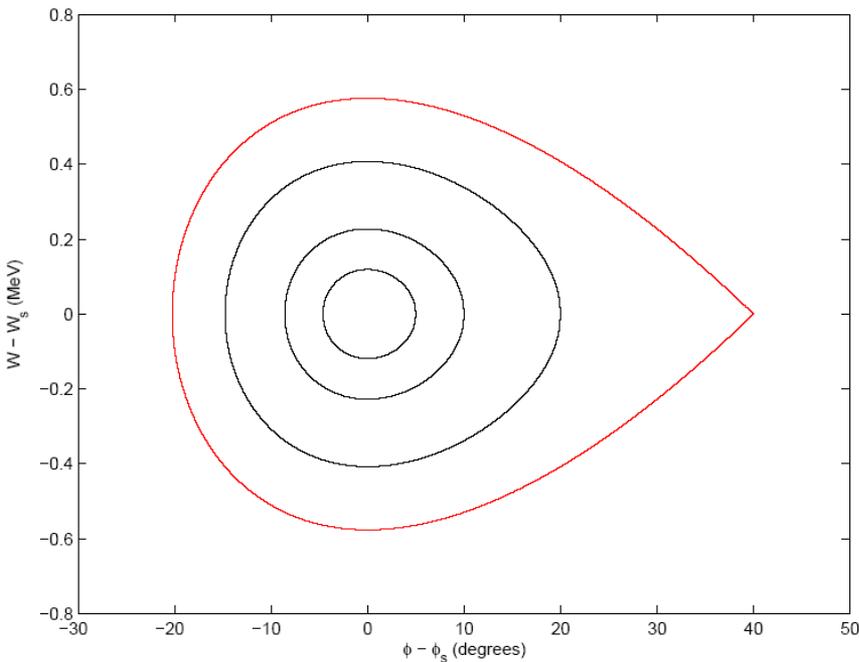
- This is the equation of an ellipse in $w, \phi - \phi_s$ *phase space*.
- Particles on a particular ellipse circulate indefinitely on that trajectory.



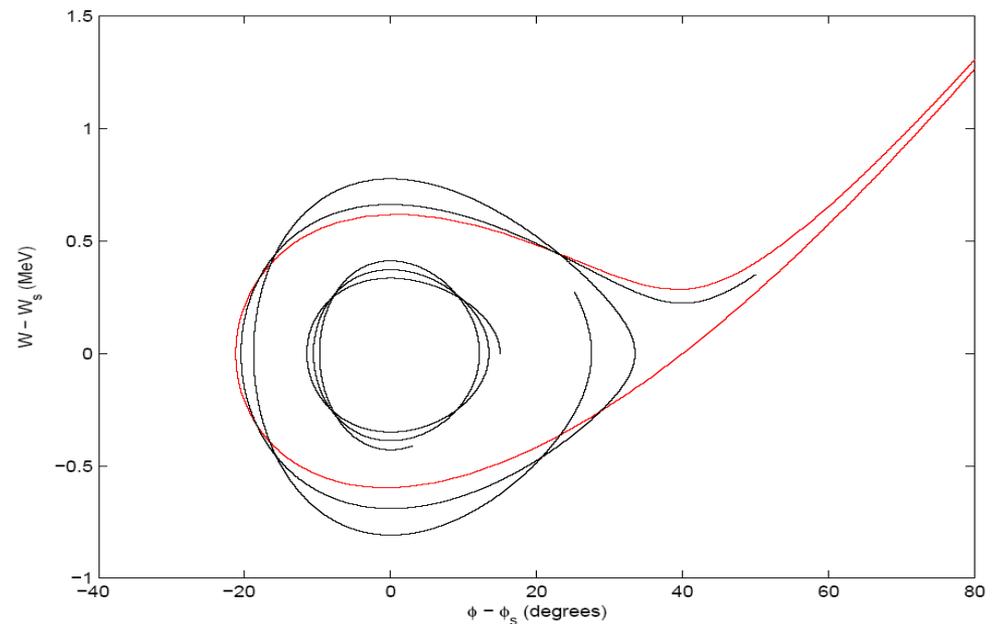
Longitudinal Phase Space Motion

- We studied the approximation of small acceleration rate, and constant velocity, synchronous phase, etc.
- In a real linac, the velocity increases, and the phase space motion and separatrix becomes more complicated.
- The “acceptance” takes a shape called the “golf-club”.

$\beta\gamma = \text{const}$



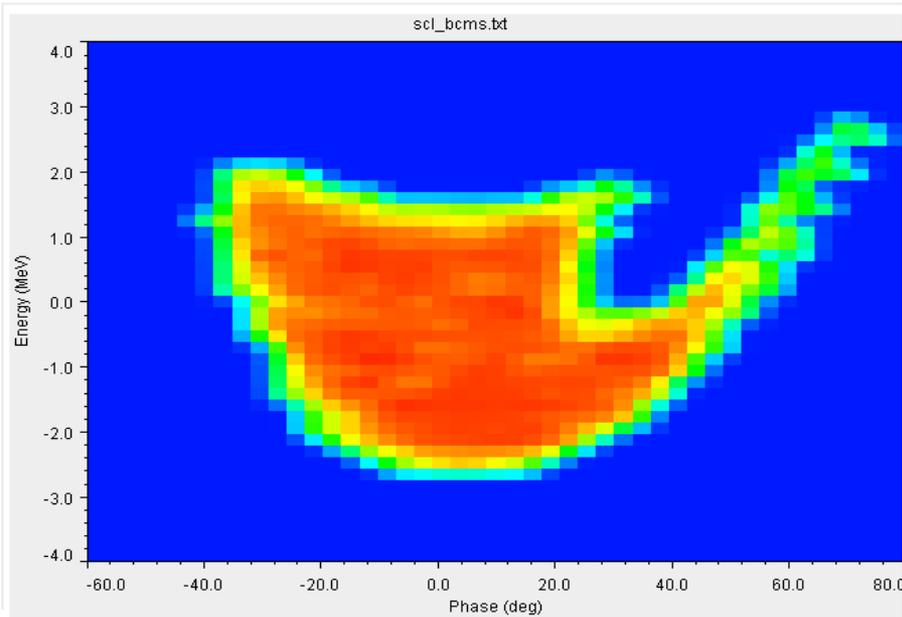
$\beta\gamma \neq \text{const}$



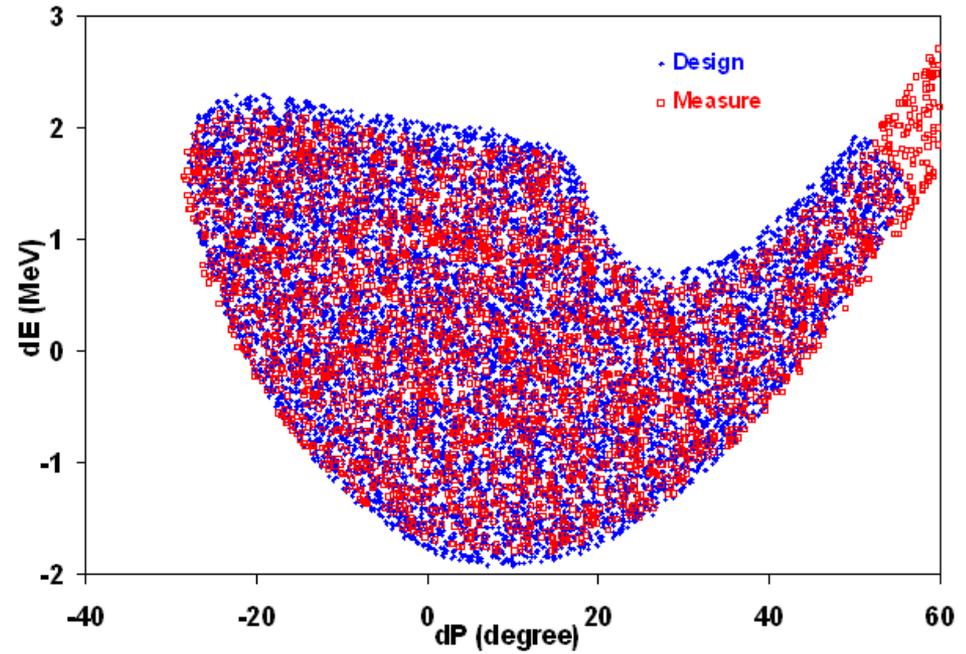


Measurement of SNS SC Linac Acceptance (Y. Zhang)

Measurement



Simulation





Adiabatic Phase Damping

- **Louville's theorem:**

The density in phase space of non-interacting particles in a conservative or Hamiltonian system measured along the trajectory of a particle is invariant.

- Or, if you prefer: **phase space area is conserved.**

- Area of ellipse:

$$\text{Area} = \pi \Delta\phi_0 \Delta W_0$$

- Which gives

$$\Delta\phi_0 = \frac{\text{const}}{(\beta_s \gamma_s)^{3/4}} \quad \Delta W_0 = \text{const} \times (\beta_s \gamma_s)^{3/4}$$

- Since area is conserved an initial distribution with phase width $(\Delta\phi)_i$ acquired a smaller phase width after acceleration:

$$\frac{(\Delta\phi_0)_f}{(\Delta\phi_0)_i} = \frac{(\beta\gamma)_i^{3/4}}{(\beta\gamma)_f^{3/4}}$$

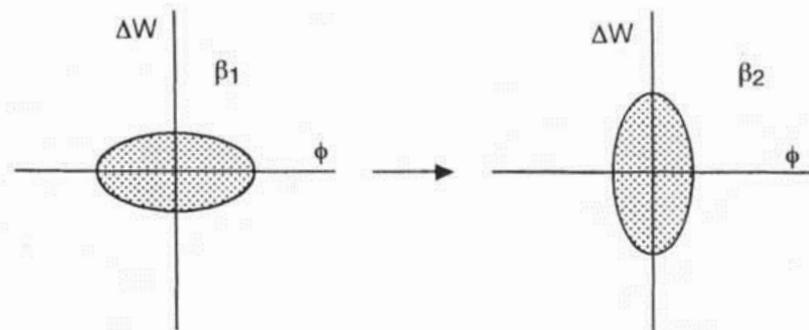


Figure 6.8. Phase damping of a longitudinal beam ellipse caused by acceleration. The phase width of the beam decreases and the energy width increases while the total area remains constant.



The End

- That concludes our whirlwind tour of Linear Accelerators
- Now, on to Rings.....



Extra Slides



Longitudinal Dynamics: Real data from SNS Drift Tube Linac

Simulated DTL1
Acceptance

- Longitudinal “Acceptance Scan”

