RF Acceleration in Linacs

Part 2
Outline

• Traveling-wave linear accelerators
• Longitudinal beam dynamics

• Material from
  – *Wiedemann, Chapters 15, 6*
  – *Wangler, Chapters 3, 6*
Can we accelerate particles by transporting EM waves in a waveguide?

Consider a cylindrical geometry. The wave equation in cylindrical coordinates for the z field component is

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{2E_z}{z^2} + \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2} = 0
\]

Assume the EM wave propagates in the Z direction. Let’s look for a solution that has a finite electric field in that same direction:

\[
E_z = E_z (r, \phi, z, t) = E_0 (r, \phi) \cos(k_z z - \omega t)
\]

The azimuthal dependence must be repetitive in \( \phi \):

\[
E_z = R(r) \cos(n\phi) \cos(k_z z - \omega t)
\]

The wave equation yields:

\[
\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r} \frac{\partial R(r)}{\partial r} + \left( \frac{\omega^2}{c^2} - k_z^2 - \frac{n^2}{r^2} \right) R(r) = 0
\]
Cylindrical Waveguides

• Which results in the following differential equation for \( R(r) \) (with \( x = k_c r \))

\[
\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{n^2}{x^2}\right)R = 0
\]

• The solutions to this equation are Bessel functions of order \( n \), \( J_n(k_c r) \), which look like this:
Cylindrical Waveguides

• The solution is:

\[ E_z = J_n(k_c r) \cos(n\phi) \cos(k_z z - \omega t) \]

• The boundary conditions require that

\[ E_z(r = a) = 0 \]

• Which requires that

\[ J_n(k_c a) = 0 \text{ for all } n \]

• Label the \( n \)-th zero of \( J_m \): \( J_m(x_{mn}) = 0 \)

• For \( m=0, \ x_{01} = 2.405 \)

\[ \frac{\omega^2}{c^2} = k_c^2 + k_z^2 = \left( \frac{2.405}{a} \right)^2 + k_z^2 \]
• The cylindrically symmetric waveguide has

\[ k_0^2 = k_c^2 + k_z^2 \quad \text{and} \quad \omega^2 = \omega_c^2 + (k_z c)^2 \]

• A plot of \( \omega \) vs. \( k \) is a hyperbola, called the Dispersion Curve.

Two cases:
• \( \omega > \omega_c \): \( k_z \) is a real number and the wave propagates.
• \( \omega < \omega_c \): \( k_z \) is an imaginary number and the wave decays exponentially with distance.
• Only EM waves with frequency above cutoff are transported!
Phase Velocity and Group Velocity

• The propagating wave solution has

\[ E_z = E_0(r, z) \cos(\phi) \quad \phi = k_z z - \omega t \]

• A point of constant \( \phi \) propagates with a velocity, called the phase velocity,

\[ v_p = \frac{\omega}{k_z} \]

• The electromagnetic wave in cylindrical waveguide has phase velocity that is faster than the speed of light:

\[ v_p = \frac{c}{\sqrt{1 - \frac{\omega^2}{\omega_c^2}}} \]

This won’t work to accelerate particles. We need to modify the phase velocity to something smaller than the speed of light to accelerate particles.

• The group velocity is the velocity of energy flow:

\[ P_{RF} = v_g U \]

• And is given by:

\[ v_g = \frac{d\omega}{dk} \]
Recall that in the cylindrical waveguide, the electromagnetic wave has phase velocity that is faster than the speed of light:

\[
v_p = \frac{c}{\sqrt{1 - \frac{\omega_c^2}{\omega^2}}} > c
\]

This won’t work to accelerate particles. We need to modify the phase velocity to the speed of light (or slower) to accelerate particles in a traveling wave.

Imagine a situation where the EM wave phase velocity equals the particle velocity.

Then the particle “rides the wave”.

A “disk-loaded waveguide” can be made to have a phase velocity equal to the speed of light. These structures are often used to accelerate electrons.

The best and largest example of such an accelerator is the SLAC two-mile long linac.
Disk-loaded waveguide structure

KEK
Define

- $E_a$: longitudinal accelerating field amplitude
- $U$: stored energy per unit length
- $P_w$: traveling wave power
- $dP_w/dz$: power dissipation per unit length
- Shunt impedance per unit length

We have

$$r_L = \frac{E_a^2}{-dP_w/dz}$$

We have

$$Q = \frac{\omega U}{-dP_w/dz}$$

$$P_w = v_g U$$

$$E_a^2 = \omega r_L P_w / Q v_g$$

$$\frac{dP_w}{dz} = -\frac{\omega}{Q v_g} P_w = -2\alpha_0 P_w$$

We have two choices for the accelerating structure, considered now in turn:
Constant Impedance Traveling Wave Structure

- Consider a disk-loaded waveguide with uniform cell geometry along the length, then $Q$, $v_g$, $r_L$, $\alpha_0$ are independent of $z$:

$$P_w(z) = e^{-2\alpha_0 z}$$

- Power decays exponentially along the length of the structure.
- The Electric field amplitude is

$$dE_a / dz = -\alpha_0 E_a$$

$$E_a(z) = E_0 e^{-\alpha_0 z}$$

- At the end of a waveguide of length $L$

$$P_w(L) = P_0 e^{-2\tau_0} \quad E_a(L) = E_0 e^{-\tau_0}$$

$$\tau_0 = \alpha_0 L = \frac{\omega L}{2Q v_g}$$

- The energy gain is

$$\Delta W = q \cos \phi \int_0^L E_a(z) dz = qE_0 L \frac{1 - e^{-\tau_0}}{\tau_0} \cos \phi$$

$$\Delta W = q \sqrt{2r_L P_0 L} \frac{1 - e^{-\tau_0}}{\sqrt{\tau_0}} \cos \phi$$
A more common design keeps the gradient constant over the length, which requires that the attenuation $\alpha_0$ depend on $z$

$$\frac{dP_w}{dz} = -2\alpha_0(z)P_w$$

Which can be integrated to yield

$$P_w(z) = P_0 \left[ 1 - \frac{z}{L} (1 - e^{-2\tau_0}) \right]$$

The attenuation factor is

$$\alpha_0(z) = \frac{1}{2L} \frac{1 - e^{-2\tau_0}}{1 - (z/L)(1 - e^{-2\tau_0})}$$

The energy gain is

$$\Delta W = q \cos \phi \int_0^L E_a(z) dz = qE_0L \cos \phi$$

$$\Delta W = q \sqrt{r_L P_0 L (1 - e^{-2\tau_0}) \cos \phi}$$

To achieve a constant gradient, the SLAC linac structure tapers from a radius of 4.2 to 4.1 cm, and the iris radii taper from 1.3 to 1.0 cm over 3 meters.

• The group velocity is

\[ v_g(z) = \frac{\omega}{2Q\alpha_0(z)} = \frac{\omega L}{Q} \frac{1 - (z/L)(1 - e^{-2\tau_0})}{1 - e^{-2\tau_0}} \]

• The filling time is

\[ t_F = \int_0^L \frac{dz}{v_g(z)} = \frac{Q}{\omega L} (1 - e^{-2\tau_0}) \int_0^L \frac{dz}{1 - (z/L)(1 - e^{-2\tau_0})} = \tau_0 \frac{2Q}{\omega} \]

• For typical parameters, the filling time is \(~1\ \mu\text{sec},\) and the beam pulse is \(1-2\ \mu\text{sec}.\)
Constant Gradient Structure Parameters

- Traveling Wave Power
- Alpha-0
- Group Velocity (x10/speed of light)
- Power Dissipation
- Accelerating Field

Z (meters)
• Largest in the world. Reached energies of 50 GeV.
Suppose we want a particle to arrive at the center of each gap at $\phi=0$. Then we would have to space the cavities so that the RF phase advanced by
- $2\pi$ if the coupled cavity array was driven in zero-mode,
- Or by $\pi$ if the coupled cavity array was driven in pi-mode.
Synchronicity Condition

**Zero-mode:**

\[ \phi = \omega t = \frac{2\pi c}{\lambda} t = \frac{2\pi c}{\lambda} \frac{l_n}{c \beta_n} = 2\pi \]

\[ l_n = \beta_n \lambda \]

- RF gaps (cells) are spaced by \( \beta \lambda \), which increases as the particle velocity increases.

**Pi-mode:**

\[ \phi = \omega t = \frac{2\pi c}{\lambda} t = \frac{2\pi c}{\lambda} \frac{l_n}{c \beta_n} = \pi \]

\[ l_n = \beta_n \lambda / 2 \]

- RF gaps (cells) are spaced by \( \beta \lambda / 2 \), which increases as the particle velocity increases.
Longitudinal Dynamics

- The drift space length between gaps is calculated for a particular particle with a very specific energy. This is the reference particle, or the synchronous particle.
- What happens to particles slightly faster or slower than the synchronous particle that the linac was designed to accelerate?
- Linacs are operated to provide longitudinal focusing to properly accelerate particles over a range in energies or arrival time.
- Slower particles arrive at the next gap later than the synchronous particle.
  - They experience a larger accelerating field.
- Faster particles arrive at the next gap earlier than the synchronous particle.
  - They experience a smaller accelerating field.
• Consider an array of accelerating cells with drift tubes and accelerating gaps.
• The array is designed at the n-th cell for a particle with synchronous phase, kinetic energy, and velocity $\phi_{sn}$, $W_{sn}$, $\beta_{sn}$. Note that the synchronous phase is not zero!
• We express the phase, energy and velocity for an arbitrary particle in the n-th cell as $\phi_n$, $W_n$, $\beta_n$.
• Assume that the particles receive a longitudinal kick at the geometric center of the cell, and drift freely to the center of the next cell.
• The half-cell length is
  \[ l_{n-1} = \frac{N\beta_{s,n-1}\lambda}{2} \]
  where N=1/2 for Pi-mode and 1 for zero-mode.
• The cell length (center of one drift tube to center of next) is therefore
  \[ L_n = N(\beta_{s,n-1} + \beta_{s,n})\lambda / 2 \]
• The RF phase changes as the particle advances from one gap to the next according to

\[ \phi_n = \phi_{n-1} + \omega \frac{2l_{n-1}}{\beta_{n-1} c} + \begin{cases} \pi & \text{\textit{\pi mode}} \\ 0 & \text{\textit{0 mode}} \end{cases} \]

• The phase change during the time an arbitrary particle travels from gap \( n-1 \) to gap \( n \), relative to the synchronous particle is

\[ \Delta(\phi - \phi_s)_n = \Delta \phi_n - \Delta \phi_{s,n} = 2\pi N \beta_{s,n-1} \left[ \frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right] \approx -2\pi N \beta_{s,n-1} \frac{\delta \beta_{n-1}}{\beta_{s,n-1}^2} \]

where we have used

\[ \frac{1}{\beta} - \frac{1}{\beta_s} = \frac{1}{\beta_s + \delta \beta} - \frac{1}{\beta_s} \approx -\frac{\delta \beta}{\beta_s^2} \text{, for } \delta \beta \ll 1 \]

• Using

\[ \delta \beta = \frac{\delta W}{mc^2 \gamma^3_s \beta_s} \]

• We get

\[ \Delta(\phi - \phi_s)_n = -2\pi N \frac{(W_{n-1} - W_{s,n-1})}{mc^2 \gamma^3_{s,n-1} \beta_{s,n-1}^2} \]
Next, derive the difference in kinetic energies of the arbitrary particle and the synchronous particle:

\[ \Delta(W - W_s)_n = qE_0 TL_n (\cos \phi_n - \cos \phi_{s,n}) \]

To figure out the dynamics, we could track particles through gaps on a computer using these difference equations.

To get a feeling for the dynamics “on paper”, we can convert these difference equations to differential equations by replacing the discrete action of the fields with a continuous field.

So we replace

\[ \Delta(\phi - \phi_s) \rightarrow \frac{d(\phi - \phi_s)}{dn} \quad \Delta(W - W_s) \rightarrow \frac{d(W - W_s)}{dn} \quad n = \frac{s}{N\beta_s \lambda} \]

giving

\[ \gamma_s^3 \beta_s^3 \frac{d(\phi - \phi_s)}{ds} = -2\pi \frac{W - W_s}{mc^2 \lambda} \quad \frac{d(W - W_s)}{ds} = qE_0 T (\cos \phi - \cos \phi_s) \]
Equations of Motion IV

• Assume acceleration rate is small, and that $E_0T$, $\phi_s$ and $\beta_s$ are constant.

• We arrive at the equations of motion:

$$w' = \frac{dw}{ds} = B(\cos \phi - \cos \phi_s) \quad \text{and} \quad \phi' = \frac{d\phi}{ds} = -Aw$$

with \[ w = \frac{W - W_s}{mc^2} \quad \text{and} \quad A = \frac{2\pi}{\beta_s^3 \gamma_s^3 \lambda} \quad B = \frac{qE_0T}{mc^2} \]

$$\frac{d^2\phi}{ds^2} = -AB(\cos \phi - \cos \phi_s)$$

• Finally

$$\frac{Aw^2}{2} + B(\sin \phi - \phi \cos \phi_s) = H_\phi$$

$$\frac{1}{2} Aw^2 + V_\phi = H_\phi$$

• Where $V$ is the potential energy term, and $H$ (the Hamiltonian) is total energy. Technically, $w$ is the canonical conjugate of $w$. 
• There is a potential well when $-\pi < \phi_s < 0$.
• There is acceleration for $-\pi/2 < \phi_s < \pi/2$.
• The stable region for phase motion is $\phi_2 < \phi < -\phi_s$.
• The “separatrix” defines the area within which the trajectories are stable.
• The stable area is called the “bucket”.
• Stable motion means that particles follow a trajectory about the stable phase, with constant amplitude given by $H_\phi$. 
Hamiltonian and Separatrix Parameters

• We can calculate the Hamiltonian to complete the discussion.
• At the potential maximum where, $\phi = -\phi_s$, $\phi' = 0$ and $w=0$

$$H_\phi = B(\sin(-\phi_s) - (-\phi_s \cos \phi_s))$$

• The points on the separatrix must therefore satisfy

$$\frac{Aw^2}{2} + B(\sin \phi - \phi \cos \phi_s) = -B(\sin \phi_s - \phi_s \cos \phi_s)$$

• We can calculate the “size” of the separatrix. We will do the energy width. The maximum energy width corresponds to $\phi = \phi_s$

$$\frac{Aw_{\text{max}}^2}{2} + B(\sin \phi_s - \phi_s \cos \phi_s) = -B(\sin \phi_s - \phi_s \cos \phi_s)$$

• Giving for the energy half-width of the separatrix. The energy acceptance is twice this value:

$$w_{\text{max}} = \frac{\Delta W_{\text{max}}}{mc^2} = \sqrt{\frac{2qE_0 T \beta_s^3 \gamma_s^3 \lambda}{\pi mc^2}} (\phi_s \cos \phi_s - \sin \phi_s)$$
• The maximum *phase width* is determined from the two solutions for \( w=0 \). One solution is \( \phi_1 = -\phi_s \). The other solution \( \phi_2 \) is given by

\[
\sin \phi_2 - \phi_2 \cos \phi_s = \phi_s \cos \phi_s - \sin \phi_s
\]

• The total phase width is \( \Psi = -\phi_s - \phi_2 \)

• The phase width is zero at \( \phi_s = 0 \) and maximum at \( \phi_s = -\pi/2 \), giving \( \psi = 2\pi \) (see Wangler figure 6.4).
Small Amplitude Oscillations

- Look at small amplitude oscillations. Letting $\phi - \phi_s$ be small,
  $$ (\phi - \phi_s) + AB \sin(\phi - \phi_s) = 0 $$

- This is an equation for simple harmonic motion with an angular frequency given by
  $$ \omega_i^2 = \frac{\omega^2 qE_0 T \lambda \sin(-\phi_s)}{2\pi mc^2 \gamma_s^3 \beta_s} $$

- Note that as the beam becomes relativistic, the frequency goes to zero.

- From the equation of motion we can calculate the trajectory of a particle:
  $$ \frac{w^2}{w_0^2} + \frac{(\phi - \phi_s)^2}{(\Delta \phi_0)^2} = 1 \quad w_0 = \frac{\Delta W}{mc^2} = \sqrt{qE_0 T \beta_s^3 \gamma_s^3 \lambda \sin(-\phi_s) \Delta \phi_0^2 / 2\pi mc^2} $$

- This is the equation of an ellipse in $w$, $\phi - \phi_s$ phase space.

- Particles on a particular ellipse circulate indefinitely on that trajectory.
- We studied the approximation of small acceleration rate, and constant velocity, synchronous phase, etc.
- In a real linac, the velocity increases, and the phase space motion and separatrix becomes more complicated.
- The “acceptance” takes a shape called the “golf-club”.

\[ \beta \gamma = \text{const} \quad \beta \gamma \neq \text{const} \]
Measurement of SNS SC Linac Acceptance (Y. Zhang)

Measurement

Simulation

- Design
- Measure
Adiabatic Phase Damping

- Louiville’s theorem:
  The density in phase space of non-interacting particles in a conservative or Hamiltonian system measured along the trajectory of a particle is invariant.

- Or, if you prefer: phase space area is conserved.

- Area of ellipse:
  \[ \text{Area} = \pi \Delta \phi_0 \Delta W_0 \]

- Which gives
  \[ \Delta \phi_0 = \frac{\text{const}}{(\beta_s \gamma_s)^{3/4}} \]
  \[ \Delta W_0 = \text{const} \times (\beta_s \gamma_s)^{3/4} \]

- Since area is conserved an initial distribution with phase width \((\Delta \phi)_i\) acquired a smaller phase width after acceleration:
  \[ \frac{(\Delta \phi_0)_f}{(\Delta \phi_0)_i} = \frac{\beta \gamma_i^{3/4}}{\beta \gamma_f^{3/4}} \]

Figure 6.8. Phase damping of a longitudinal beam ellipse caused by acceleration. The phase width of the beam decreases and the energy width increases while the total area remains constant.
That concludes our whirlwind tour of Linear Accelerators

Now, on to Rings…. 
Longitudinal Dynamics: Real data from SNS Drift Tube Linac

- Longitudinal “Acceptance Scan”

Simulated DTL1 Acceptance

Data

FWHM=24 deg