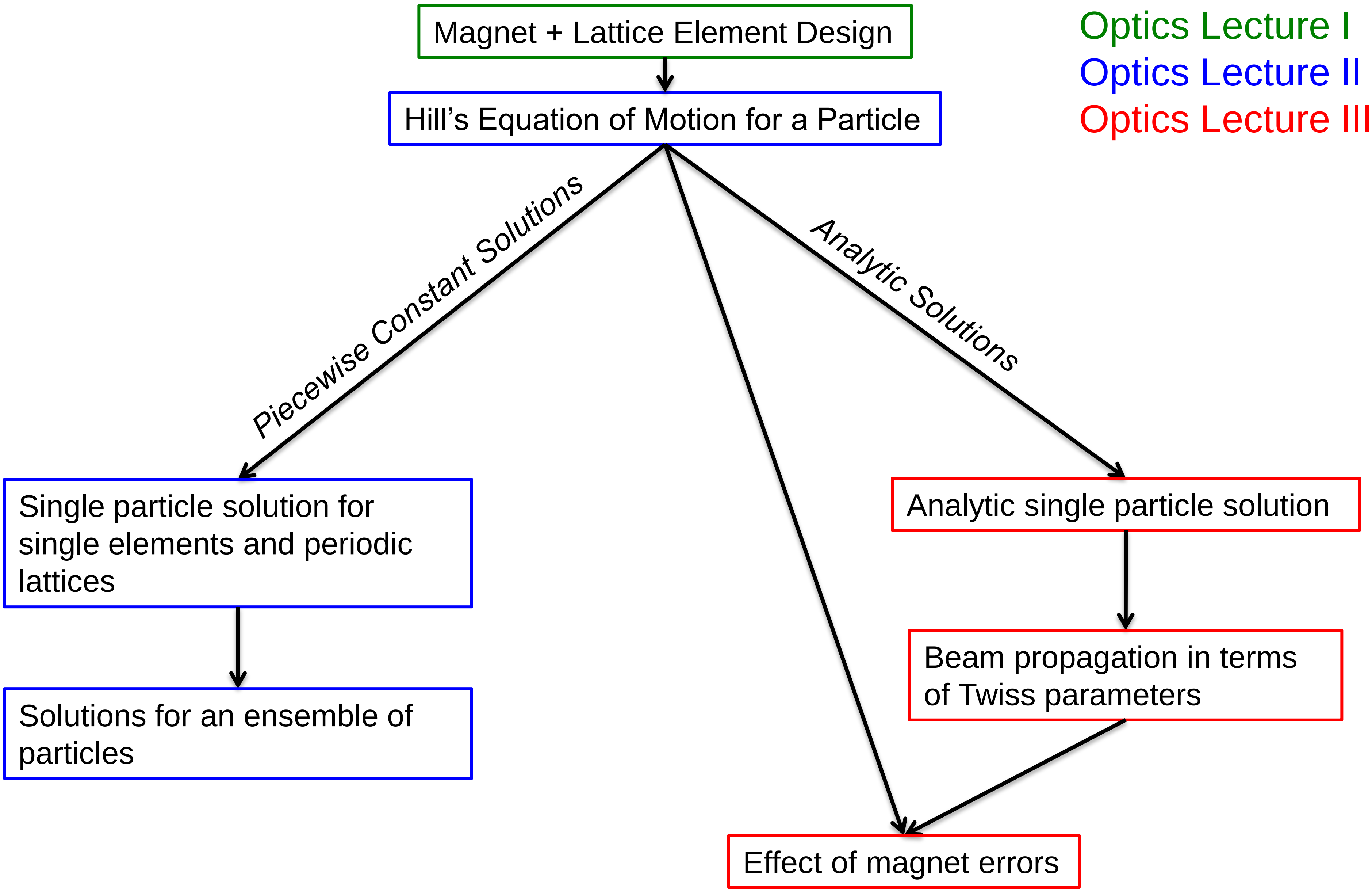
Lecture 6

Transverse Beam Optics, Part II

Sarah Cousineau, Jeff Holmes, Robert Potts,
Yan Zhang

USPAS
January, 2014

Map of Optics Lectures I - III



Goals for this Lecture

PART I

Single Particle Transverse Equation of Motion and the Piece-wise Constant Solutions

Phase Space and Units

In transverse particle dynamics, we are concerned with the effect of external magnetic fields on the phase space coordinates of a particle or beam. We call the phase space coordinates (u, u'), where (u, u') can be either (x, x') or (y, y').

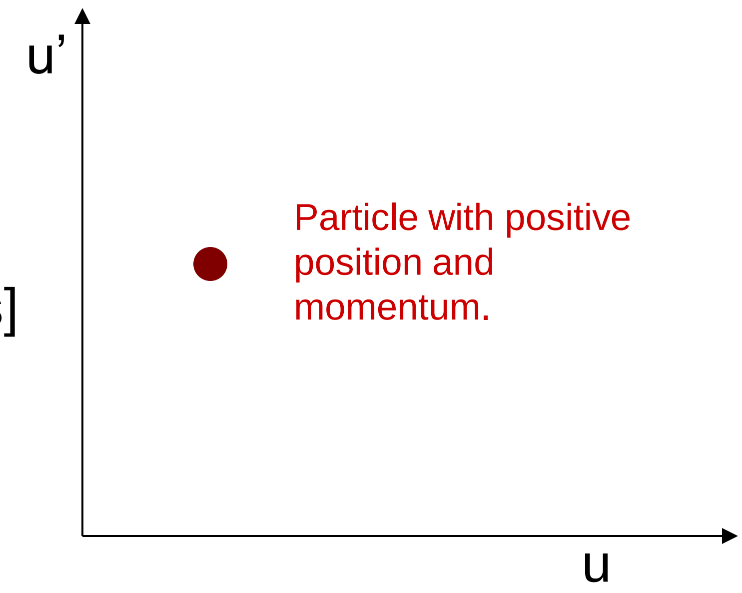
Coordinates and units:

s distance along reference trajectory, [m], ("time" coordinate)

$u \ (x \text{ or } y)$ position [meters]

$u' = \frac{du}{ds} \quad \left(\frac{dx}{ds} \text{ or } \frac{dy}{ds} \right)$ transverse momentum [radians]

$u'' = \frac{d^2u}{ds^2}$ transverse acceleration [m^{-1}]



What does the phase space path of a harmonic oscillator look like? (Mass on a pendulum, child on a swing, etc)?

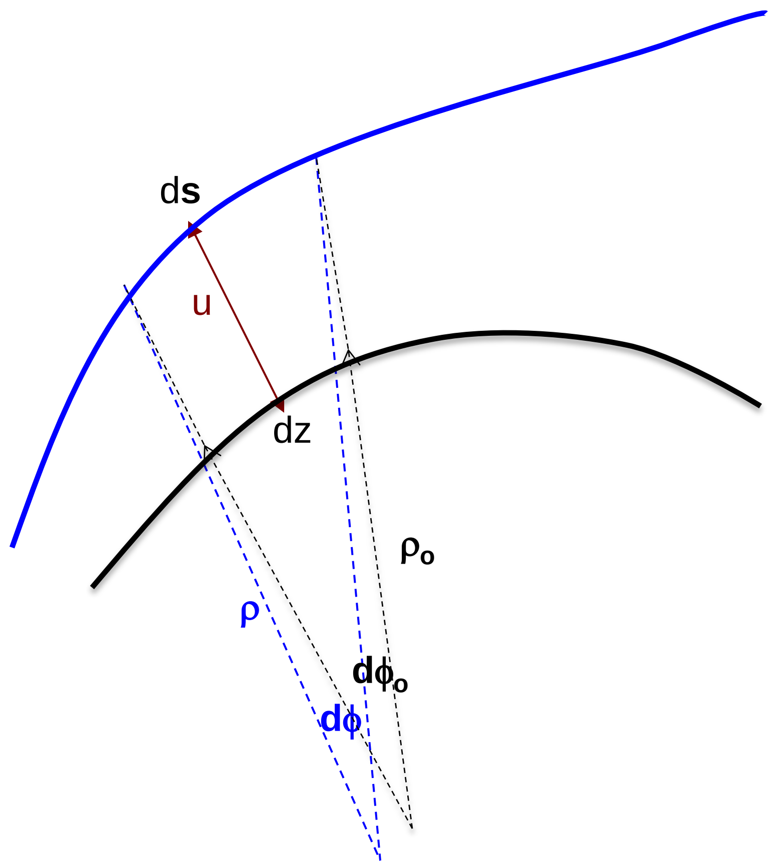
Motion about the Reference Trajectory

The first thing to do in describing an accelerator is to define a reference orbit. Only the *ideal particle* actually follows the reference orbit. All other particles will follow trajectories about the reference orbit.

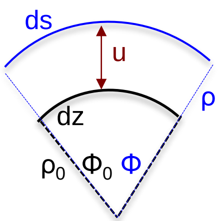
What we really need is an equation which governs a particle's deviation from the reference trajectory.

The idea is to subtract the reference from the actual trajectory, and then make the desired approximation by discarding all terms of higher order than those of interest. We'll study linear terms here.

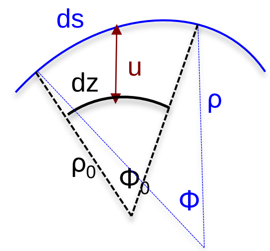
This gives us equations for x" and y", which are the equations of motion about the reference trajectory.



From geometrical arguments:



If $\phi=\phi_0$, then $\dfrac{du}{dz}=0$



Otherwise, $\dfrac{du}{dz}=-\phi+\phi_0$

Then, $u''=\dfrac{d^2u}{dz^2}=-\left(\dfrac{d\phi}{dz}-\dfrac{d\phi_0}{dz}\right)$ is an equation of motion for u.

Also, for small angles and path lengths:

$$\left.\begin{aligned}\frac{d\phi_0}{dz}&=k_0=\frac{1}{\rho_0}\\ \frac{d\phi}{ds}&=k=\frac{1}{\rho}\\ \frac{ds}{dz}&=(1+uk_0)\end{aligned}\right\}\xrightarrow{\text{Insert into EOM}}\quad u''=-\left(\frac{d\phi}{dz}-\frac{d\phi_0}{dz}\right)=-\frac{d\phi}{ds}\frac{ds}{dz}-\frac{d\phi_0}{dz}=-k(1+uk_0)+k_0$$

(Wiedemann 2.28)

Derivation of Transverse Equations of Motion (Wiedemann)

Assume we are in the horizontal plane and expand the field. Also consider the slightly off momentum particle.

$$\kappa_x = \frac{e}{p} B_y = \frac{e}{p}\left[B_{y0} + gx + \ldots \right]$$

$$\frac{1}{p} = \frac{1}{p_0(1+\delta)} \approx \frac{1}{p_0}(1 - \delta + \ldots)$$

(Wiedemann 2.29, 2.30)

Assuming the reference orbit is in the horizontal plane, and substituting above into our EOM and simplifying:

$$x'' + (k + \kappa_{0x}^2)x = \kappa_{0x}(\delta - \delta^2) + (k + \kappa_{0x})x\delta - \frac{1}{2}mx^2 - k\kappa_0 x^2 + O(3)$$

(Wiedemann 2.31)

≈0

where $\kappa_{0x} = \frac{e}{p_0} B_{y0}$ and $k = \frac{e}{p_0} g$

And for the vertical plane:

$$y'' - ky = -ky\delta + mxy + O(3)$$

≈0

(Wiedemann 2.33)

Assuming a linear lattice and and on energy beam, we lose all terms on the LHS!
We'll come back for them later on...

Linear Equations of Motion - Summary

Finally (!), after discarding nonlinear terms, and assuming an *on-energy beam*, the equations of motion about the reference trajectory are:

$$x'' + (k_0(s) + k_{0x}^2(s))x = 0$$
$$y'' - k_0(s)y = 0$$

(Wiedemann 4.2a, 4.2b)

$$k = \frac{1}{r} = \frac{e}{p_0}B_0 \quad \text{dipole focusing strength}$$

$$k = \frac{e}{p_0}\frac{dB_y}{dx} \quad \text{quadrupole focusing strength}$$

Note that the dipole term here is second order and provides a geometric dipole focusing term caused by certain dipole magnets. The first order dipole term was built into the reference trajectory.

Linear Equation of Motion

These are known as Hill's (homogeneous) Equations of Motion, more succinctly written as:

$$x'' + K_x(s)x = 0$$
$$y'' + K_y(s)y = 0$$

These equations are the foundation of single particle transverse motion in an accelrator!

where we have defined…

$$K_x(s) = k_{0x}^2(s) + k_0(s)$$
$$K_y(s) = -k_0(s)$$

Before closing the book on the derivation of the equation of motion, note that there is first principals derivation at the end of this lecture.

A Closer Look at Hill's Equation

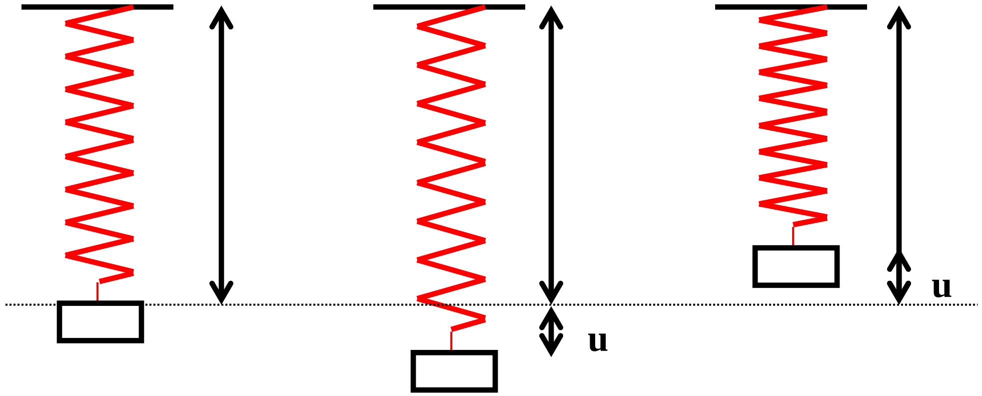
What does it tell us? Look at the general form.

$$u'' + K(s)u = 0$$ (Wiedemann 2.65)

✓ Particle motion about the reference trajectory is caused by normal dipoles and quadrupoles, whose strength varies with s.

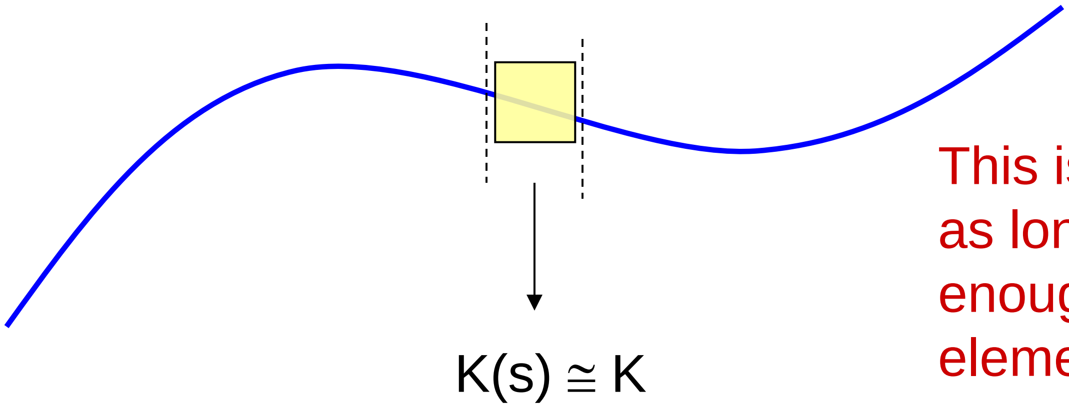
✓ If k and ρ are constant (or vary slowly), the motion is essentially harmonic. Therefore we won't be surprised later to find that the motion has a "frequency"… The total motion, with s-dependence, is "quasi-harmonic".

✓ The equation acts like a spring with "spring constant" or restoring force, K(s). But K(s) changes over the distance of the accelerator.



Peicewise Constant Approximation

The s-dependence of K(s) complicates the solution to the equation. To make life a little easier, lets consider a single piece of the accelerator having constant K. In doing so, we are making a “piecewise constant” approximation.



This is a good approximation as long as we select small enough pieces (<= 1 lattice element)

Hill’s equation becomes: $$u'' + Ku = 0$$

And this is a problem we know how to solve!

Solving Hill's Equation

Solve the piece-wise constant Hill's equation with appropriate initial conditions:

Solve: $u'' + Ku = 0$ with initial conditions: $u(0) = u_o;\ u'(0) = u_o'$

(**Derivation**)

The solution is: $$u(s) = C(s)u_o + S(s)u_o'$$

K>0: $C(s) = \cos(\sqrt{K}s)$

$$S(s) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}s)$$

K<0: $C(s) = \cosh(\sqrt{|K|}s)$

$$S(s) = \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}s)$$

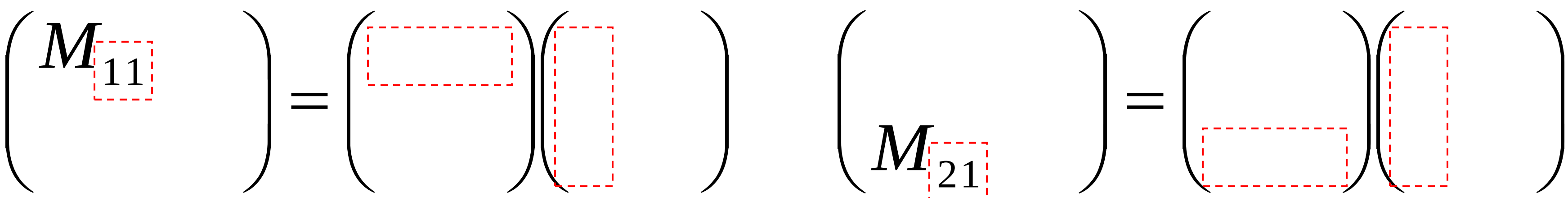
Quick Review of Matrix Multiplication

Suppose we multiply two matrices, M1 and M2. The (row m, column n) element of the final matrix is the vector dot product of the m row from M1 and the n column from m2:

$M_{n,m}$=(row n from M1) x (column m from M2)

Example: M_{11}=(row 1 from M1) x (column 1 from M2)

M_{21}=(row 2 from M1) x (column 1 from M2)



$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

In general, matrix multiplication is not commutative: M1 x M2 $\neq$ M2 x M1. The order of multiplication is very important!

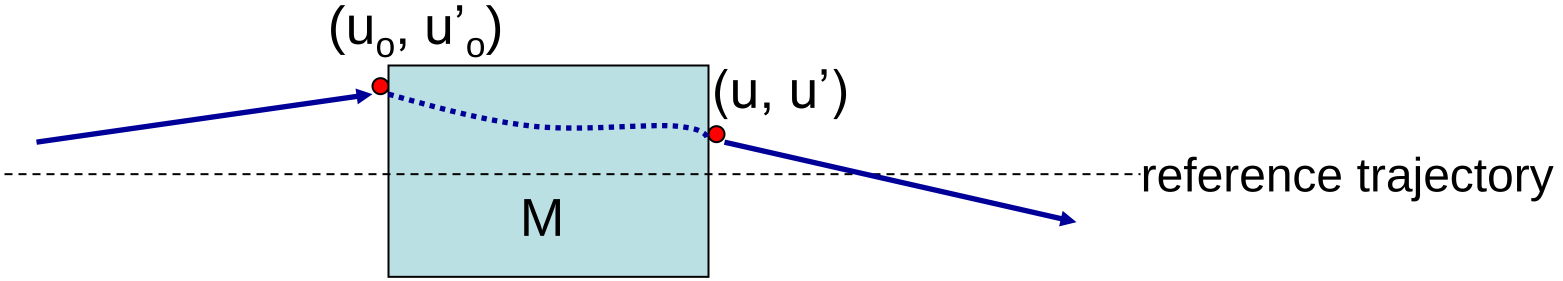
First take derivative:

$$u(s) = C(s)u_o + S(s)u_0'$$

$$u'(s) = C'(s)u_o + S'(s)u_0'$$

Then we can write the transport equation as a matrix:

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \underbrace{\begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}}_{\text{M}} \begin{pmatrix} u_o \\ u_0' \end{pmatrix} = M \begin{pmatrix} u_o \\ u_0' \end{pmatrix}$$



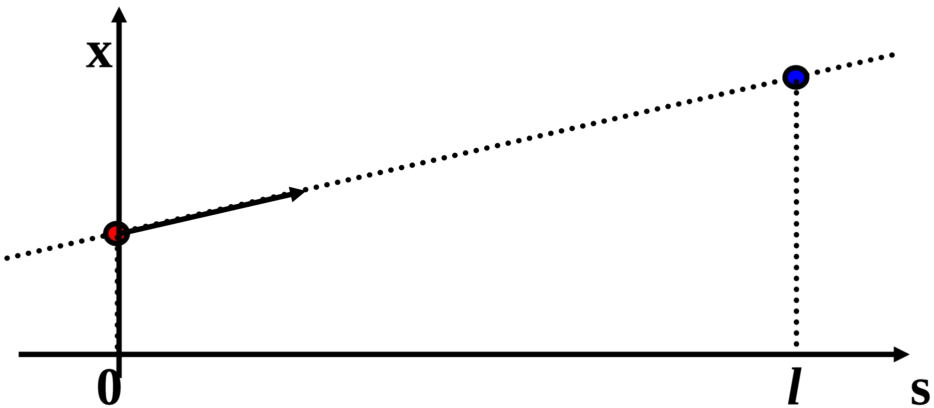
Transport Through a Drift

In a drift space, there is no change in the momentum of the particle. We take the limit of M as K-> 0.

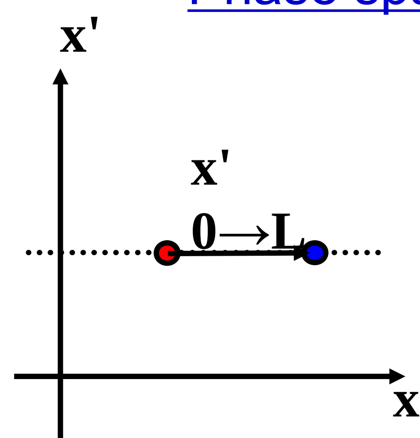
$$M_{drift} = \begin{pmatrix} \cos(\sqrt{K}l) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}l) \\ -\sqrt{K}\sin(\sqrt{K}l) & \cos(\sqrt{K}l) \end{pmatrix} \xrightarrow{K=0} M_{drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_o \\ u_0' \end{pmatrix} \qquad \begin{aligned} u &= u_o + l u_0' \\ u' &= u_0' \end{aligned}$$

Real space (s,x)



Phase space (x, x')



Transport Through a Quadrupole

In the case of a quadrupole, there is no bending, so the only remaining term is the quad strength term.

$$K = K_n + \frac{1}{\rho^2} \xrightarrow{\rho=\infty} K_n$$

Focusing:

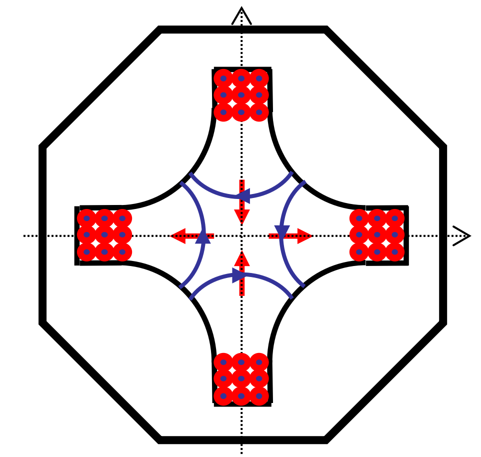
$$M_{QF} = \begin{pmatrix} \cos(\sqrt{K_n}\, l) & \frac{1}{\sqrt{K_n}} \sin(\sqrt{K_n}\, l) \\ -\sqrt{K_n} \sin(\sqrt{K_n}\, l) & \cos(\sqrt{K_n}\, l) \end{pmatrix}$$

Defocusing:

$$M_{QD} = \begin{pmatrix} \cosh(\sqrt{|K_n|}\, l) & \frac{1}{\sqrt{|K_n|}} \sinh(\sqrt{|K_n|}\, l) \\ \sqrt{|K_n|} \sinh(\sqrt{|K_n|}\, l) & \cosh(\sqrt{|K_n|}\, l) \end{pmatrix}$$

Finite Length Quad Transport.

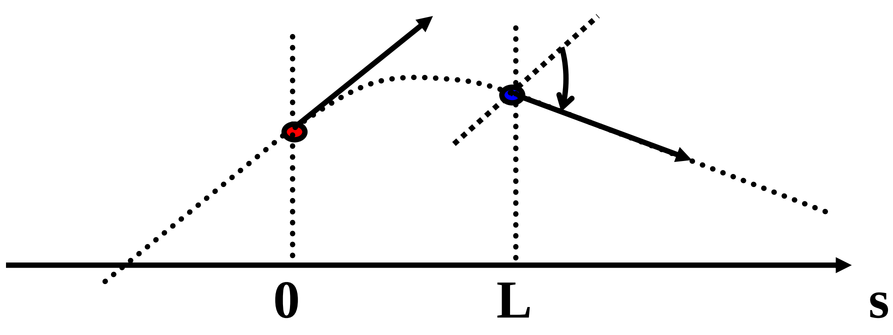
Now consider again the quadrupole with finite length, L. The angle is changed through the length, and the position as well. For instance, for K>0:



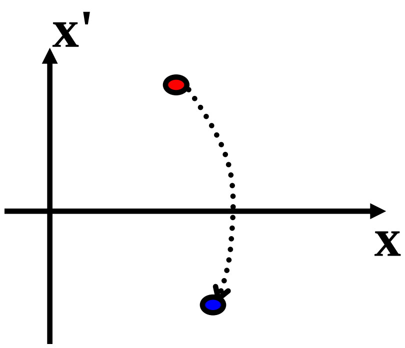
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K_n}\, l) & \frac{1}{\sqrt{K_n}} \sin(\sqrt{K_n}\, l) \\ -\sqrt{K_n} \sin(\sqrt{K_n}\, l) & \cos(\sqrt{K_n}\, l) \end{pmatrix} \begin{pmatrix} x_o \\ x_0' \end{pmatrix}$$

(**Examples**)

Real space (s, x):



Phase space (x,x'):



Thin Lens Approximation for a Quadrupole

In the "thin lens approximation", we let the length of the quadrupole approach zero while holding the focal length constant: L→0 as 1/f=KL=constant.

(**Derivation/Example**)

$$M_{Quad} = \begin{pmatrix} 1 & 0 \\ \mp \frac{1}{f} & 1 \end{pmatrix}$$

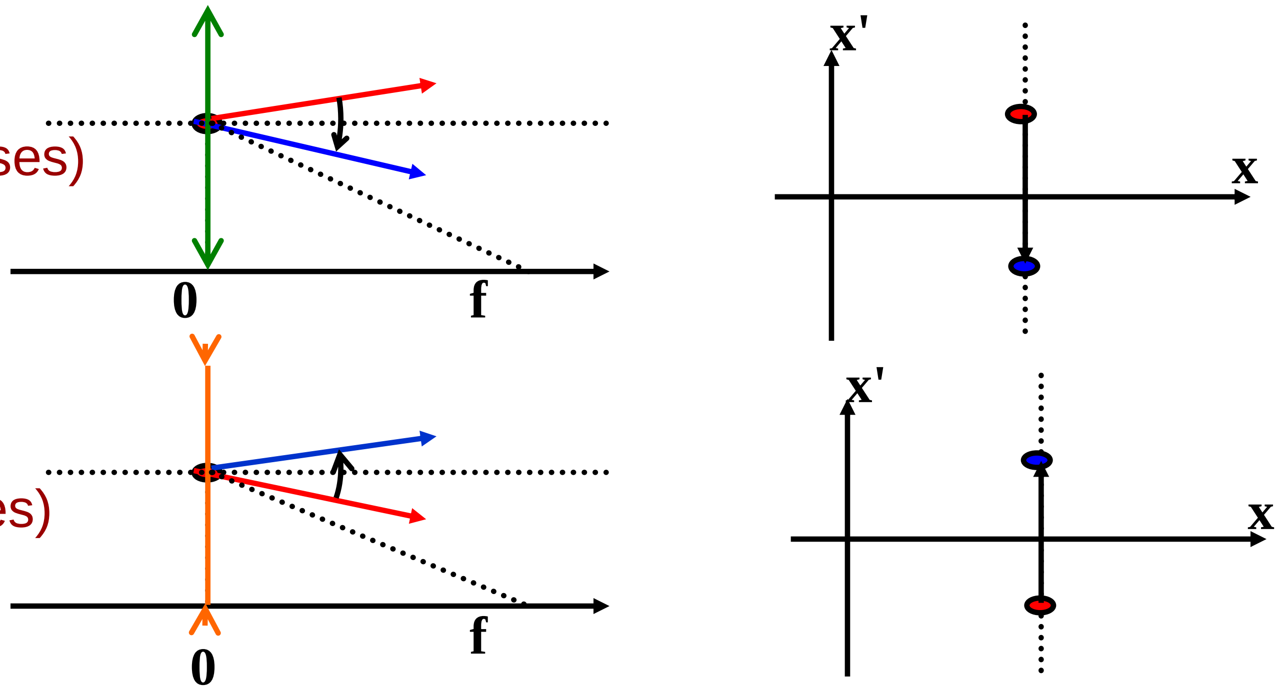
In this approximation, the position remains fixed, but the momentum changes:

Real space (s, x):

Phase space (x, x'):

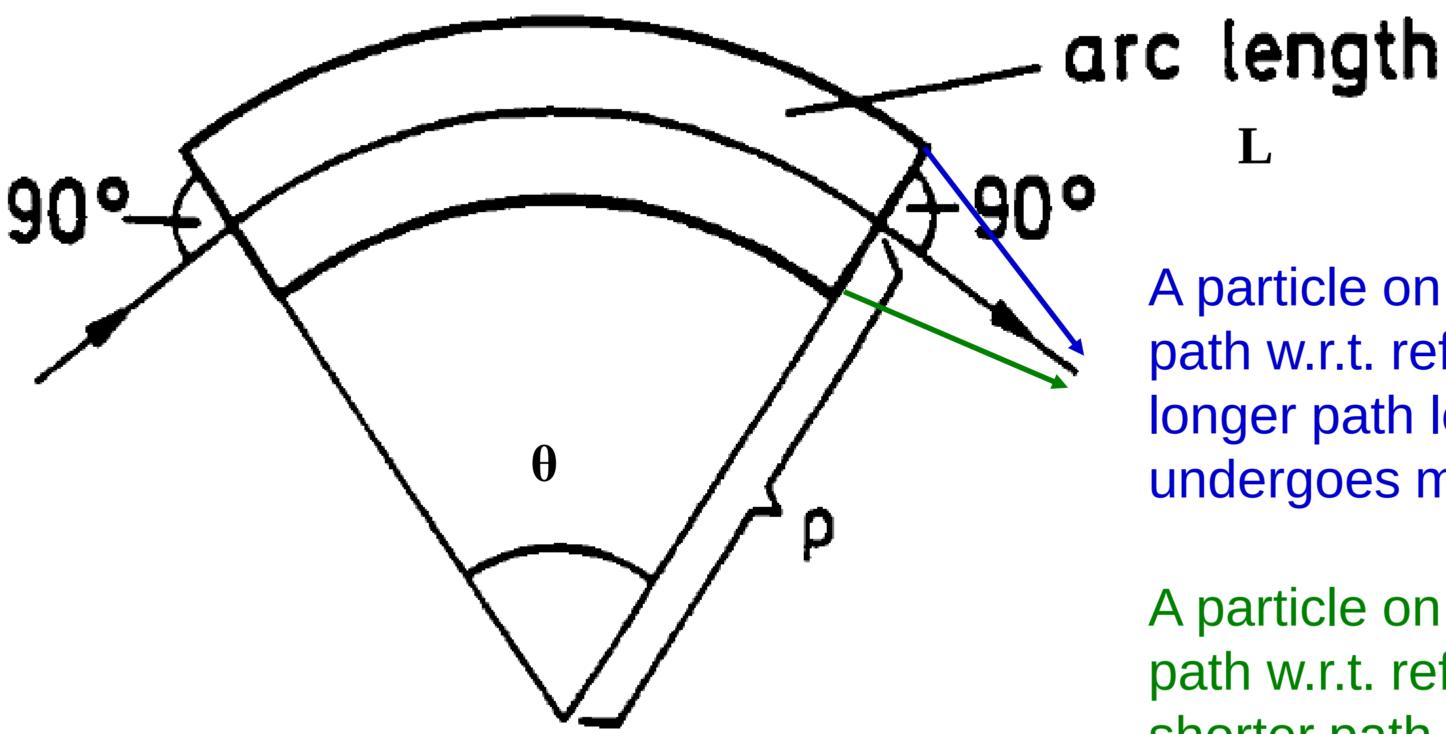
Focusing:
(slope decreases)

Defocusing:
(slope increases)



Focusing in a Sector Dipole

The axis of a sector dipole usually corresponds to the reference trajectory. In the plane of the bend, off-axis particles are focused by the dipole, as seen in the $1/\rho^2$ contribution to K in Hill's equation: $K=k+1/\rho^2$



A particle on an exterior path w.r.t. reference has a longer path length and undergoes more bending.

A particle on an interior path w.r.t. reference has a shorter path and undergoes less bending.

Transport in Pure Dipole Sector Magnet

In a pure sector dipole, we take the quad strength k, to be zero, k=0. In the deflecting plane, i.e, the plane of the bend (usually horizontal), we have:

$$M_{\mathrm{x,sector}} = \begin{pmatrix} \cos(\theta) & \rho_o \sin(\theta) \\ -k_o \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\theta = k_o l, \quad k_{\mathrm{o}} = \frac{1}{\rho_o}$$

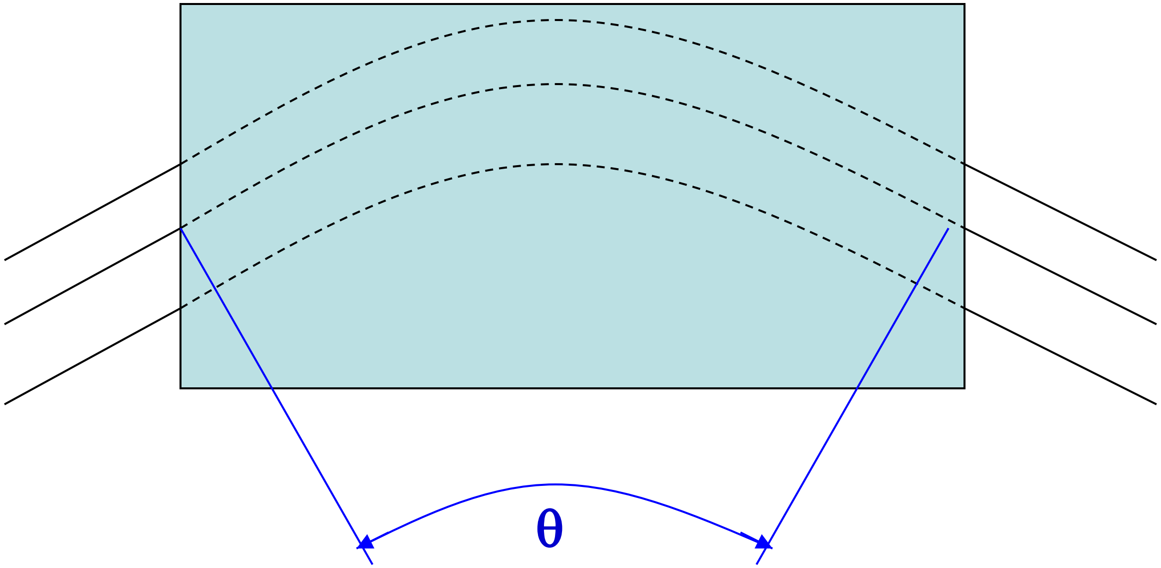
(Wiedemann 4.41)

And in the non-deflecting plane, $\rho \rightarrow 0$, and we are left with a drift:

$$M_{\mathrm{y,sector}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

Transport in Rectangular Dipoles

In a rectangular dipole, the particle path in the horizontal direction is the same for all trajectories, so there is no focusing in the horizontal direction.

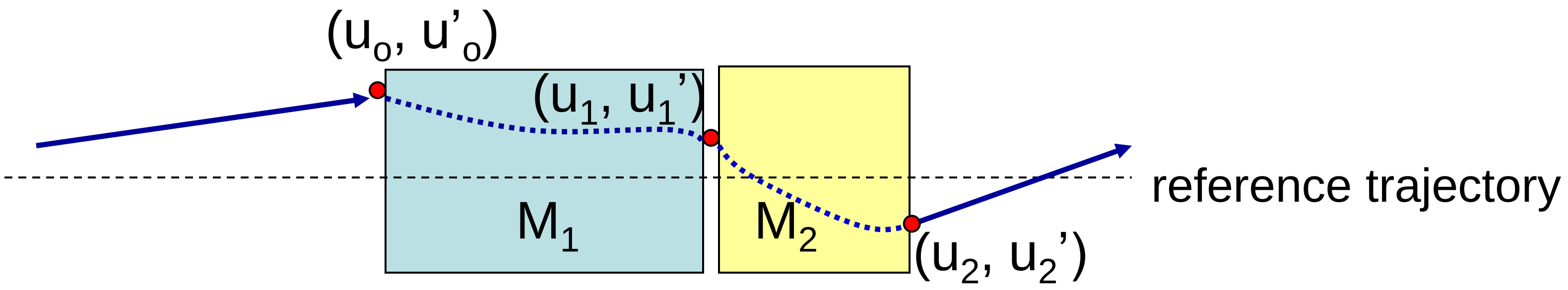


$$M_{\mathrm{x,rect}} = \begin{pmatrix} 1 & \rho \sin\theta \\ 0 & 1 \end{pmatrix}$$

In the horizontal direction the magnet transforms like a drift with length equal to the path length $\rho \sin\theta$.

Piecewise Constant Transport: Two Elements

The matrix representation is very convenient. For instance, what if we had two consecutive elements, with strengths K_1 and K_2? What is the final equation of transport for a particle through both elements?



The solution for the first element becomes the initial condition for the second element…

First element: $\begin{pmatrix} u_1 \\ u_1' \end{pmatrix} = M_1 \begin{pmatrix} u_o \\ u_0' \end{pmatrix}$ Second element: $\begin{pmatrix} u_2 \\ u_2' \end{pmatrix} = M_2 \begin{pmatrix} u_1 \\ u_1' \end{pmatrix}$

Putting them together gives: $M_2 \left(M_1 \begin{pmatrix} u_o \\ u_0' \end{pmatrix} \right) = M_2 M_1 \begin{pmatrix} u_o \\ u_0' \end{pmatrix}$

More succinctly

$$\begin{pmatrix} u_2 \\ u_2' \end{pmatrix} = M(s_2|s_0) \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}, \text{ where } M(s_2|s_1) = M_2 M_1$$

Piecewise Constant Transport: n Elements

For an arbitrary number of transport elements, each with a constant, but different, K_n, we have:

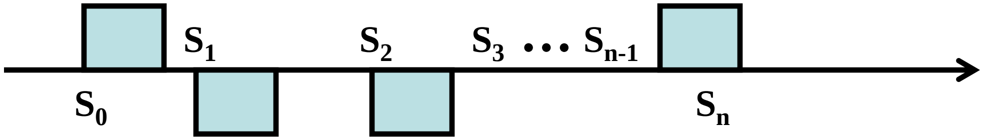
$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1})\ldots\mathcal{M}(s_3|s_2)\cdot\mathcal{M}(s_2|s_1)\cdot\underbrace{\mathcal{M}(s_1|s0)}_{\text{from } s_0 \text{ to } s_1}$$

from $\mathbf{s_0}$ to $\mathbf{s_1}$

from $\mathbf{s_0}$ to $\mathbf{s_2}$

from $\mathbf{s_0}$ to $\mathbf{s_3}$

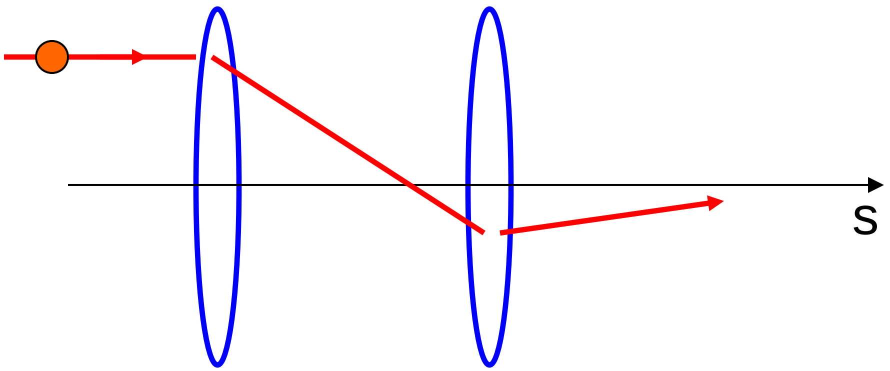
from $\mathbf{s_0}$ to $\mathbf{s_n}$



$$\Rightarrow \begin{pmatrix} u_n \\ u_n' \end{pmatrix} = M(s_n|s_o)\begin{pmatrix} u_o \\ u_o' \end{pmatrix}$$

Thus by breaking up the parameter K(s) into piecewise constant chunks, K(s)={K_1, K_2, … K_n}, we have found a useful method for finding the particle transport equation through a long section of beamline with many elements.

Let's consider quadrupole doublet sequence separated by a drift L, in the thin lens approximation:



(**Derivation/Examples**)

Answer:

$$M_{\text{Doublet}} = \begin{pmatrix} 1-\frac{L}{f_1} & L \\ -\frac{1}{f^*} & 1-\frac{L}{f_2} \end{pmatrix}$$

$$\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$$

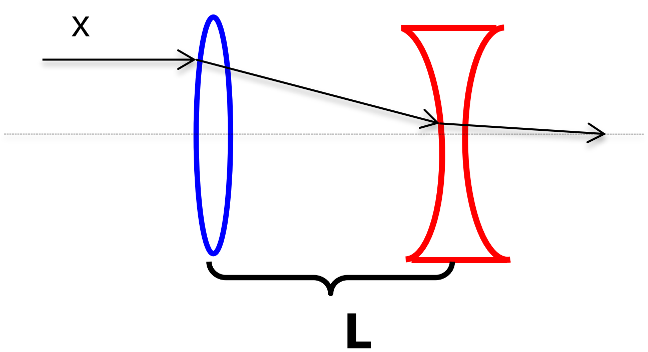
is the total focal length of the system.

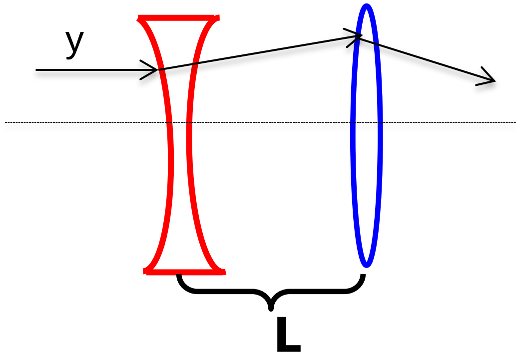
Why don't we use sequences of …FOFOFOFO… magnets to create lattices in an accelerator?

FOD Channel

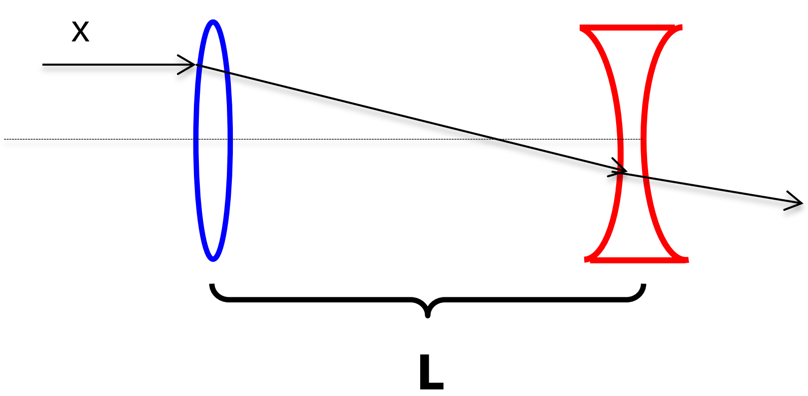
Consider a particle x'=y'=0 and an FOD lattice sequence:

Focusing in both planes



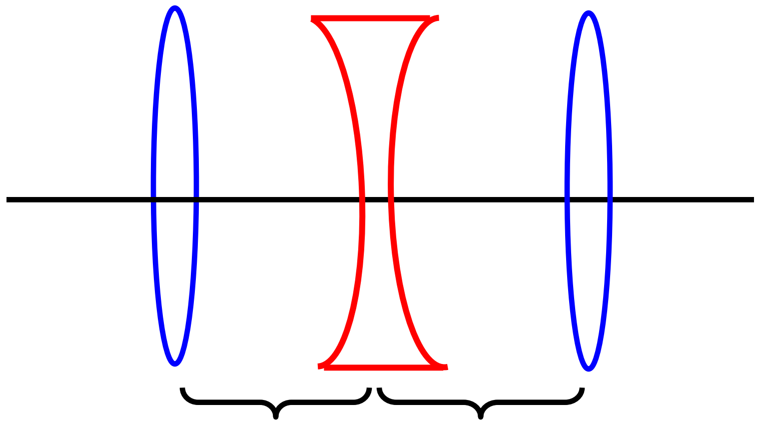


Defocusing horizontal



The system is net focusing for certain conditions, depending on the initial particle parameters, the quad strengths, and the drift length.

FODO Channel



- Consider a defocusing quadrupole "sandwiched" by two focusing quadrupoles with focal lengths **f**.
- The symmetric transfer matrix is taken from center to center of focusing quads (thus one full focusing quad and one full defocusing quad)

$$M_{\mathrm{FODO}} = M_{\mathrm{Half\ QF}} M_{\mathrm{Drift}} M_{\mathrm{QD}} M_{\mathrm{Drift}} M_{\mathrm{Half\ QF}}$$

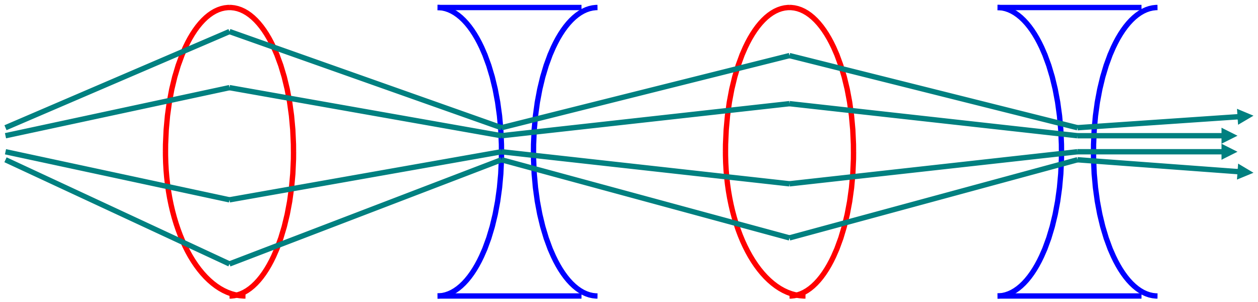
$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{2f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{2f_1} & 1 \end{pmatrix}$$

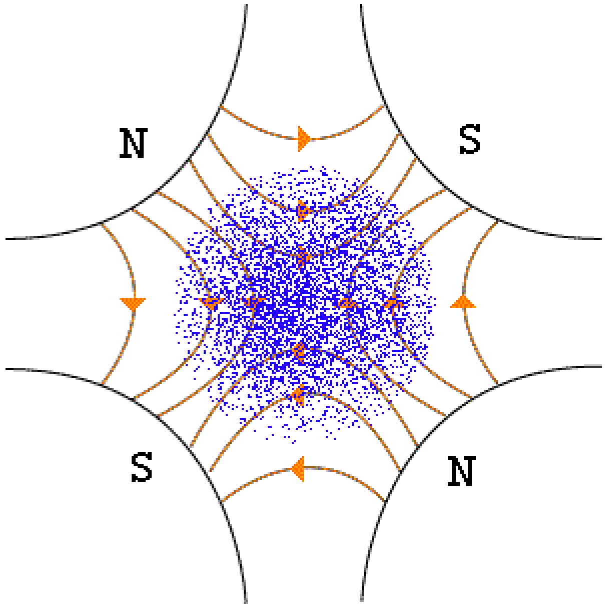
$$= \begin{pmatrix} 1-2\frac{L}{f^*} & 2L(1-\frac{L}{2f_2}) \\ -\frac{2}{f^*}(1-\frac{L}{2f_1}) & 1-2\frac{L}{f^*} \end{pmatrix}$$

$$\frac{1}{f^*} = \frac{1}{2f_1} + \frac{1}{2f_2} - \frac{L}{4f_1 f_2}$$

More on Focusing Particles …

The key is to alternate focusing and defocusing quadrupoles. This is called a FODO lattice (Focus-Drift-Defocus-Drift). :





FODO Channel

The general expression for a FODO lattice with focal lengths f_1 and f_2, separated by a distance L is:

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1-2\dfrac{L}{f*} & 2L(1-\dfrac{L}{2f_2}) \\ -\dfrac{2}{f*}\left(1-\dfrac{L}{2f_1}\right) & 1-2\dfrac{L}{f^*} \end{pmatrix}$$

(Wiedemann 7.32)

with,

$$\frac{1}{f*} = \frac{1}{2f_1} + \frac{1}{2f_2} - \frac{L}{4f_1 f_2}$$

And the special case where $f_1=-f_2=f$ is:

$$\frac{1}{f^*} = \frac{L}{4f^2}$$

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1-\dfrac{L^2}{2f^2} & 2L(1+\dfrac{L}{2f}) \\ -\dfrac{L}{2f^2}\left(1-\dfrac{L}{2f}\right) & 1-\dfrac{L^2}{2f^2} \end{pmatrix}$$

This arrangement is very common in beam transport lines.

Stability Condition

Most of the time we deal with lattices where the arrangements of magnets repeats, i.e., periodic systems.

Q: How do we know if our lattice produces stable particle motion?
A: After finding the composite transport matrix, M, of the lattice, we determine the "stability condition" for the matrix:

If: $$M_{\text{Lattice}} = M_n M_{n-1} M_{n-2} \ldots M_2 M_1$$

Stability condition: $$\left| Tr(M_{\text{Lattice}}) \right| < 2$$ (Wiedemann 7.37)

Where "Tr(M)" means the "Trace" of the matrix, which is the sum of the diagonal elements. And for N repetitions of this lattice sequence, we generalize to:

$$\left| Tr(M_{Lattice}^{N}) \right| < 2$$

What is the stability condition for a FODO lattice?

Stability Condition for a FODO Lattice ($f_2=-f_1$)

The stability condition for a FODO lattice is found by taking the trace and applying the stability condition. So, for the thin lens approximation of a FODO cell with equal focal length quadrupoles:

Transfer matrix:

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1-\dfrac{L^2}{2f^2} & \cdots \\ \cdots & 1-\dfrac{L^2}{2f^2} \end{pmatrix}$$

Stability condition:

$$\left|Tr(M_{FODO})\right| = \left|2-\frac{L^2}{f^2}\right| < 2$$

Result for FODO:

$$0 < \frac{L}{2f} < 1$$

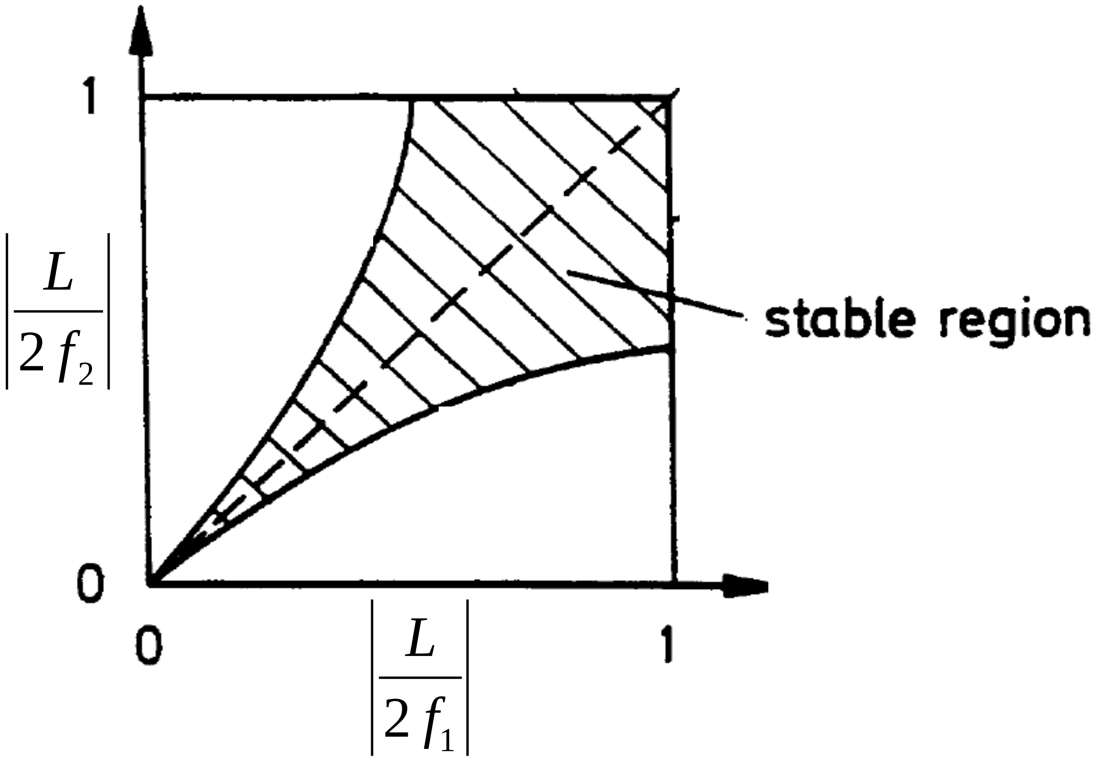
For a thin lens FODO lattice, the distance between magnets should be less than twice the focal length.

Stability Condition for a FODO Lattice (general)

For the general FODO cell with unequal focal lengths, the condition is more complicated. We have:

$$0 < \frac{L}{f^*} = \frac{L}{2f_1} + \frac{L}{2f_2} - \frac{L^2}{4f_1 f_2} < 1$$

The allowed magnitudes are given by the "Necktie Diagram"



PART II

Parameterization and Linear Transport for an Ensemble of Particles

The Problem of Real Beam Distributions

So far we have learned how to write the transport equations for a single particle in a beamline.

The problem: In a real machine, we rarely have any information about a single particle!

We do have information about the entire beam, i.e, the ensemble of all of the particles. We could solve separate transport equations for each particle in the bunch (PIC simulations do something like this)... Very impractical for analytic work!

Solution:
We parameterize the entire particle distribution, and write the transport equations for the parameters. Thus we can write transport equations for the whole beam, not just one particle!

Twiss Parameters

A good approximation for the beam shape in phase space is an ellipse. Any ellipse can be defined by specifying:

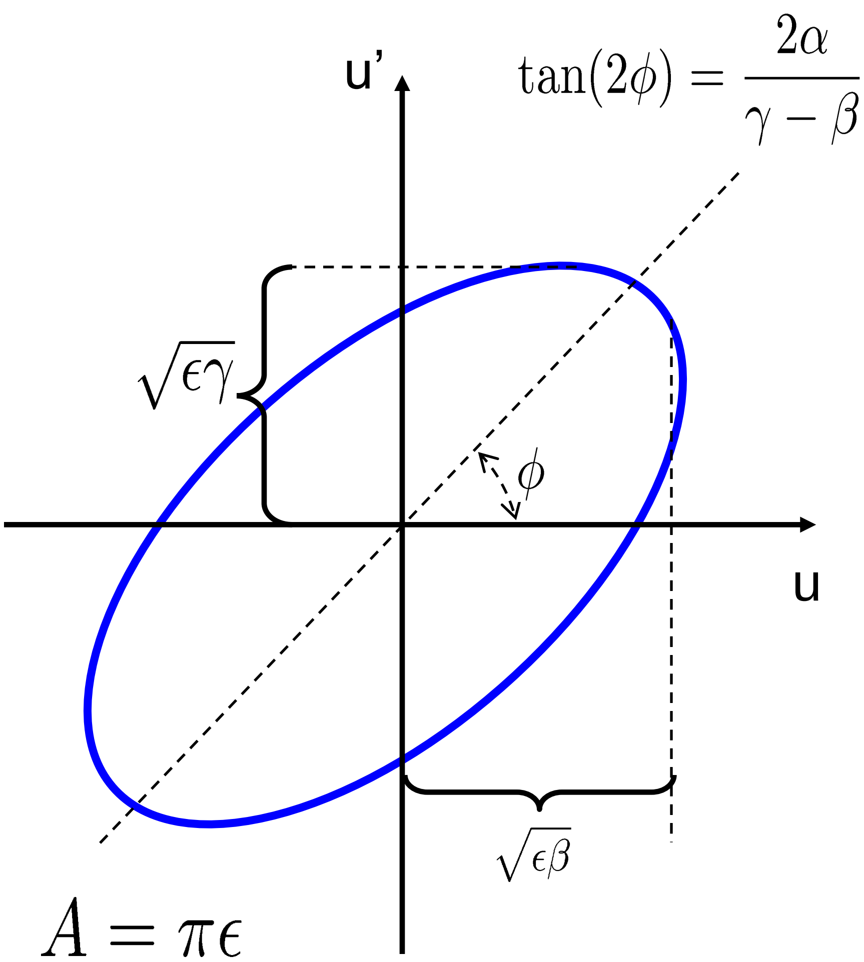
✓ Area
✓ Shape
✓ Orientation

We choose 4 parameters –
3 independent, 1 dependent:

α - related to beam tilt
β - related to beam shape and size
ε - related to beam size
γ - dependent on α and **β**.

These are the “Twiss Parameters” (or “Courant-Snyder Parameters”)

Beam Ellipse in Phase Space:



The equation for the beam ellipse, with our Twiss parameterization can be written as:

$$e = gu^2 + 2auu' + bu'^2$$

$$\gamma = \frac{1+\alpha^2}{\beta}$$

(Weidemann 5.18, 5.19)

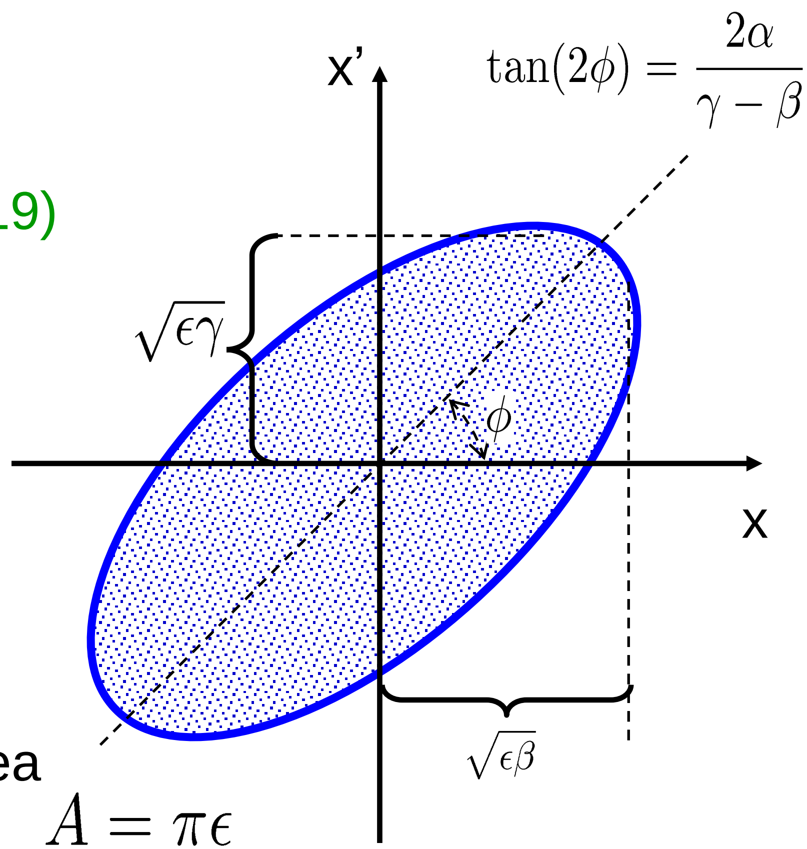
And the ellipse has area:

$$\grave{O} du'\,du = pe$$

$$\varepsilon = \text{beam emittance}$$

The beam emittance is the phase space area of the beam (to within π). Emittance is a parameter used to gauge beam quality.

Beam Ellipse in Phase Space:

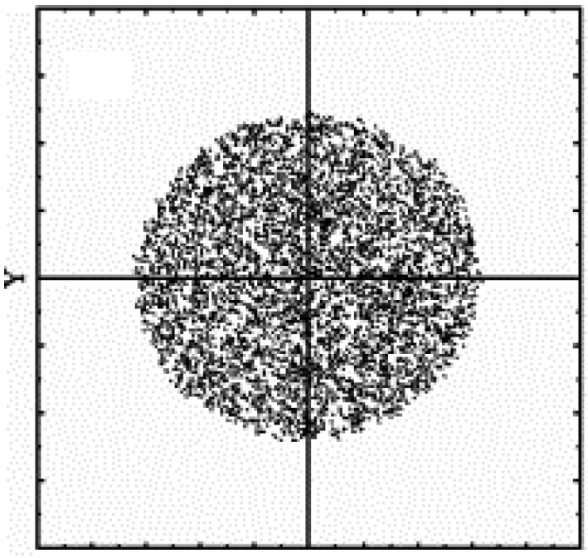


$$A = \pi\epsilon$$

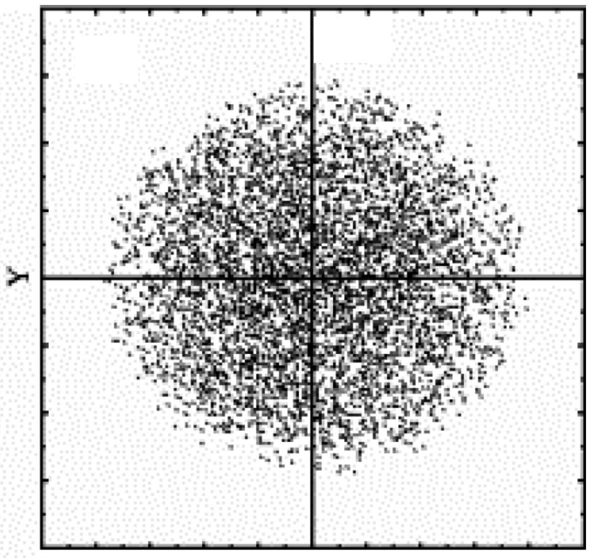
Real Beam Distributions

In reality, real beam distributions are not uniform in phase space and, in practice, it can be difficult to locate the beam edge.

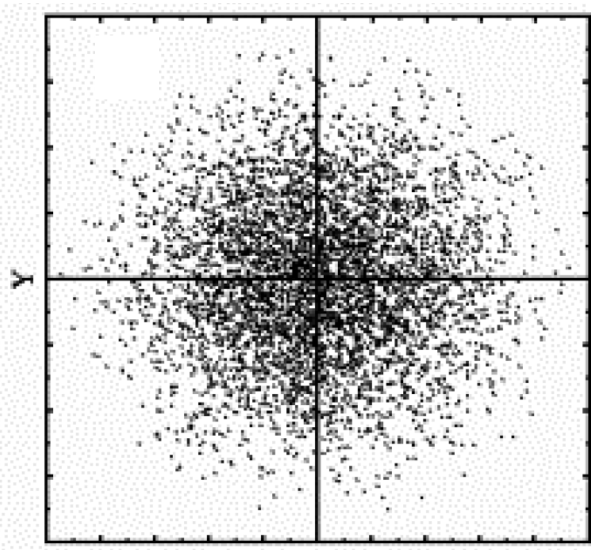
"KV"



"Waterbag"



Gaussian



RMS Quantities

Most often, we will deal with RMS quantities. The RMS size of a beam with N particles is defined as:

$$u_{\mathrm{RMS}} = \sqrt{\frac{1}{N}\sum_{i}^{N}(u_i - u_{\mathrm{avg}})^2}$$

And the RMS momentum spread, is:

$$u'_{\mathrm{RMS}} = \sqrt{\frac{1}{N}\sum_{i}^{N}(u_i' - u'_{\mathrm{avg}})^2}$$

For most common distributions, the RMS is some fraction of the total beam size. For example, for a KV distribution, the RMS beam size is half the total beam size.

RMS Quantities

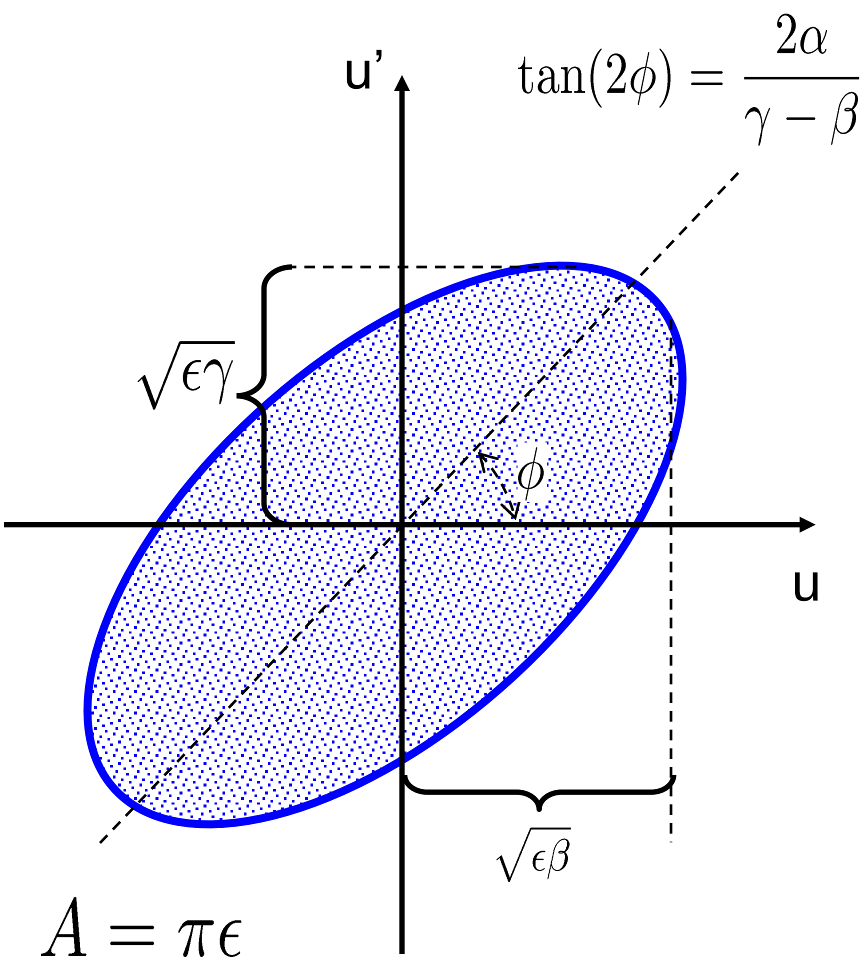
We can relate our Twiss Parameters for the beam to RMS quantities, as well:

$$\beta = \frac{u^2_{\mathrm{RMS}}}{\varepsilon_{\mathrm{RMS}}}$$

$$\gamma = \frac{u'^2_{\mathrm{RMS}}}{\varepsilon_{\mathrm{RMS}}}$$

$$\alpha = \frac{-(uu')_{RMS}}{\varepsilon_{\mathrm{RMS}}}$$

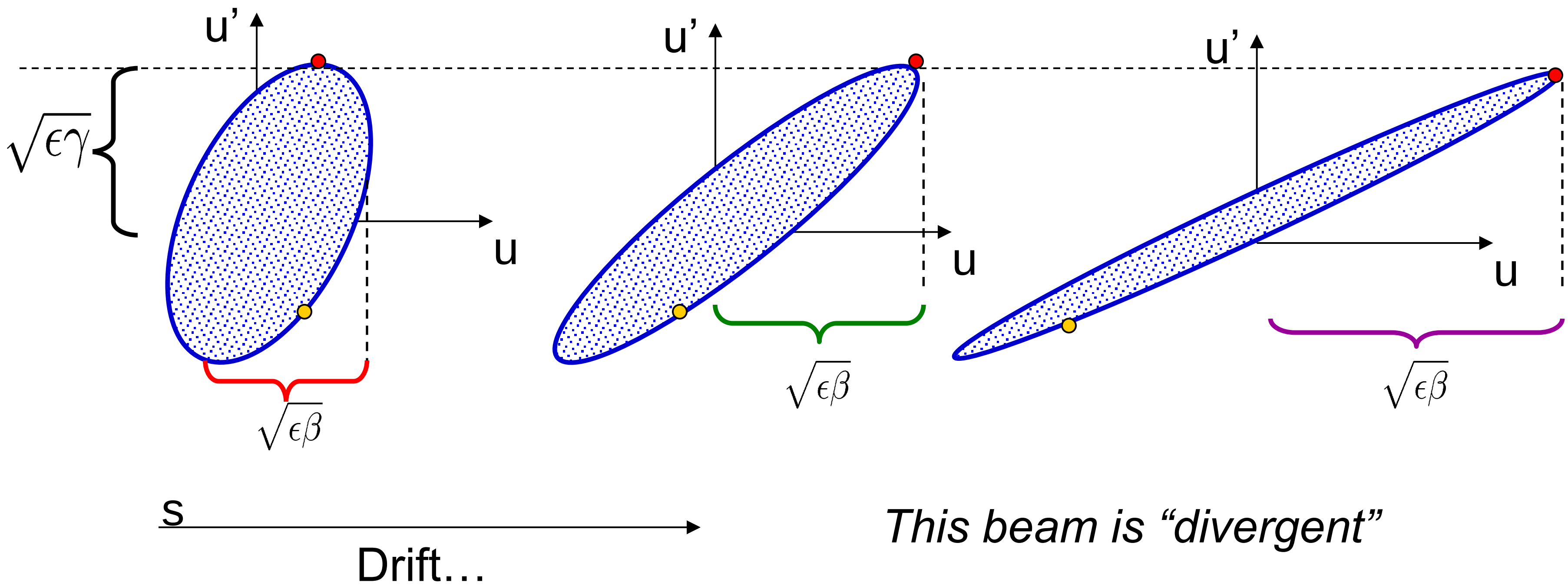
Beam Ellipse in Phase Space:



The Beam Ellipse in a Drift

How does this "beam ellipse" transform through a drift space?

Drift: $u = u_o + u'_o l$

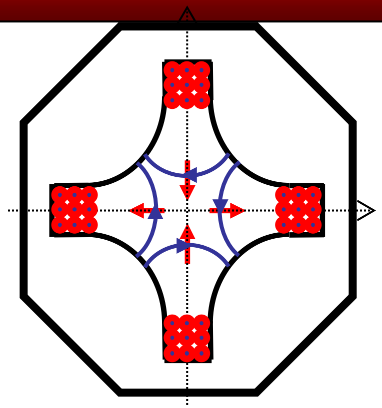


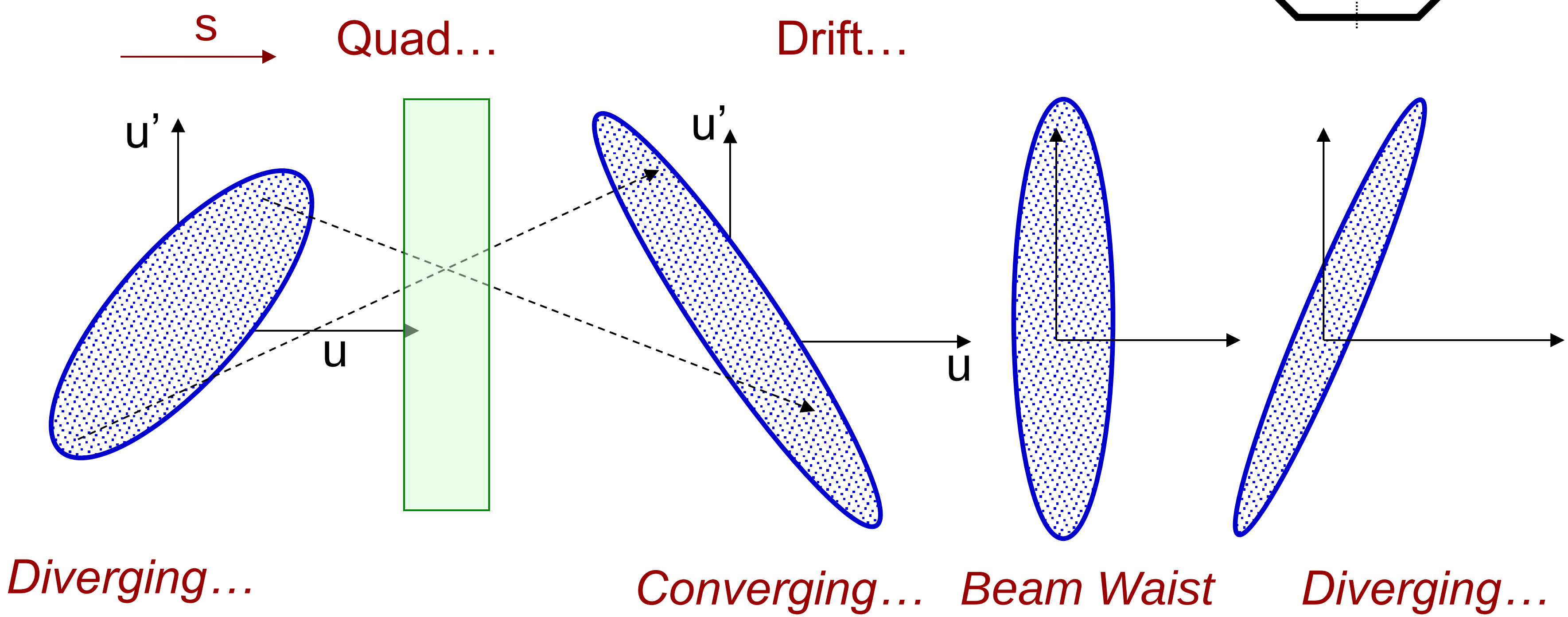
This beam is "divergent"

Analogous to a single particle, u increases while u' remains fixed.
Observation: Without focusing, any beam would spread out…

The Beam Ellipse in a Quadrupole

Recall that in a focusing quadrupole, the force of the *kick is opposite to the sign of the particle's position, and proportional to the distance from the axis.* So, for a distribution of particles:





A focusing quadrupole causes a diverging beam to converge. In reality, the scenario is more complicated because we focus in one plane while defocusing in the other.

According to *Louiville's Theorem*, the phase space area of the beam does not change under linear transformations. This means that the beam emittance is conserved in a linear transport system.

For our homogenous Hill's equation, the emittance between two points is conserved, regardless of the change in beam shape and orientation.

With this fact, we find that for a piece-wise constant lattice, the Twiss parameters transform as:

(**Derivation**)

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & (S'C + SC') & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_o \\ \alpha_o \\ \gamma_o \end{pmatrix}$$

(Wiedemann 5.22)

Twiss Parameters through a Drift

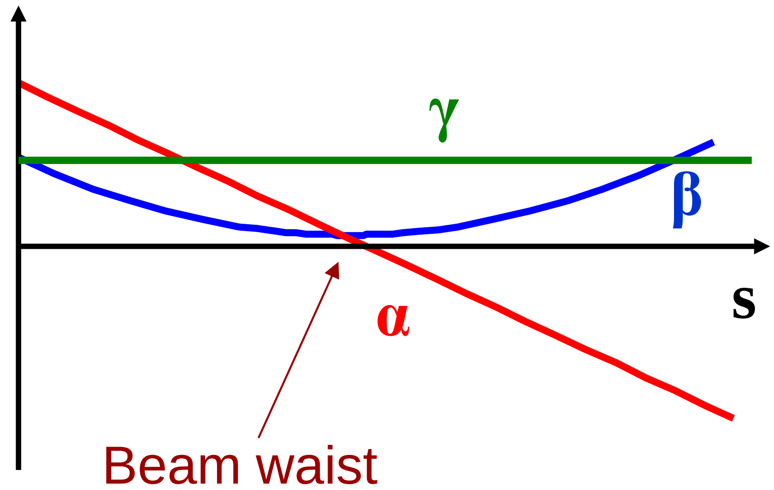
$$M_{\text{twiss,drift}} = \begin{pmatrix} 1 & -2L & L^2 \\ 0 & 1 & -L \\ 0 & 0 & 1 \end{pmatrix}$$

Recall that $\sqrt{\beta\varepsilon}$ is a measure of beam size. So clearly, the beam size always eventually grows in the absence of focusing.

$$\beta = \beta_o - 2L\alpha_o + L^2\gamma_o$$

$$\alpha = \alpha_o - L\gamma_o$$

$$\gamma = \gamma_o$$



(**Derivation**)

We will attach more physical meaning to these parameters soon!

Summary

✓ We found the equation of motion with respect to the reference trajectory.

✓ We solved the equation of motion for the case of K=constant.

✓ We represented the solution in matrix form.

✓ We learned how to transport a particle through an arbitrarily long piecewise constant lattice, by multiplying the individual transport matrices in the correct order.

✓ We parameterized the entire distribution of particles using Twiss parameters.

✓ We learned how to transport the Twiss parameters – and therefore the shape and orientation of the beam - through a piece-wise constant lattice.

Derivation of Transverse Equations of Motion - 1

There are several ways to derive the transverse equations of motion. A particularly elegant method involves Hamiltonian dynamics, but is beyond the present scope. Here, we will make some ordering approximations and then substitute into the Lorentz force equation:

$$\frac{d\vec{p}}{dt} = e(\vec{v} \times \vec{B})$$

Assumptions:

Orthogonal right hand coordinate system (x,y,s) where

- s is the distance along the reference particle orbit,
- x is “horizontal” coordinate in direction of reference orbit curvature, and
- y is “vertical” coordinate perpendicular to s and x.
- ρ is the bending radius of the reference orbit

$$\frac{1}{\rho B_0} = \frac{e}{p_0}$$

$$\vec{r} = (\rho + x)\hat{x} + y\hat{y} = \vec{r}_0 + x\hat{x} + y\hat{y}$$

$$\vec{p} = (p_0 + \delta p)\hat{s} + p_x\hat{x} + p_y\hat{y} = \gamma m\left(v\hat{s} + \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y}\right) = \gamma m\vec{v} = \gamma m\frac{d\vec{r}}{dt}$$

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = \frac{v}{(1+\frac{x}{\rho})}\frac{d}{ds}, \quad \frac{d\hat{s}}{ds} = -\frac{\hat{x}}{\rho}, \quad \frac{d\hat{x}}{ds} = \frac{\hat{s}}{\rho}, \quad \frac{d\hat{y}}{\mathrm{ds}} = 0$$

$$\vec{B} = B_0\hat{y} + \delta\vec{B} = B_0\hat{y} + \delta B_x\hat{x} + \delta B_y\hat{y} + \delta B_s\hat{s}$$

Ordering: Assume

$$x << \rho \qquad y << \rho$$

$$p_x << p_0 \qquad p_y << p_0 \qquad \delta p << p_0$$

$$\delta B_x << B_0 \qquad \delta B_y << B_0 \qquad \delta B_s << B_0$$

Now plug into Lorentz force equation, do algebra, and get:

$$x'' = \frac{x'^2}{\rho(1+\frac{x}{\rho})} + \frac{1+\frac{x}{\rho}}{1+\frac{\delta p}{p_0}}\{-[\frac{x}{\rho^2} + (1+\frac{x}{\rho})\frac{\delta B_y}{\rho B_0}] + y'\frac{\delta B_s}{\rho B_0} + \frac{1}{\rho}\frac{\delta p}{p_0}\}$$

$$y'' = \frac{x'y'}{\rho(1+\frac{x}{\rho})} + \frac{1+\frac{x}{\rho}}{1+\frac{\delta p}{p_0}}\{(1+\frac{x}{\rho})\frac{\delta B_x}{\rho B_0} - x'\frac{\delta B_s}{\rho B_0}\}$$

Potential V(x,y) for B_x and B_y:

$$V(x,y) = -\rho B_0\{(\frac{1}{\rho} + \frac{\delta B_{0y}}{\rho B_0})y + \frac{\delta B_{0x}}{\rho B_0}x + K_n xy + K_s\frac{x^2-y^2}{2} + \dots$$

$$\vec{B} = -\nabla V + \delta B_s = B_0\hat{y} + \delta\vec{B} = \rho B_0\{\hat{x}(\frac{\delta B_{0x}}{\rho B_0} + K_n y + K_s x + \dots) + \hat{y}(\frac{1}{\rho} + \frac{\delta B_{0y}}{\rho B_0} + K_n x - K_s y + \dots) + \frac{\delta B_s}{\rho B_0}\}$$

Derivation of Transverse Equations of Motion - 3

We plug in B to get:

$$x'' = \frac{1}{1+\frac{\delta p}{p_0}}\left\{-\frac{\delta B_{y0}}{\rho B_0} + \frac{1}{\rho}\frac{\delta p}{p_0} - \left[\frac{1}{\rho^2} + K_n + \frac{2}{\rho}\frac{\delta B_{y0}}{\rho B_0} - \frac{1}{\rho^2}\frac{\delta p}{p_0}\right]x + K_s y + y'\frac{\delta B_s}{\rho B_0} + \cdots\right\}$$

$$y'' = \frac{1}{1+\frac{\delta p}{p_0}}\left\{\frac{\delta B_{x0}}{\rho B_0} + K_n y + \left(K_s + \frac{2}{\rho}\frac{\delta B_{x0}}{\rho B_0}\right)x - x'\frac{\delta B_s}{\rho B_0} + \cdots\right\}$$

Comments:

In the algebra (not shown), the reference orbit terms cancelled out due to $\frac{1}{\rho B_0} = \frac{e}{cp_0}$. This happens in the x-equation. The "zoo" of terms are interpreted as follows:

K_n and $\frac{1}{\rho^2}$ terms - focusing from normal quadrupoles and from dipoles, respectively.

$\frac{\delta B_{(x,y)0}}{\rho B_0}$ and K_s terms - skew and normal dipole errors and skew quad linear coupling, respectively. IGNORE HERE!

δB_s term - solenoid or small error field or fringe field. IGNORE HERE!

$\frac{1}{\rho}\frac{\delta p}{p_0}$ term - dispersion. Will discuss next week. IGNORE HERE!

Other $\frac{\delta p}{p_0}$ terms - chromatic effects. IGNORE HERE!

$\cdots$ terms - nonlinear fields (sextupoles, octupoles, etc.) IGNORE HERE!

Example of transport through Quad

Example: For a quad with L=0.1m, (Bρ)=5 Tm, and dB_y/dx=3 T/m, calculate the transport matrix for:

(a) Thin lens approximation.
(b) Full thick lens treatment.

$$K = \pm \frac{1}{\rho B}\frac{dB_y}{dx} = \pm \frac{3}{5} m^{-2}$$

$$L = 0.1m$$

$$f = \frac{1}{KL} = 16.667m$$

$$M_F = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{\sin(\sqrt{K}L)}{\sqrt{K}} \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \qquad M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$M_D = \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{\sinh(\sqrt{K}L)}{\sqrt{K}} \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh(\sqrt{K}L) \end{pmatrix} \qquad M_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$