Off-Momentum Effects and Longitudinal Motion in Rings
• Dispersion (Sections 2.5.4, 5.4)
• Momentum Compaction (Section 5.4)
• Chromaticity (Section 12.2)
• Longitudinal dynamics in rings (Chapter 6)
Equation of Motion

- Go back to full equation of motion for \( x \):

\[
 x'' + (k_0 + \kappa_{x0}^2) x = \kappa_{x0}(\delta - \delta^2) + (k_0 + \kappa_{x0}^2) x \delta - k_0 \kappa_{x0} x^2 - \frac{1}{2} m(x^2 - y^2) + \ldots
\]

- We solved the simplest case, the homogeneous differential equation, with all terms on the r.h.s equal to zero

\[
x'' + (k_0 + \kappa_{x0}^2) x = 0
\]

- And found the solution

\[
x(s) = C(s)x_0 + S(s)x'_0
\]

\[
x'(s) = C'(s)x_0 + S'(s)x'_0
\]

- We will now look at the highest-order energy (momentum)-dependent perturbation term:

\[
x'' + (k_0 + \kappa_{x0}^2) x = \kappa_{x0}\delta = \delta / \rho_0(s)
\]

\[
\delta = \frac{p - p_0}{p_0} = \frac{\Delta p}{p_0}
\]
The general solution of the equation of motion is the sum of the two principal solutions of the homogeneous part, and a particular solution for the inhomogeneous part, where we call the particular solution $\delta D(s)$

$$x(s) = C(s)x_0 + S(s)x'_0 + \delta D(s)$$

$$x'(s) = C'(s)x_0 + S'(s)x'_0 + \delta D'(s)$$

where

$$D(s) = \int_0^s \frac{1}{\rho(\tilde{s})} \left[ S(s)C(\tilde{s}) - C(s)S(\tilde{s}) \right] d\tilde{s}$$

The function $D(s)$ is called the dispersion function.

We can write this solution as the sum of two parts:

$$x(s) = x_\beta(s) + x_\delta(s)$$

From which we conclude the the particle motion is the sum of the betatron motion ($x_\beta$) plus a displacement due to the energy error ($x_\delta$).

We can write the trajectory above in terms of a 3x3 matrix that includes the off-momentum term

$$\begin{bmatrix} x(s) \\ x'(s) \\ \delta \end{bmatrix} = \begin{bmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(s_0) \\ x'(s_0) \\ \delta \end{bmatrix}$$
Examples of trajectories

- No betatron motion:  \( x_\beta = 0: \ x(s) = x_\delta = \delta D(s) \)

- with betatron motion:  \( x' = x'_\beta + x'_\delta \)

\[ x(s) = \delta D(s) \]
Where Does Dispersion Come From?

- Imagine a particle entering a sector bending magnet with an energy that is a little lower than the design energy:

\[
\frac{1}{\rho[m]} = 0.3 \frac{B[T]}{cp[GeV]}
\]

\[
\frac{\rho}{\rho_0} = \frac{cp}{cp_0} = \frac{p_0 + \Delta p}{p_0} = 1 + \delta
\]

\[
\delta D(s) = y_\delta - y_0 = (\rho_0 - \rho) \cos \theta_0 + \rho \cos(\theta - \theta_0) - \rho_0
\]

\[
\delta D(s) = -\delta \rho_0 \cos \theta_0 + (1 + \delta) \rho_0 \cos(\theta - \theta_0) - \rho_0
\]

\[
\delta D(s) \approx \delta \rho_0 (1 - \cos \theta_0)
\]
• Use the transport matrix for a sector bending magnet to calculate the dispersion

\[ M_{SB} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} = \begin{bmatrix} \cos(s / \rho_0) & \rho_0 \sin(s / \rho_0) \\ -\frac{1}{\rho_0} \sin(s / \rho_0) & \cos(s / \rho_0) \end{bmatrix} \]

\[ D(s) = \frac{1}{\rho_0} \int_0^s \left[ \rho_0 \sin \frac{s}{\rho_0} \cos \frac{s}{\rho_0} - \rho_0 \cos \frac{s}{\rho_0} \sin \frac{s}{\rho_0} \right] d\bar{s} \]

\[ D(s) = \rho_0 \left( 1 - \cos \frac{s}{\rho_0} \right) \]

\[ D'(s) = \sin \frac{s}{\rho_0} \]

• Giving the 3x3 transport matrix for a sector bend:

\[ M_{s,\rho} = \begin{bmatrix} \cos \theta & \rho_0 \sin \theta & \rho_0 (1 - \cos \theta) \\ -\frac{1}{\rho_0} \sin \theta & \cos \theta & \sin \theta \\ 0 & 0 & 1 \end{bmatrix} \]

\[ M_{s,0} = \begin{bmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
3x3 Transport Matrices for Drifts and Quadrupoles

- Dispersion is generated in bending magnets
- Quadrupoles and drifts are not sources of dispersion, although they influence the dispersion function because the off-momentum trajectory is bent by quadrupoles

\[ M_{\text{drift}} = \begin{bmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad M_{\text{thinquad}} = \begin{bmatrix} 1 & 0 & 0 \\ -1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
Propagation of Dispersion

- We can write the coordinate vector as
  \[
  \begin{bmatrix}
  x(s) \\
  x'(s) \\
  \delta
  \end{bmatrix} = M
  \begin{bmatrix}
  x(s_0) \\
  x'(s_0) \\
  \delta
  \end{bmatrix} = M
  \begin{bmatrix}
  x_\beta(s_0) + x_\delta(s_0) \\
  x'_\beta(s_0) + x'_\delta(s_0) \\
  \delta
  \end{bmatrix}
  \]

- Suppose we set the starting betatron amplitude and slope equal to zero, that is, make \(x_\beta=0\).
  \[
  \begin{bmatrix}
  x(s) \\
  x'(s) \\
  \delta
  \end{bmatrix} = \begin{bmatrix}
  \delta D(s) \\
  \delta D'(s) \\
  \delta
  \end{bmatrix} = M
  \begin{bmatrix}
  x_\delta(s_0) \\
  x'_\delta(s_0) \\
  \delta
  \end{bmatrix} = M
  \begin{bmatrix}
  \delta D(s_0) \\
  \delta D'(s_0) \\
  \delta
  \end{bmatrix}
  \]

- And dividing by \(\delta\) we have
  \[
  \begin{bmatrix}
  D(s) \\
  D'(s) \\
  1
  \end{bmatrix} = M
  \begin{bmatrix}
  D(s_0) \\
  D'(s_0) \\
  1
  \end{bmatrix}
  \]

- This means that if we know the 3x3 transport matrices, and the starting dispersion functions, we can calculate the dispersion anywhere downstream.
• What is the dispersion in a FODO lattice?

• Construct a simple FODO lattice from this sequence
  \( \frac{1}{2}Q\text{-Bend} - \frac{1}{2}Q \frac{1}{2}Q\text{-Bend} - \frac{1}{2}Q \)

  Where for simplicity the “Bend” has \( \theta \ll 1 \)

\[
\mathcal{M}_{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 1/f & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & L^2/2\rho \\ 0 & 1 & L/\rho \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-L/f & L & L^2/2\rho \\ -L/f^2 & 1+L/f & L \left(1 + \frac{L}{2f}\right) \\ 0 & 0 & 1 \end{bmatrix}
\]

• We look for a periodic solution to the dispersion function in a FODO, that is, a function \( \eta(s) \) that repeats itself

• With that constraint, the \( \eta(s) \) must reach a point of maximum or minimum at a quadrupole, that is \( \eta' = 0 \).

\[
\begin{bmatrix} \eta^- \\ 0 \\ 1 \end{bmatrix} = \mathcal{M}_{1/2} \begin{bmatrix} \eta^+ \\ 0 \\ 1 \end{bmatrix}
\]

• Which gives with \( \kappa = f/L \)

\[
\eta^+ = \frac{f^2}{\rho} \left(1 + \frac{L}{2f}\right) = \frac{L^2}{2\rho} \kappa(2\kappa + 1) \quad \eta^- = \frac{f^2}{\rho} \left(1 - \frac{L}{2f}\right) = \frac{L^2}{2\rho} \kappa(2\kappa - 1)
\]
Periodic Dispersion

• Can solve the equation of motion:

\[ \eta'' + K\eta = 1/\rho \]

• To arrive at the solution for \( \eta(s) \)

\[
(s) = \frac{\sqrt{(s)}}{2\sin(s)} \int_{s}^{s+L_p} \frac{1}{\sqrt{(s)}} \cos[(s) + (s)]d
\]

• Finally, the rms beamsize at a given location has two components, one from the betatron motion of the collection of particles, and another from the finite energy spread in the beam:

\[
\sigma_u(s) = \sqrt{\epsilon_u \beta(s) + \eta^2(s)\sigma^2_\delta}
\]

• Likewise for the \textit{angular beam divergence}

\[
\sigma'_u(s) = \sqrt{\epsilon_u \gamma_u(s) + \eta'^2(s)\sigma^2_\delta}
\]
• Suppose one location in a lattice has a horizontal beta-function = 20 meters, vertical beta-function = 10 meters, and peak dispersion = 8 meters with $\varepsilon_x = \varepsilon_y = 1$ mm-mrad, and $\sigma_\delta = 0.0007$,
  – calculate the horizontal and vertical rms beamsizes
Achromaticity

- Suppose we want to arrange the lattice so that \( D = D' = 0 \) at some particular location in the beamline
- Having established \( D = D' = 0 \) at some location, the lattice has \( D = 0 \) everywhere downstream, up to the next bending magnet
- Such a lattice, or section of lattice is termed *achromatic*
- Start with the integral equation for \( D(s) \)

\[
D(s) = \int_{0}^{s} \frac{1}{\rho(\tilde{s})} [S(s)C(\tilde{s}) - C(s)S(\tilde{s})] d\tilde{s}
\]

- The dispersion and dispersion derivative can be written

\[
D(s) = -S(s)I_c + C(s)I_s \\
D'(s) = -S'(s)I_c + C'(s)I_s
\]

- In terms of the integrals

\[
I_c = \int_{0}^{s} \frac{1}{\rho_0(\tilde{s})} C(\tilde{s}) d\tilde{s} = 0
\]

\[
I_s = \int_{0}^{s} \frac{1}{\rho_0(\tilde{s})} S(\tilde{s}) d\tilde{s} = 0
\]
The integrals can be made to vanish in a lattice segment with $360^\circ$ horizontal phase advance through a FODO section with Bends.
Accelerator Lattices: SNS Accumulator Ring

Four-fold symmetry
Arc: four FODO cells
S.S.: doublets

Working point (6.40,6.30)
Path length and momentum compaction

- The path length is given by
  \[ L = \int (1 + \kappa x) ds = \int (1 + \frac{1}{\rho} \delta D(s)) ds \quad \kappa = 1/\rho \]

- The deviation from the ideal path length is
  \[ \Delta L = L - L_0 = \delta \int \frac{D(s)}{\rho(s)} ds = \delta L_0 \alpha_c \]

- With the momentum compaction factor defined as
  \[ \alpha_c = \frac{\Delta L / L_0}{\delta} \]

- The travel time around the accelerator is
  \[ \tau = L / c\beta \]
  \[ \frac{\Delta \tau}{\tau} = \frac{\Delta L}{L} - \frac{\Delta \beta}{\beta} \]
  \[ \frac{\Delta \tau}{\tau} = \left( \alpha_c - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p} = \eta_c \frac{\Delta p}{p} \]

- The momentum compaction is \( \eta_c \) and the transition-gamma is
  \[ \gamma_t = \frac{1}{\sqrt{\alpha_c}} \]
Path length and momentum compaction

\[
\frac{\Delta \tau}{\tau} = \eta_c \frac{\Delta p}{p} = \left( \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p}
\]

- Three cases:
  - \(\gamma > \gamma_t\), \(\eta_c > 0\), and \(\Delta \tau\) increases with energy, revolution frequency decreases with energy
  - \(\gamma < \gamma_t\), \(\eta_c < 0\), and \(\Delta \tau\) decreases with energy, revolution frequency increases with energy
  - \(\gamma = \gamma_t\), \(\Delta \tau = 0\), independent of energy. Such a ring is called \textit{isochronous}

- This behaviour is a result of the fact that the dispersion function causes higher energy particles to follow an orbit with slightly larger radius than the ideal orbit
- All electron rings operate above transition
- Many proton/hadron synchrotrons must pass through transition as the beam is accelerated
The focusing strength of a quadrupole is

\[ k[m^{-2}] = 0.3 \frac{\partial B / \partial x[T]}{cp[GeV]} \]

A beam particle with momentum error \( \delta \) sees a focusing strength slightly different from that of a particle at the design energy

\[ k[m^{-2}] = 0.3 \frac{\partial B / \partial x[T]}{(1 + \delta)cp[GeV]} \]

In addition to dispersion, we would also expect some effect to the weakened or strengthened quadrupole focusing seen by off-momentum particles

This is the particle-beam equivalent of the chromatic aberration from light optics, which arises from the dependence of the index of refraction of a glass lens on the wavelength of light.

Special optical materials can be made in a telescope to make the image achromatic.
Chromaticity

• Go back to the equations of motion for x and y

\[ x'' + (k_0 - \kappa_{x0}^2) x = \kappa_{x0} (\delta - \delta^2) + (k_0 - \kappa_{x0}^2) x \delta - k_0 \kappa_{x0} x^2 - \frac{1}{2} m(x^2 - y^2) + \ldots \]

\[ y'' - (k_0 + \kappa_{y0}^2) y = \kappa_{y0} (\delta - \delta^2) - (k_0 - \kappa_{y0}^2) y \delta + k_0 \kappa_{y0} y^2 + mxy + \ldots \]

• Plug in \( x = x_\beta + x_\delta = x_\beta + \delta \eta \) \quad \( y = y_\beta \)

• We arrive at the equations of motion for the betatron amplitude, neglecting terms proportional to \( \delta^2 \) or \( x_\beta^2 \) or \( y_\beta^2 \)

\[ x''_\beta + (k + \kappa_{x0}^2) x_\beta = (k + \kappa_{x0}^2) x_\beta \delta - mx_\beta \delta \eta \]

\[ y''_\beta - (k + \kappa_{x0}^2) y_\beta = -(k + \kappa_{x0}^2) y_\beta \delta - my_\beta \delta \eta \]

• or

\[ x''_\beta + Kx_\beta = (K - m\eta) \delta x_\beta \]

\[ y''_\beta - Ky_\beta = -(K - m\eta) \delta y_\beta \]

Modified focusing strength due to momentum error \( \delta \)

Additional focusing from displaced closed orbit in sextupoles due to dispersion
Chromaticity

• In the last lecture we studied gradient errors. This new term is just another type of gradient error, as we anticipated, which will modify the beta-functions and therefore also the betatron tunes of a circular accelerator

• We calculated the betatron tune shift due to gradient errors:

\[ \Delta \nu_x = -\frac{1}{4\pi} \int \beta_x (\Delta k) ds \]

• With the gradient error \((k-m\eta)\), this gives

\[ \Delta \nu_x = -\delta \frac{1}{4\pi} \int \beta_x (k-m\eta) ds = \delta \xi_x \]

\[ \Delta \nu_y = \delta \frac{1}{4\pi} \int \beta_y (k-m\eta) ds = \delta \xi_y \]

• In an accelerator without sextupoles, or with sextupoles turned off, the resulting chromaticity is that due solely to the slightly different focusing seen by off-energy particles. This value of chromaticity is called the natural chromaticity, which always has a negative value!

\[ \xi_{x0} = -\frac{1}{4\pi} \int \beta_x k ds \]

\[ \xi_{y0} = \frac{1}{4\pi} \int \beta_y k ds \]
1. Non-zero chromaticity means that each particle’s tune depends on energy. If there is a range in energies, there will be a range in tunes.
   • A beam with a large range in tunes, or tune-spread occupies a large area on the tune-plane. This opens the possibility of a portion of the beam being placed on a resonance line.

2. The value of the chromaticity, as it turns out, is an important variable that determines whether certain intensity-dependent motion is stable or unstable.
How Sextupoles Work

- The field of a sextupole, in the horizontal plane is this:
  \[
  \frac{e}{cp} B_x = mxy \\
  \frac{e}{cp} B_y = \frac{1}{2} m(x^2 - y^2)
  \]

- The vertical field gradient is:
  \[
  \frac{e}{cp} \frac{\partial B_y}{\partial x} = mx = m\delta\eta
  \]

- Where the coordinates for off-momentum particles \((y=0, x=\delta\eta)\) has been taken.
- Therefore, the sextupole provides quadrupole focusing in the horizontal plane, with focusing strength proportional to \(\delta\):
  - particles with higher momentum are focused in the horizontal plane, and
  - particles with lower momentum are defocusing in the horizontal plane.

- This is exactly what is needed to counteract the dependence of quadrupole focusing on energy.
Chromaticity Correction: Sextupole Magnets

- We can use this feature of the sextupole field to correct the chromaticity, that is, make \( \xi_x = \xi_y = 0 \)

\[
\xi_x = \xi_{x0} + \frac{1}{4\pi} \int m\beta_x \eta ds
\]

\[
\xi_y = \xi_{y0} \quad - \quad \frac{1}{4\pi} \int m\beta_y \eta ds
\]

- We need at least two sextupole magnets to simultaneously make both chromaticities zero. Let’s place two sextupoles in the lattice, with strength \( m_1, m_2 \) and length \( l \).

\[
\xi_x = \xi_{x0} + \frac{1}{4\pi} \left( m_1 l \eta_1 \beta_{x1} + m_2 l \eta_2 \beta_{x2} \right) = 0
\]

\[
\xi_y = \xi_{y0} \quad - \quad \frac{1}{4\pi} \left( m_1 l \eta_1 \beta_{y1} + m_2 l \eta_2 \beta_{y2} \right) = 0
\]

- Sextupoles placed at locations with large dispersion are more effective. We also need \( \beta_x >> \beta_y \) at one location and \( \beta_y >> \beta_x \) at another.
Chromaticity in FODO Cells

- The natural chromaticity in one-half FODO cell becomes:

\[ \xi_{x0} = -\frac{1}{4\pi} \int k_x ds = -\frac{1}{4\pi} \left( \beta^+ \int k^+ ds + \beta^- \int k^- ds \right) \]

\[ \xi_{x0} = -\frac{1}{4\pi} \left( \beta^+ - \beta^- \right) \int k \ ds \]

- Giving for a full FODO cell:

\[ \xi_{x0} = -\frac{1}{\pi} \frac{1}{\sqrt{k^2 - 1}} = -\frac{1}{\pi} \tan(\varphi_x / 2) \]

- So a FODO channel with 90 degrees phase advance/cell has natural chromaticity -1/\pi
The formulation of longitudinal motion in linacs holds also for rings. The synchronous phase is set according to the need to accelerate, and according to the sign of the momentum compaction so that phase stability is achieved.
Phase Stability

• Electron storage rings and Synchrotrons: \( \pi /2 < \phi_s < \pi \)
• Proton storage rings and synchrotrons below transition: \( 0 < \phi_s < \pi /2 \)
• Proton storage rings and synchrotrons above transition: \( \pi /2 < \phi_s < \pi \)
• Proton synchrotrons may start with \( \gamma < \gamma_{\text{tr}} \), but since the energy increases, eventually \( \gamma \) crosses the transition-energy to reach \( \gamma > \gamma_{\text{tr}} \)
• This is called “transition-crossing”. During this event, the synchronous phase of the RF system must jump by 180° so that the higher energy beam remains phase-stable.
• Proton accelerators often have a “gamma-t jump” system consisting of a set of pulsed-quadrupole magnets that momentarily varies the momentum compaction by perturbing the dispersion function so that the lattice \( \gamma_{\text{tr}} \) is pushed below the proton \( \gamma \).
• Same analysis that we followed for the linac case can be repeated for the circular case

• Results in the equation of motion for the particle phase:

\[ \ddot{\phi} + \Omega^2 \phi = 0 \]

• With an oscillation frequency given by:

\[ \Omega^2 = \omega_{rev}^2 \frac{h \eta_c e \hat{V}_0 \cos \phi_s}{2\pi \beta cp} \]

• Where
  – \( h \) is the harmonic number, defined by \( f_{RF} = hf_{rev} \)
  – The particle’s energy gain in one ring revolution is:

\[ e \hat{V}_0 \sin \phi_s \]

• The oscillation frequency is called the **synchrotron frequency**, and the ratio of synchrotron frequency to revolution frequency is the **synchrotron tune**

\[ \nu_s = \frac{\Omega}{\omega_{rev}} \]
Longitudinal Motion

• This should equal the result we obtained previously for a linac:
  \[ \omega_t^2 = \frac{\omega_0^2 qE_0 T \lambda \sin(-\phi_s)}{2\pi m c^2 \gamma_s^3 \beta_s} \]

• We can see that these two are equal by noting that,
  – The convention for linacs is  \( V_{RF} = V_0 \cos \omega t \)
  – Whereas that for rings is  \( V_{RF} = V_0 \sin \omega t \)
  
  therefore,  \( \phi_s^{ring} = \phi_s^{linac} + \pi/2 \), so  \( \cos(\phi_s^{ring}) = \cos(\phi_s^{linac} + \pi/2) = \sin(-\phi_s^{linac}) \)

  – The momentum compaction in the linac is just:
    \[ \eta_c = \left( \frac{1}{\gamma^2 - \alpha_c} \right) = \frac{1}{\gamma^2} \]
  
  – Since  \( \alpha_c = (\Delta L/L)/(\Delta p/p) = 0 \) since there are no bending magnets, and therefore no dispersion in a linac
  – The energy gain in one ring revolution is:  \( e\hat{V}_0 = qE_0 TC = qE_0 T (h\beta\lambda) \)
  – Putting all this together, we arrive at the same frequency that we calculated for the linac.
  – The longitudinal dynamics that we learned in the linac applies directly to the ring case as well
  – The various parameters expressed for the ring contain the momentum compaction factor, which is zero in a linac