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High Intensity RF Linear Accelerators

2-1. Preliminaries of high-intensity beam dynamics

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Content

Maxwell's Equations

Electromagnetic field created by external sources, and by particle beam is described by set of Maxwell's equations:

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad rot\,\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}$$

$$div\,\vec{D} = \rho \qquad div\,\vec{B} = 0$$

where $\vec{E}$ is the electric field, $\vec{D}$ is the electric displacement field, $\vec{B}$ is the magnetic field, $\vec{H}$ is the magnetic field strength. In vacuum, $\vec{D} = \varepsilon_o \vec{E}$, $\vec{B} = \mu_o \vec{H}$, where ε_o = 8.85 x 10^{-12} F/m is the electric permittivity, and μ_o = 4π 10^{-7} H/m is the magnetic permeability of free space.

Electric, $\vec{E}$, and magnetic, $\vec{B}$, fields are expressed through vector potential $\vec{A}$ and scalar potential U

$$\vec{E} = -q\frac{\partial \vec{A}}{\partial t} - grad\,U\,, \qquad \vec{B} = rot\,\vec{A}\,.$$

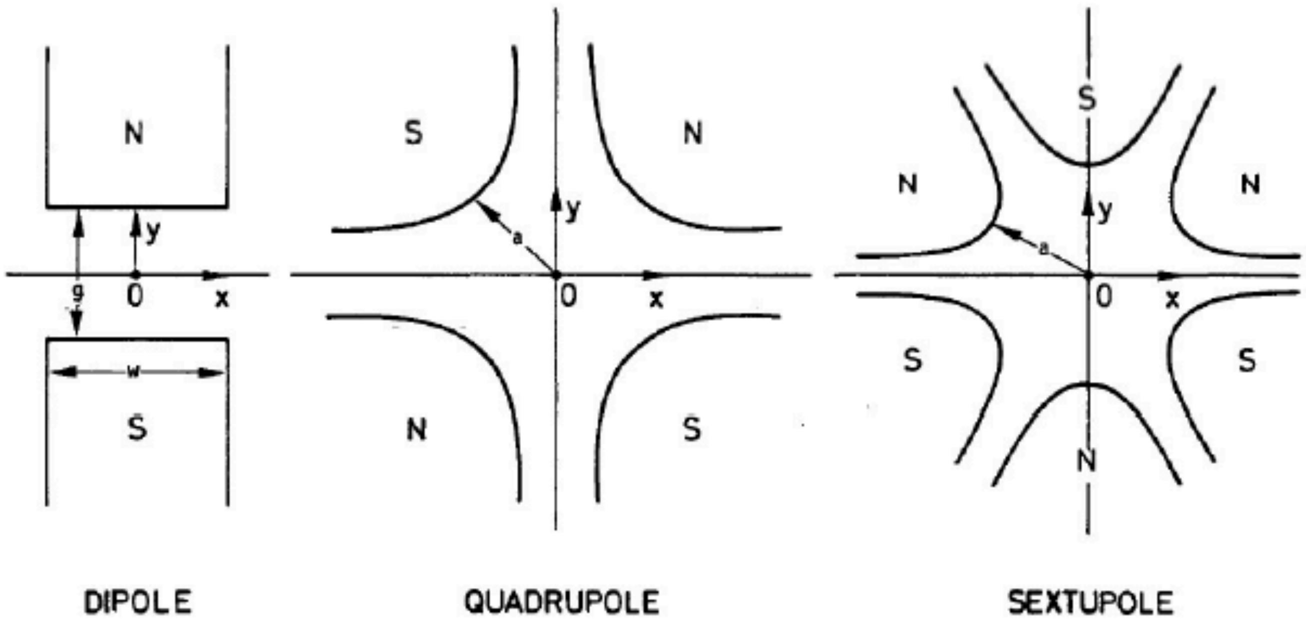
Substitution into Maxwell's equations together with Lorenz gauge

$$div\,\vec{A} + \frac{1}{c^2}\frac{\partial U}{\partial t} = 0$$

gives equations for potentials of the field:

$$\Delta U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = -\frac{\rho}{\varepsilon_o}\,, \qquad \Delta \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_o \vec{j}$$

Fields in Focusing Channels



Focusing magnets used in acceleratior facilities:
dipole, quadrupole, sextupole.

Magnetostatic and Electrostatic Fields

Equations describing magnetostatic field are obtained from Maxwell equations assuming $\partial / \partial t = 0$ in equations for magnetic field:

$$rot\,\vec{H} = \vec{j}$$

$$div\,\vec{B} = 0$$

$$\vec{B} = \mu \vec{H}$$

Inside aperture, in the absense of currents, $\vec{j} = 0$, magnetic field can be equally determined using magnetic scalar potential, U_{magn}, or vector potential, $\vec{A}_{magn}$:

$$\vec{B} = -grad\,U_{magn}, \qquad \vec{B} = rot\,\vec{A}_{magn}$$

Magnetic scalar potential is convenient to determine ideal pole contour, while vector potential is convenient to determine magnetic field shape. Electrostatic field is expressed through electrostatic potential:

$$\vec{E} = -grad\,U_{el}$$

Formally, both magnetic and electrostatic multipole fields are derived from Laplace equation with appropriate boundary conditions:

$$\frac{1}{r}\frac{\partial U}{\partial r}+\frac{\partial^2 U}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}+\frac{\partial^2 U}{\partial z^2}=0$$

and similar for A_z component of vector – potential:

$$\frac{1}{r}\frac{\partial A_z}{\partial r}+\frac{\partial^2 A_z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 A_z}{\partial \theta^2}+\frac{\partial^2 A_z}{\partial z^2}=0$$

General solution of 3-dimensional Laplace equation in cylindrical coordinates is [W.Glazer, Grundlagen Der Electronenoptik, Wien, Springer-Verlag, p.102 (1952).]

$$\Pi(r,\theta,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n m!}{4^n n!(m+n)!}r^{m+2n}(\Phi_m^{(2n)}\cos m\theta+\Psi_m^{(2n)}\sin m\theta)$$

$$=\Phi-\frac{1}{4}r^2\Phi''+\frac{1}{64}r^4\Phi^{(4)}-.....+(\Phi_1-\frac{1}{8}r^2\Phi_1'')r\cos\theta+(\Psi_1-\frac{1}{8}r^2\Psi_1'')r\sin\theta+$$

$$+(\Phi_2-\frac{1}{12}r^2\Phi_2'')r^2\cos 2\theta+(\Psi_2-\frac{1}{12}r^2\Psi_2'')r^2\sin 2\theta$$

$$+\Phi_3 r^3\cos 3\theta+\Psi_3 r^3\sin 3\theta+...+\Phi_4 r^4\cos 4\theta+\Psi_4 r^4\sin 4\theta+\ldots$$

where $\Phi_m(z)$ and $\Psi_m(z)$ are functions of longitudinal coordinate z and m is the order of multipole:

m = 0 for axial-symmetric filed, m =1 for dipole, m = 2 for quadrupole, m = 3 for sextupole, m = 4 for octupole, m = 5 for decapole , m = 6 for dodecapole.

Here solution $\Pi(r,\theta,z)$ stands for either A_z, or U_{magn}, or U_{el}.

Number of poles to excite the multipole lens of the order m is $$N_{poles} = 2m$$

In most of cases, it is possible to substitute actual z-dependence of the field by "step" function. For such representation, solution of Laplace equation is

$$\Pi(r,\theta) = \sum_{m=0}^{\infty} r^m (\Phi_m \cos m\theta + \Psi_m \sin m\theta)$$

Field is periodic determined by condition $m\theta = 2\pi$. Solutions for magnetic field can be represented as a combination of multipoles with field:

$$A_z = -\frac{G_m}{m} r^m \cos[m(\theta - \theta_o)]$$

$$U_{magn} = -\frac{G_m}{m} r^m \sin[m(\theta - \theta_o)]$$

where G_m is the strength of the multipole of order m: $$G_m = \frac{B(r_o)}{r_o^{m-1}}$$

and $B(r_o)$ is the absolute value of magnetic field at certain radius r_o.

Normal and Skew Magnets

"Normal" multipole corresponds to $\theta_o = 0$, while skew multipole is achieved by rotating multipole at the angle $\theta_o = \dfrac{\pi}{2m}$

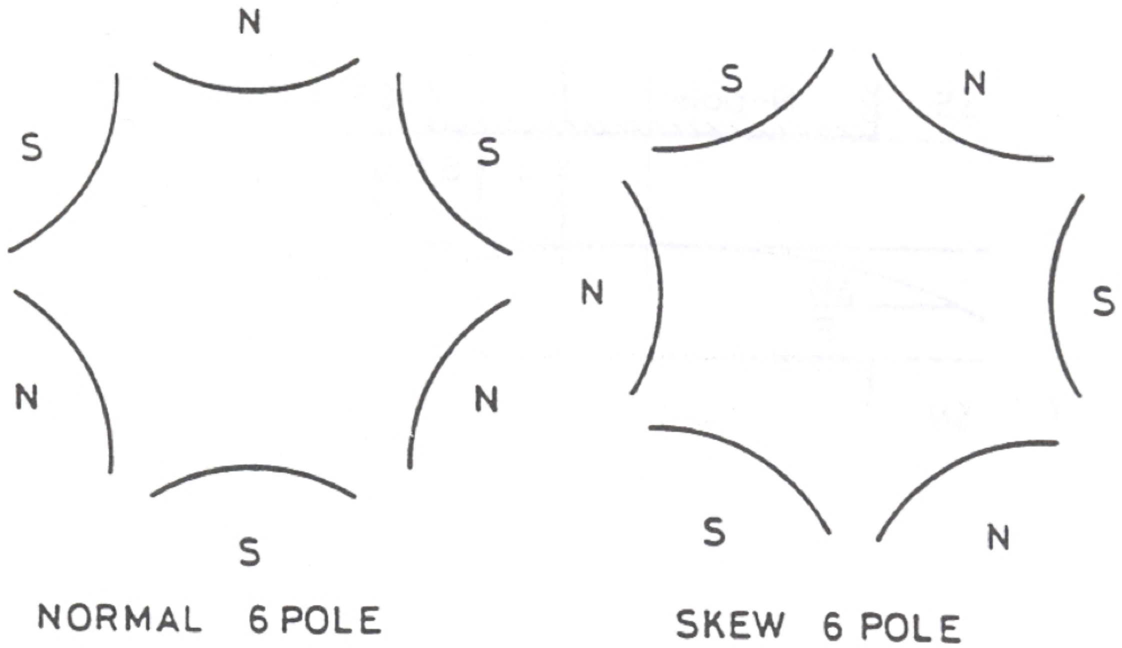


Fig. 2 Pole configurations for a regular sextupole and a skew sextupole

Field Components

Components of magnetic field are determined by

$$B_r = -\frac{\partial U_m}{\partial r} = \frac{1}{r}\frac{\partial A_z}{\partial \theta} = G_m r^{m-1} \sin[m(\theta - \theta_o)]$$

$$B_\theta = -\frac{1}{r}\frac{\partial U_m}{\partial \theta} = -\frac{\partial A_z}{\partial r} = G_m r^{m-1} \cos[m(\theta - \theta_o)]$$

Absolute value of the field is azimuth-independent:

$$B(r) = \sqrt{B_r^2 + B_\theta^2} = G_m r^{m-1}$$

Components of electric field are:

$$E_r = -\frac{\partial U_{el}}{\partial r} = -G_m r^{m-1} \cos[m(\theta - \theta_o)]$$

$$E_\theta = -\frac{1}{r}\frac{\partial U_{el}}{\partial \theta} = G_m r^{m-1} \sin[m(\theta - \theta_o)]$$

Expressions for Multipole Fields

Expressions for magnetic vector potential $-A_z$ and electrostatic potential U_{el} of "normal" multipole:

m=1 Dipole $G_1 r \cos\theta = G_1 x$

m=2 Quadrupole $\frac{G_2}{2} r^2 \cos 2\theta = \frac{G_2}{2}(x^2 - y^2)$

m=3 Sextupole $\frac{G_3}{3} r^3 \cos 3\theta = \frac{G_3}{3}(x^3 - 3xy^2)$

m=4 Octupole $\frac{G_4}{4} r^4 \cos 4\theta = \frac{G_4}{4}(x^4 \text{-} 6x^2y^2 + y^4)$

m=5 Decapole $\frac{G_5}{5} r^5 \cos 5\theta = \frac{G_5}{5}(x^5 - 10x^3y^2 + 5xy^4)$

m=6 Dodecapole $\frac{G_6}{6} r^6 \cos 6\theta = \frac{G_6}{6}(x^6 - y^6 - 15x^4y^2 + 15x^2y^4)$

Quadrupole Pole Shapes and Higher Order Harmonics

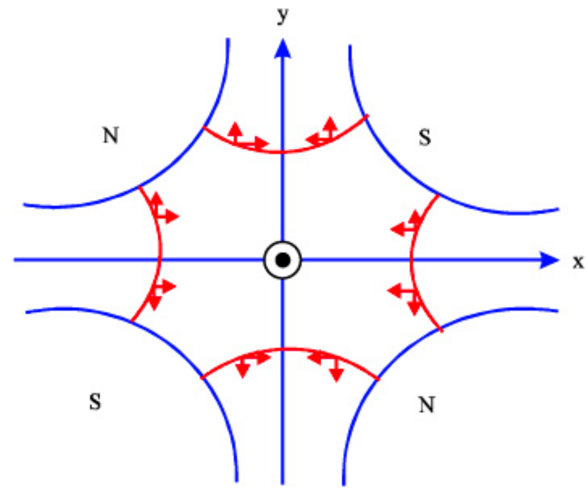
Pole contours are determined by lines of equal values of scalar potentials

$$U_{magn}(r,\theta) = const\ , \qquad U_{el}(r,\theta) = const\ .$$

For example, shape of "normal" quadrupole poles are described by infinite hyperbolas:

$x^2 - y^2 = \pm a^2$ for electrostatic quadrupole

$2xy = \pm a^2$ for magnetostatic quadruple



In practice, actual pole shapes are different from that determined above. Solution of Laplace equation for multipole is antysymmetric after angle π / m because of separation of neighbour poles with alternative polarity:

$$\Pi(r,\theta) = -\Pi(r,\theta + \frac{\pi}{m})$$

It results in the following equations to determine numbers of higher harmonics k with respect to fundamental harmonic with number m:

$$\cos k(\theta + \frac{\pi}{m}) = -\cos k\theta\ , \qquad \sin k(\theta + \frac{\pi}{m}) = -\sin k\theta$$

which are satisfied when $\cos(k\frac{\pi}{m}) = -1\ ,$ $\sin(k\frac{\pi}{m}) = 0$

Both equations are valid for $k = m(1+2l)$, $l = 0, 1, 2, 3, \ldots$.. Particularly, field of quadrupole lens contains the following multipole harmonics:

$$A_z(r,\theta) = -(\frac{G_2}{2} r^2 \cos 2\theta + \frac{G_6}{6} r^6 \cos 6\theta + \frac{G_{10}}{10} r^{10} \cos 10\theta + \ldots)$$

Hamiltonian dynamics

Hamiltonian of charged particle with charge q and mass m

$$H = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} + q\,U$$

$x,\ y,\ z$	position in real space
$P_x,\ P_y,\ P_z$	components of canonical momentum
$A_x,\ A_y,\ A_z$	components of the vector – potential
$U(x,y,z)$	scalar potential of the electromagnetic field

Equations of motion: $$\frac{d\vec{x}}{dt} = \frac{\partial H}{\partial \vec{P}} \qquad \frac{d\vec{P}}{dt} = -\frac{\partial H}{\partial \vec{x}}$$

Canonical momentum $\vec{P} = (P_x, P_y, P_z)$ and mechanical momentum $\vec{p} = (p_x, p_y, p_z)$ are related:

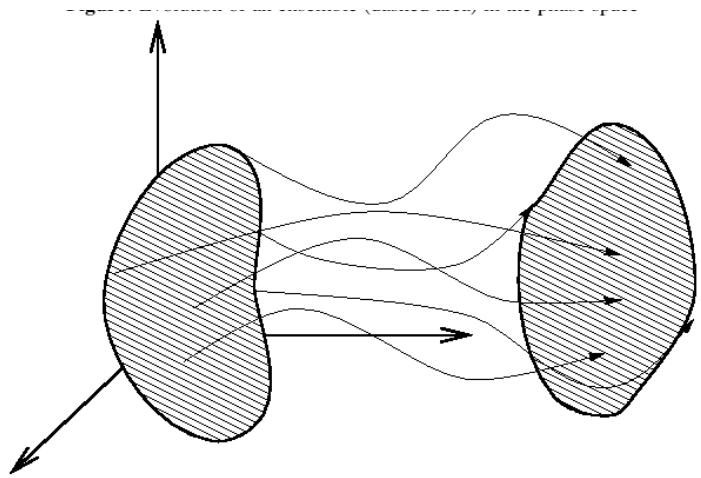
$$\vec{p} = \vec{P} - q\,\vec{A}$$

Element of phase space: $dV = dx\,dy\,dz\,dP_x\,dP_y\,dP_z$

Phase space density (beam distribution function):

$$f(x, y, z, P_x, P_y, P_z) = \frac{dN}{dx\,dy\,dz\,dP_x\,dP_y\,dP_z}$$

Liouville’s Theorem



Conservation of phase space volume occupied by particles in Hamiltonian systems.

Liouville’s theorem: if the motion of a system of mechanical particles obeys Hamilton’s equations, then phase space density remains constant along phase space trajectories and phase space volume occupied by the particles is invariant (Liouville's Equation):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{x}}\frac{d\vec{x}}{dt} + \frac{\partial f}{\partial \vec{P}}\frac{d\vec{P}}{dt} = 0$$

Hamiltonian equations of motion

Motion of a charged classical particle in an electromagnetic field is described by Hamiltonian dynamics. The three corresponding canonical conjugate variable pairs are $(x,\ P_x)$, $(y,\ P_y)$, $(z,\ P_z)$. The equations of motion then follow from Hamilton's equations:

$$\frac{dx}{dt} = \frac{\partial H}{\partial P_x}, \qquad \frac{dy}{dt} = \frac{\partial H}{\partial P_y}, \qquad \frac{dz}{dt} = \frac{\partial H}{\partial P_z}, \tag{1.27}$$

$$\frac{dP_x}{dt} = -\frac{\partial H}{\partial x}, \qquad \frac{dP_y}{dt} = -\frac{\partial H}{\partial y}, \qquad \frac{dP_z}{dt} = -\frac{\partial H}{\partial z}. \tag{1.28}$$

As an example, taking a partial derivative of the Hamiltonian with respect to P_x yields the equation for the rate of change of the particle's x-position

$$\frac{dx}{dt} = \frac{c\ (P_x - q\ A_x)}{\sqrt{m^2c^2 + (P_x - q\ A_x)^2 + (P_y - q\ A_y)^2 + (P_z - q\ A_z)^2}}\ . \tag{1.29}$$

Canonical momentum $\vec{P} = (P_x, P_y, P_z)$ is related to mechanical momentum $\vec{p} = (p_x, p_y, p_z)$ via the expression:

$$\vec{p} = \vec{P} - q\vec{A} \tag{1.30}$$

Note that the denominator in Eq.(1.29) is actually $mc\gamma$, where the relativistic factor γ is:

$$\gamma = \sqrt{1 + \frac{(P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2}{m^2c^2}}\ . \tag{1.31}$$

Analogously, the equations for the rates of change of the y- and z - positions of the particle can be derived. So, the set of equations for the rate of change of the particle's position is

$$\frac{dx}{dt} = \frac{(P_x - qA_x)}{m\gamma}, \quad \frac{dy}{dt} = \frac{(P_y - qA_y)}{m\gamma}, \quad \frac{dz}{dt} = \frac{(P_z - qA_z)}{m\gamma}. \tag{1.32}$$

Taking partial derivatives of the Hamiltonian with respect to the particle's positions, the equations for the rate of change of the canonical momentum vector are:

$$\frac{dP_x}{dt} = \frac{q}{m\gamma}\,[(P_x - qA_x)\frac{\partial A_x}{\partial x} + (P_y - qA_y)\frac{\partial A_y}{\partial x} + (P_z - qA_z)\frac{\partial A_z}{\partial x}] - q\frac{\partial U}{\partial x}\,, \tag{1.33}$$

$$\frac{dP_y}{dt} = \frac{q}{m\gamma}\,[(P_x - qA_x)\frac{\partial A_x}{\partial y} + (P_y - qA_y)\frac{\partial A_y}{\partial y} + (P_z - qA_z)\frac{\partial A_z}{\partial y}] - q\frac{\partial U}{\partial y} \tag{1.34}$$

$$\frac{dP_z}{dt} = \frac{q}{m\gamma}\,[(P_x - qA_x)\frac{\partial A_x}{\partial z} + (P_y - qA_y)\frac{\partial A_y}{\partial z} + (P_z - qA_z)\frac{\partial A_z}{\partial z}] - q\frac{\partial U}{\partial z}\,. \tag{1.35}$$

It is more common to integrate the equations of motion for mechanical momentum, and use electric, $\vec{E}$, and magnetic, $\vec{B}$, fields instead of vector potential $\vec{A}$ and scalar potential U:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - gradU \ , \quad \vec{B} = rot\ \vec{A} \ . \tag{1.36}$$

The left-hand side of the equation for the rate of change of the x-component of the canonical momentum, $P_x = p_x + qA_x$, can be represented as follows:

$$\frac{dP_x}{dt} = \frac{dp_x}{dt} + q\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt}\right). \tag{1.37}$$

A combination of this equation with Eq. (1.33), gives:

$$\frac{dp_x}{dt} = q\left(-\frac{\partial A_x}{\partial t} - \frac{\partial U}{\partial x}\right) + q\left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + v_z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right], \tag{1.38}$$

Applying the same derivations for p_y and p_z, the final set of equations in Cartesian coordinates is:

$$\frac{dx}{dt} = \frac{p_x}{m\,\gamma}\,, \tag{1.39}$$

$$\frac{dy}{dt} = \frac{p_y}{m\,\gamma}\,, \tag{1.40}$$

$$\frac{dz}{dt} = \frac{p_z}{m\,\gamma}\,, \tag{1.41}$$

$$\frac{d\,p_x}{dt} = q\,(E_x + \frac{p_y}{m\,\gamma}B_z - \frac{p_z}{m\,\gamma}B_y)\,, \tag{1.42}$$

$$\frac{d\,p_y}{dt} = q\,(E_y - \frac{p_x}{m\,\gamma}B_z + \frac{p_z}{m\,\gamma}B_x)\,, \tag{1.43}$$

$$\frac{d\,p_z}{dt} = q\,(E_z + \frac{p_x}{m\,\gamma}B_y - \frac{p_y}{m\,\gamma}B_x)\,, \tag{1.44}$$

or

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m\gamma} \qquad \frac{d\vec{p}}{dt} = q\{\vec{E} + [\vec{v}\vec{B}]\}$$

Canonical Transformations

In Hamiltonian mechanics, a canonical transformation is a change of canonical coordinates (q,p,*t*) → (Q,P,*t*) that preserves the form of Hamilton's equations. Hamiltonian equations of motions are

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}$$

New variables also obey canonical equations of motion

$$\frac{dQ_i}{dt}=\frac{\partial H'}{\partial P_i}, \qquad \frac{dP_i}{dt}=-\frac{\partial H'}{\partial Q_i} \tag{5.1}$$

where H' is a new Hamiltonian. New variables can be considered as functions of old variables and time $Q_i = Q_i(p_i, q_i, t)$, $P_i = P_i(p_i, q_i, t)$. Transformations from old variables to new variables, which keep canonical structure of the equation of motion (5.1) are called <u>canonical transformations</u>.

From classical mechanics it follows, that both new and old variables obey principle of least action :

$$\delta \int \left(\sum p_i \, dq_i - H dt \right) = 0 \tag{5.2}$$

$$\delta \int \left(\sum P_i \, dQ_i - H' \, dt \right) = 0 \tag{5.3}$$

That means, that integrands in eqs. (5.2), (5.3) are different as total differential of arbitrary function F of coordinates, momentum and time:

$$\sum p_i \, dq_i - H \, dt = \sum P_i \, dQ_i - H' \, dt + dF, \text{ or} \tag{5.4}$$

$$dF = \sum p_i \, dq_i - \sum P_i \, dQ_i + (H' - H) \, dt \tag{5.5}$$

Function F is called generating function of transformation.

Type 1 generating function

To be a total differential, equation (5.5) has to have the following form:

$$dF = \sum \frac{\partial F}{\partial q_i} dq_i + \sum \frac{\partial F}{\partial Q_i} dQ_i + \frac{\partial F}{\partial t} dt \tag{5.6}$$

From comparison of equations (5.5) and (5.6) it is clear, that the variables and the new Hamiltonian have to obey the following equatons:

$$p_i = \frac{\partial F}{\partial q_i}, \qquad P_i = -\frac{\partial F}{\partial Q_i} \qquad (H' - H)\, dt = \frac{\partial F}{\partial t} dt \tag{5.7}$$

Therefore new Hamiltonian is connected with the old one via relationship

$$\boxed{H' = H + \frac{\partial F}{\partial t}} \tag{5.8}$$

Equations (5.7) provide canonical transformation from old variables to new variables, if generating function depends on old and new coordinates:

$$\boxed{p_i = \frac{\partial F_1}{\partial q_i}} \qquad \boxed{P_i = -\frac{\partial F_1}{\partial Q_i}} \qquad \boxed{F_1 = F_1(q, Q, t)} \tag{5.9}$$

Type 2 generating function

Let us rewrite eq. (5.5) as follow:

$$dF = \sum p_i\, dq_i - \sum P_i\, dQ_i + \sum Q_i\, dP_i - \sum Q_i\, dP_i + (H' - H)\, dt \tag{5.11}$$

Let us introduce new generating function F_2

$$F_2 = F + \sum P_i Q_i, \qquad dF_2 = dF + \sum P_i\, dQ_i + \sum Q_i\, dP_i \tag{5.12}$$

For new generting function the following equation is valid:

$$dF_2 = \sum p_i\, dq_i + \sum Q_i\, dP_i + (H' - H)\, dt \tag{5.13}$$

Equation (5.13) indicates, that generating function of the second type is a function of old coordinates and new momentum $F_2 = F_2(q, P, t)$. Relationship between new Hamiltonian and the old one is given by equation (5.8). Again, to be a total differential, the following eqautions have to be valid, which form the second canonical transformation:

$$\boxed{p_i = \frac{\partial F_2}{\partial q_i}} \qquad \boxed{Q_i = \frac{\partial F_2}{\partial P_i}} \qquad \boxed{F_2 = F_2(q, P, t)} \tag{5.14}$$

Type 3 generating function

To find third canonical transformation, let us add and subtract $\sum q_i \, dp_i$ from eq. (5.5):

$$dF = \sum p_i \, dq_i - \sum P_i \, dQ_i + \sum q_i \, dp_i - \sum q_i \, dp_i + (H' - H) \, dt \qquad (5.16)$$

Introducing generating function of the 3rd type

$$F_3 = F - \sum p_i \, q_i, \qquad dF_3 = dF - \sum p_i \, dq_i - \sum q_i \, dp_i \qquad (5.17)$$

the eqution for total differential of the generating function is as follow:

$$dF_3 = - \sum P_i \, dQ_i - \sum q_i \, dp_i + (H' - H) \, dt \qquad (5.18)$$

Last equation forms the canonical transformation of the 3rd type:

$$\boxed{P_i = - \frac{\partial F_3}{\partial Q_i}} \qquad \boxed{q_i = - \frac{\partial F_3}{\partial p_i}} \qquad \boxed{F_3 = F_3 \, (Q, p, t)} \qquad (5.19)$$

Type 4 generating function

Forth canonical transformation is attained via adding and subtracting of the $\sum Q_i \, dP_i$ from Eq. (5.5):

$$dF = \sum p_i \, dq_i - \sum P_i \, dQ_i + \sum q_i \, dp_i - \sum q_i \, dp_i + \sum Q_i \, dP_i - \sum Q_i \, dP_i + (H' - H) \, dt$$

Generating function of the 4th ype is defined as follow:

$$F_4 = F - \sum p_i \, q_i + \sum P_i \, Q_i \qquad (5.22)$$

It results in the eqution for total differential of the generating function:

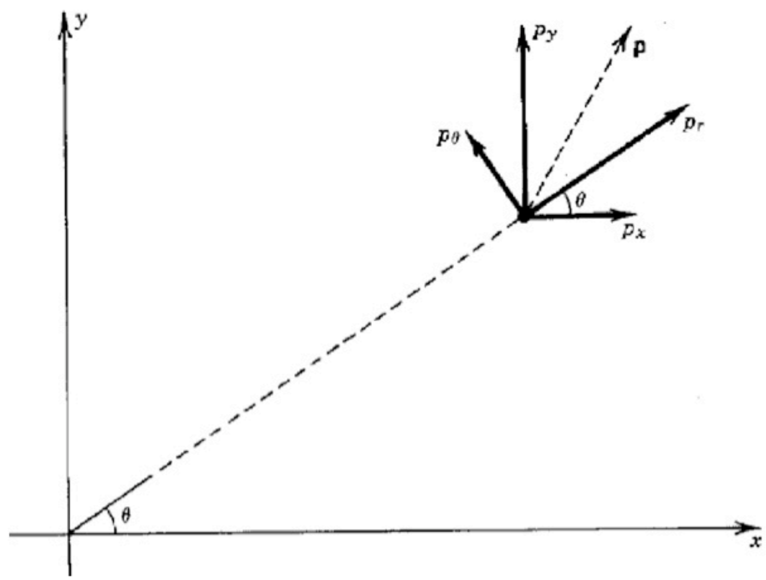
$$dF_4 = - \sum q_i \, dp_i + \sum Q_i \, dP_i + (H' - H) \, dt \qquad (5.23)$$

Canonical transformation of the 4th type are descibed by equations:

$$\boxed{q_i = - \frac{\partial F_4}{\partial p_i}} \qquad \boxed{Q_i = \frac{\partial F_4}{\partial P_i}} \qquad \boxed{F_4 = F_4 \, (p, P, t)} \qquad (5.24)$$

Example: Canonical transformation from Cartesian to cylindrical coordinates

Very often, particle dynamics in accelerators is described in a cylindrical system of coordinates (r, θ, z), because of axial symmetry inherent to accelerating structures.



Relationship between cylindrical and Cartesian coordinates.

A canonical transformation of the Hamiltonian from Cartesian to cylindrical system of coordinates is accomplished by selecting a generating function of the transformation, as a function of new position variables and old momentum:

$$F_3\,(r, \theta, z, P_x, P_y, P_z) = -\,r\,P_x \cos\theta\ -\,r\,P_y \sin\theta - z\,P_z\,. \tag{1.45}$$

The relationships between new and old variables in a canonical transformation are obtained using the equations

$$x = -\frac{\partial F_3}{\partial P_x}, \qquad y = -\frac{\partial F_3}{\partial P_y}, \qquad z = -\frac{\partial F_3}{\partial P_z}, \tag{1.46}$$

$$P_r = -\frac{\partial F_3}{\partial r}, \qquad P_\theta = -\frac{\partial F_3}{\partial \theta}, \quad P_z = -\frac{\partial F_3}{\partial z}. \tag{1.47}$$

Calculation of the partial derivatives, Eqs. (1.46), (1.47), gives the relationship between Cartesian and cylindrical coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \tag{1.48}$$

$$P_r = P_x\cos\theta + P_y\sin\theta, \tag{1.49}$$

$$P_\theta = r\,(-P_x\sin\theta + P_y\cos\theta), \tag{1.50}$$

$$P_z = P_z\,. \tag{1.51}$$

Inverse transformation of Eqs. (1.49) (1.50), (1.52), (1.53) gives

$$P_x = P_r \cos\theta - \frac{P_\theta}{r} \sin\theta \,, \tag{1.56}$$

$$P_y = P_r \sin\theta + \frac{P_\theta}{r} \cos\theta \,, \tag{1.57}$$

$$P_z = P_z \,. \tag{1.51}$$

$$A_x = A_r \cos\theta - A_\theta \sin\theta \,, \tag{1.58}$$

$$A_y = A_r \sin\theta + A_\theta \cos\theta \,. \tag{1.59}$$

$$A_z = A_z \tag{1.54}$$

After a canonical transformation, the new Hamiltonian is expressed in terms of the old one as

$$K = H + \frac{\partial F_3}{\partial t} \, . \tag{1.55}$$

Since the generating function, Eq. (1.45), does not depend on time explicitly, the new Hamiltonian equals the old one, $K = H$:

$$H = c\sqrt{(mc)^2 + \left(\frac{P_\theta}{r} - qA_\theta\right)^2 + (P_r - qA_r)^2 + (P_z - qA_z)^2} + qU \, . \tag{1.60}$$

Hamilton’s equations in cylindrical coordinates read

$$\frac{dr}{dt} = \frac{\partial H}{\partial P_r}, \qquad \frac{d\theta}{dt} = \frac{\partial H}{\partial P_\theta}, \qquad \frac{dz}{dt} = \frac{\partial H}{\partial P_z}, \tag{1.61}$$

$$\frac{dP_r}{dt} = -\frac{\partial H}{\partial r}, \qquad \frac{dP_\theta}{dt} = -\frac{\partial H}{\partial \theta}, \qquad \frac{dP_z}{dt} = -\frac{\partial H}{\partial z} \, . \tag{1.62}$$

Calculating the partial derivatives, Eqs. (1.61), the equations for particle position are

$$\frac{dr}{dt} = \frac{P_r - qA_r}{m\gamma}, \tag{1.63}$$

$$\frac{d\theta}{dt} = \frac{1}{m\gamma\, r}\left(\frac{P_\theta}{r} - qA_\theta\right), \tag{1.64}$$

$$\frac{dz}{dt} = \frac{P_z - qA_z}{m\gamma}, \tag{1.65}$$

Again, instead of canonical momentum, it is more common to use mechanical momentum, components of which are obtained from Eqs. (1.63) – (1.65) by

$$p_r = m\gamma \frac{dr}{dt} = P_r - qA_r, \tag{1.69}$$

$$p_\theta = m\gamma\, r \frac{d\theta}{dt} = \frac{P_\theta}{r} - qA_\theta, \tag{1.70}$$

$$p_z = m\gamma \frac{dz}{dt} = P_z - qA_z. \tag{1.71}$$

Equations of motion in cylindrical coordinates are

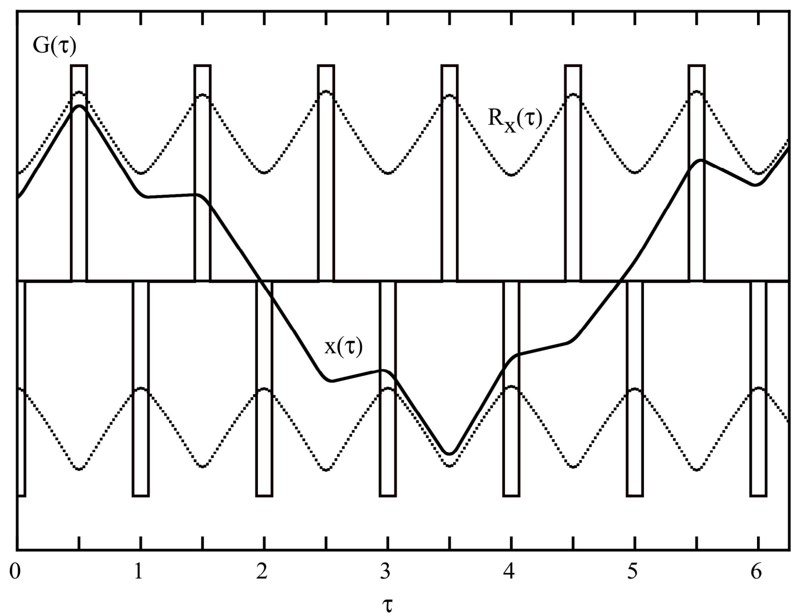
$$\frac{dr}{dt}=\frac{p_r}{m\gamma}, \qquad \frac{d\theta}{dt}=\frac{p_\theta}{m\gamma\, r} \qquad \frac{dz}{dt}=\frac{p_z}{m\gamma} \tag{1.81}$$

$$\frac{dp_r}{dt}=\frac{p_\theta^2}{m\gamma\, r}+q\,(E_r+\frac{p_\theta}{m\gamma}B_z-\frac{p_z}{m\gamma}B_\theta), \tag{1.84}$$

$$\frac{1}{r}\frac{d(rp_\theta)}{dt}=q\,(E_\theta+\frac{p_z}{m\gamma}B_r-\frac{p_r}{m\gamma}B_z), \tag{1.85}$$

$$\frac{dp_z}{dt}=q\,(E_z+\frac{p_r}{m\gamma}B_\theta-\frac{p_\theta}{m\gamma}B_r). \tag{1.86}$$

Averaged Particle Trajectories



Field gradient $G(\tau)$, particle trajectory $x(\tau)$, and beam envelope $R_x(\tau)$ as functions of longitudinal coordinate $\tau = z/L$ in an alternating-gradient focusing structure.

Consider one-dimensional particle motion in the combination of constant field *U(x)* and fast oscillating field

$$f(x,t) = f_1(x)\cos\omega t + f_2(x)\sin\omega t$$

Fast oscillations means that frequency $\omega >> \frac{1}{T}$, where T is the time period for particle motion in the constant field U only. Equation of particle motion:

$$m\frac{d^2x}{dt^2} = -\frac{dU}{dx} + f_1\cos\omega t + f_2\sin\omega t$$

Let us express expected solution is a combination of slow variable *X(t)* and fast oscillation $\xi(t)$:

$$x(t) = X(t)+\xi(t)$$

where $|\xi(t)| << |X(t)|$

Fields can be expressed as:
$$U(x) = U(X) + \frac{dU}{dX}\xi$$

$$f(x) = f(X) + \frac{df}{dX}\xi$$

Substitution of the expected solution into equation of motion gives:

$$m\ddot{X} + m\ddot{\xi} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + f(X,t) + \xi\frac{df}{dX}$$

For fast oscillating term: $m\ddot{\xi} = f(X,t)$

After integration: $\xi = -\frac{f}{m\omega^2}$

Let us average all terms over time, where averaging means mean value over period $T = \frac{2\pi}{\omega}$

$$< g(t) > = \frac{1}{T}\int_0^T g(t)dt$$

$$< m\ddot{X} > + < m\ddot{\xi} > = - < \frac{dU}{dX} > - < \xi\frac{d^2U}{dX^2} > + < f(X,t) > + < \xi\frac{df}{dX} >$$

Average value of $\xi(t)$ at the period of $T = \frac{2\pi}{\omega}$ is zero, while function *X(t)* is changing slowly during that time. Taking into account that

$$< \ddot{X} > \approx \ddot{X} \qquad < \ddot{\xi} > = 0$$

$$m\ddot{X} = -\frac{dU}{dX} + < \xi \frac{df}{dX} > = -\frac{dU}{dX} - \frac{1}{m\omega^2} < f \frac{df}{dX} >$$

Taking into account that $$< f \frac{df}{dX} >= \frac{1}{2} < \frac{df^2}{dX} >$$

$$< \frac{df^2}{dX} > = \frac{1}{2}(\frac{df_1^2}{dX} + \frac{df_2^2}{dX})$$

equation for slow particle motion is $$m\ddot{X} = -\frac{dU_{eff}}{dX}$$

where effective potential is $$U_{eff} = U + \frac{1}{4m\omega^2}(f_1^2 + f_2^2)$$

3D Averaging Method

Equations of motion: $\ddot{\vec{r}} = \frac{q}{m}[\vec{E}(\vec{r},t) - \frac{\partial U(\vec{r})}{\partial \vec{r}}]$

Fast oscillating field: $\vec{E}(\vec{r}, t) = \sum_{k=1}^{\infty} \vec{E}_k(\vec{r}) \sin \omega_k t$

Particle trajectory (slow + fast components): $\vec{r}(t) = \vec{R}(t) + \vec{\xi}(t)$

Equation for slow component $\ddot{\vec{R}} = -\frac{q}{m}\frac{\partial U_{eff}(\vec{R})}{\partial \vec{r}}$

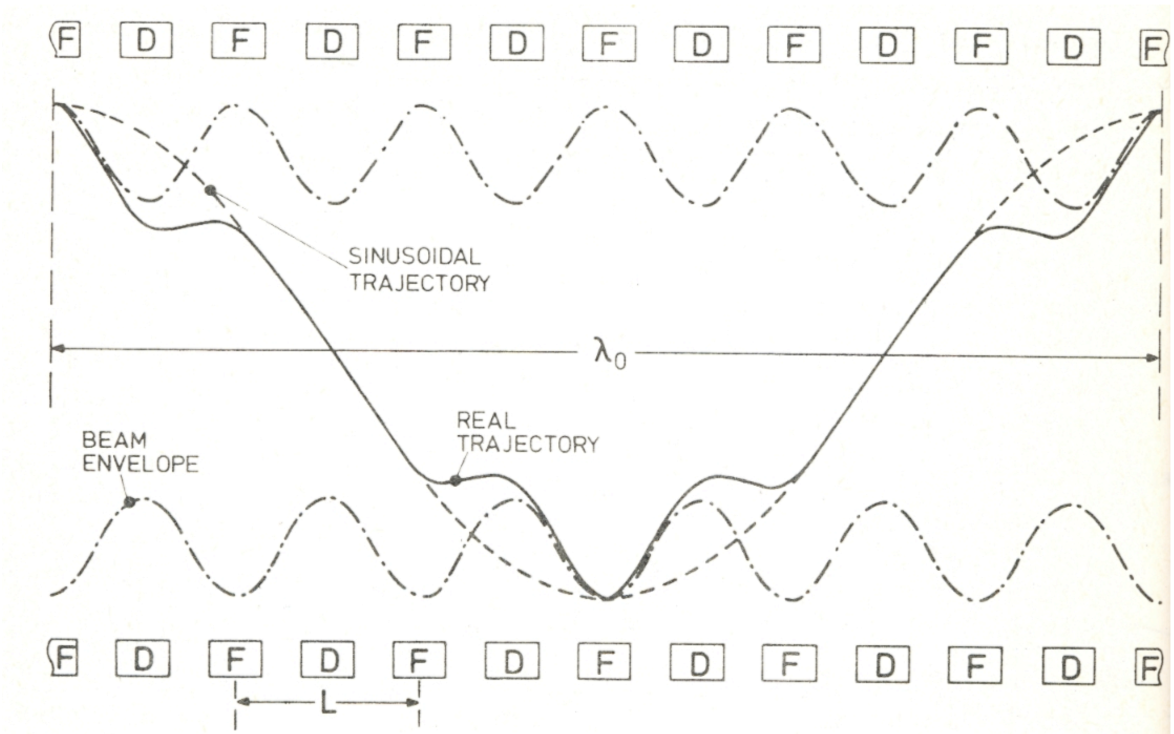
Effective potential: $U_{eff}(\vec{R}) = U(\vec{R}) + \frac{q}{4m}\sum_{k=1}^{\infty}\frac{\vec{E}_k^2(\vec{R})}{\omega_k^2}$

Fast component: $\vec{\xi}(t) = -\frac{q}{m}\sum_{k=1}^{\infty}\frac{\vec{E}_k(\vec{R})}{\omega_k^2}\sin \omega_k t$

Let us apply averaging method to quadrupole channel. Single particle equations of motion in quadrupole channel is

$$\frac{d^2x}{dt^2} = \frac{q}{m\gamma} G(z)\, x$$

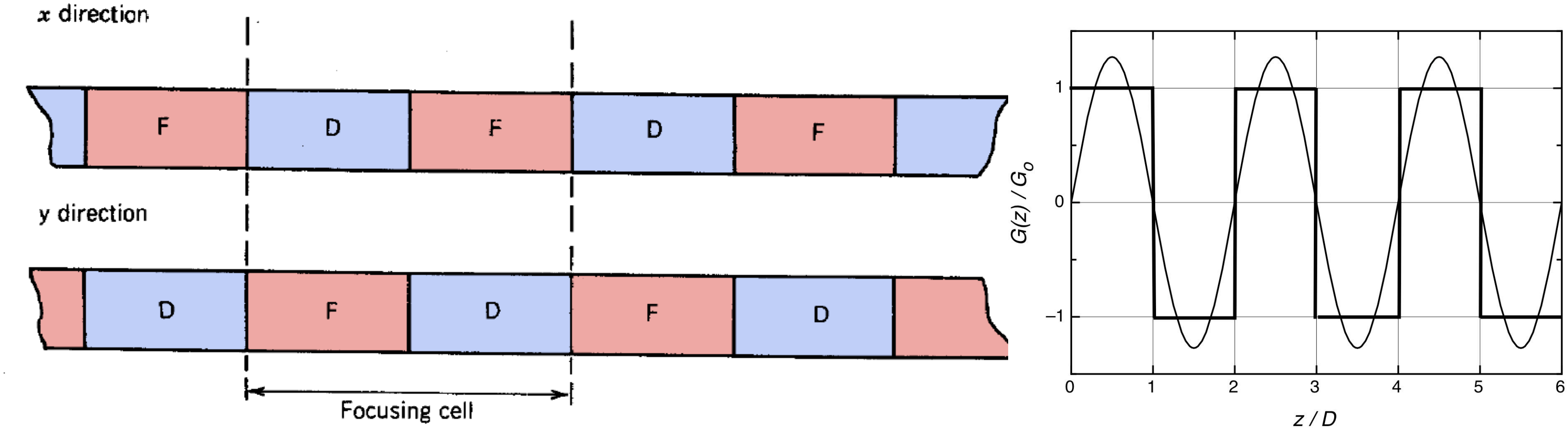
where $z = \beta ct$, focusing gradient $G(z) = G_{el}(z)$ for electrostatic quadrupole, and $G(z) = \beta c G_{magn}(z)$ for magnetic quadrupole.



(Solid line) typical particle trajectory and (dashed line) the sine approximation to that trajectory.

Consider periodic FD structure of quadrupole lenses with length of $D = L/2$, and field gradient in each lens G_o. In FD structure, focusing-defocusing lenses follow each other without any gap. Let us expand focusing function $G(z)$ in Fourier series:

$$G(z) = \frac{4G_o}{\pi}\left[\sin\left(\frac{\pi z}{D}\right) + \frac{1}{3}\sin\left(\frac{3\pi z}{D}\right) + \frac{1}{5}\sin\left(\frac{5\pi z}{D}\right) + \ldots\right]$$



FD focusing structure and approximation of field gradient.

Let us keep only first term: $$m\frac{d^2x}{dt^2} = x\frac{q}{\gamma}\frac{4G_o}{\pi}\sin(\frac{\pi\beta c}{D}t)$$

Equation of particle motion in fast oscillating field $$m\frac{d^2x}{dt^2} = f_1(x)\sin\omega t$$

can be substituted by slow motion in an effective potential $$U_{eff} = \frac{f_1^2}{4m\omega^2} = \frac{1}{4m}(\frac{q}{\gamma}\frac{4G_oD}{\pi^2\beta c})^2X^2$$

Equation for slow particle motion $$m\ddot{X} = -\frac{dU_{eff}}{dX}$$

$$\frac{d^2X}{dt^2} = -\frac{1}{2m^2}(\frac{q}{\gamma}\frac{4G_oD}{\pi^2\beta c})^2X$$

$$\frac{d^2X}{dt^2} + \Omega_r^2X = 0$$

Let us introduce new variable $\tau = \frac{t\beta c}{L}$ where for FD structure $L = 2D$

Equation of motion in new variables

$$\frac{d^2X}{d\tau^2} + \mu_o^2 X = 0$$

Frequency of smoothed transverse oscillations in the scale of the period of focusing structure

$$\mu_o = \frac{q}{\gamma m}\frac{4\sqrt{2}G_o D^2}{\pi^2(\beta c)^2}$$

Taking into account, that $\frac{4\sqrt{2}}{\pi^2} \approx \frac{1}{\sqrt{3}}$ and $G_o = \beta c G_{magn}$, frequency can be written as

$$\boxed{\mu_o = \frac{1}{\sqrt{3}}\frac{qG_{magn}D^2}{m\gamma\beta c}}$$

Compare with matrix method for FODO period with *L =2D* :

$$\boxed{\mu_o = \frac{L}{2D}\sqrt{1-\frac{4}{3}\frac{D}{L}}\frac{qG_{magn}D^2}{m\gamma\beta c} = \frac{1}{\sqrt{3}}\frac{qG_{magn}D^2}{m\gamma\beta c}}$$

Equation for fast component: $\xi = -\frac{f}{m\omega^2}$ $\quad f = x\frac{q}{\gamma}\frac{4G_o}{\pi}\sin(\frac{\pi\beta c}{D}t)$ $\quad \omega = \frac{\pi\beta c}{D}$

Solution for fast component:

$$\boxed{\xi = -x\frac{q}{\gamma m}\frac{4G_o D^2}{\pi^3(\beta c)^2}\sin(\frac{\pi\beta c}{D}t)}$$

Amplitude of small fast oscillations in FD structure:

$$\frac{\xi_{\max}}{X} = \frac{4\sqrt{3}}{\pi^3}\mu_o$$

For typical values of $\mu_o=\pi/3\ldots\pi/2$ in transport channels, this ratio is of the order of $|\xi_{max}| / X \approx$ 0.2....0.3.

Beam Emittance

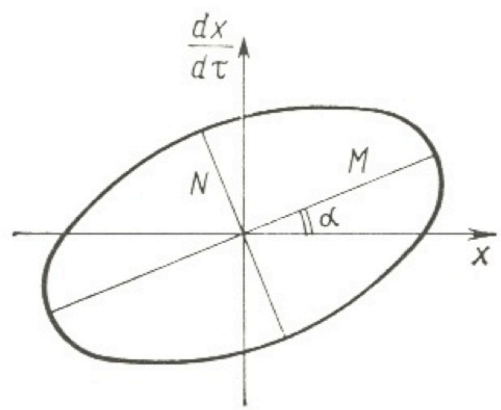
In the phase plane, the beam is usually approximated by an ellipse. The area of ellipse with semi-axes M and N is simply

$$S = \pi M N$$

The general ellipse equation can be written as

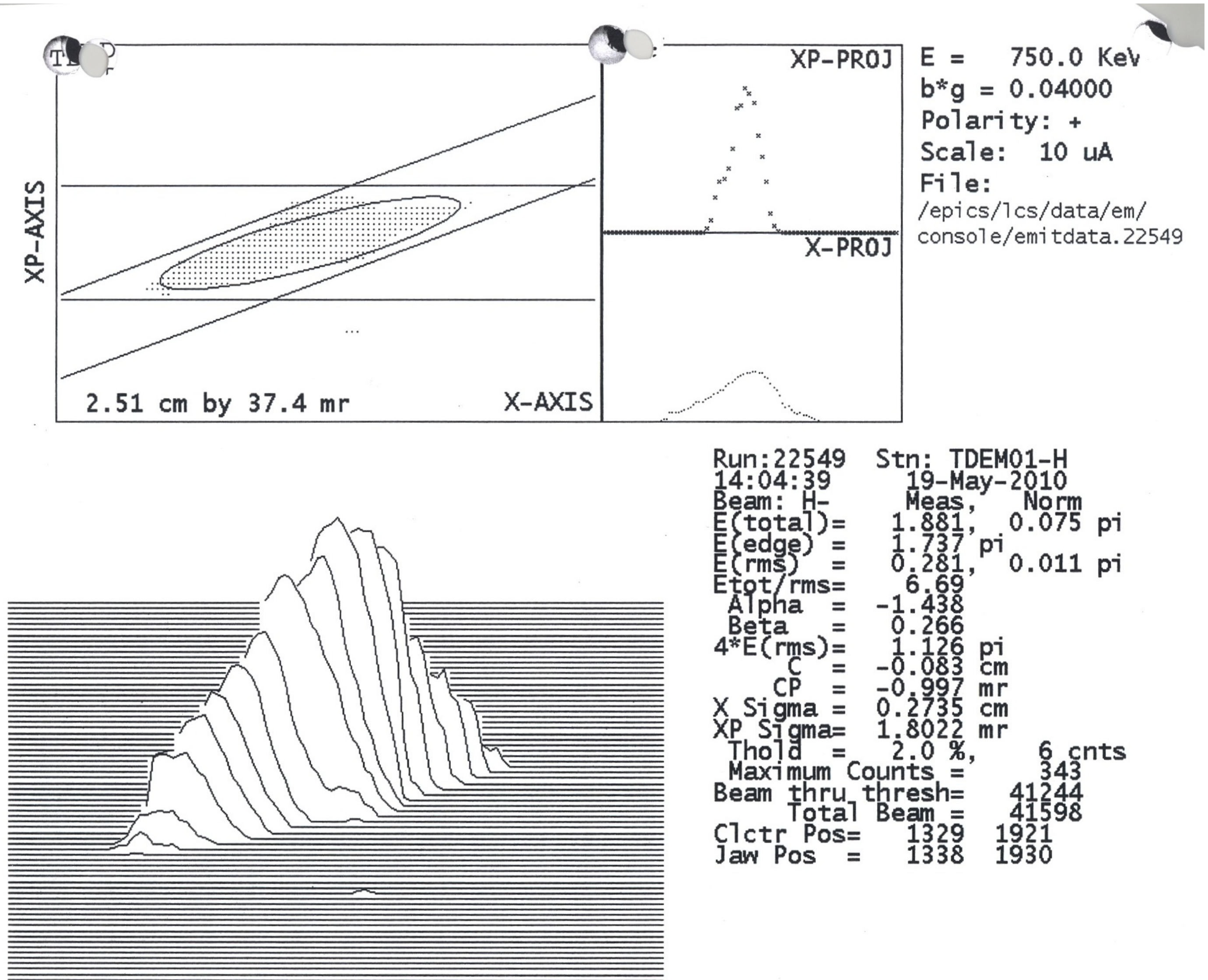
$$\gamma x^2 + 2\,\alpha x\, x' + \beta\, x'^2 = э$$

parameters α, β, γ are called Twiss parameters



Ellipse of the beam at phase plane of transverse oscillations.

Root-mean-square (rms) beam emittance



Realistic beam distribution in phase space.

Consider a beam with a distribution function $f(\vec{x}, \vec{P}, t)$ and let $g(\vec{x}, \vec{P}, t)$ be an arbitrary function of position, momentum, and time. The average value of the function $g(\vec{x}, \vec{P}, t)$ is defined as:

$$\langle g \rangle = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\vec{x}, \vec{P}, t)\, f(\vec{x}, \vec{P}, t)\, d\vec{x}\, d\vec{P}}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{x}, \vec{P}, t)\, d\vec{x}\, d\vec{P}}$$

The integral in the denominator is just the total number of particles. Now, let us consider some examples of physically significant average values. For $g(\vec{x}, \vec{P}, t) = x$, the average value

$$\langle x \rangle = \frac{1}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f(\vec{x}, \vec{P}, t)\, d\vec{x}\, d\vec{P}$$

gives the center of gravity of the beam in the *x*-direction.

Analogously, for $g(\vec{x}, \vec{P}, t) = x^2$, the average value of x^2 is defined as

$$<x^2> = \frac{1}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 \, f(\vec{x}, \vec{P}, t) \, d\vec{x} \, d\vec{P}$$

and is called the mean-square value of x. The correlation between variables x and P_x is given by the following expression: taking $g(\vec{x}, \vec{P}, t) = x \, P_x$

$$< xP_x > = \frac{1}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \, P_x \, f(\vec{x}, \vec{P}, t) \, d\vec{x} \, d\vec{P}$$

An expression of the form $<x^{n1} \, y^{n2} \, z^{n3} \, P_x^{n4} \, P_y^{n5} \, P_z^{n6}>$ is referred to as the n^{th} order moment, $M_{n_1,\, n_2,\, n_3,\, n_4,\, n_5,\, n_6}$, of the distribution function, where $n = n_1 + n_2 + n_3 + n_4 + n_5 + n_6$:

$$<x^{n1} \, y^{n2} \, z^{n3} \, P_x^{n4} \, P_y^{n5} \, P_z^{n6}> = \frac{1}{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \, dz \, dP_x \, dP_y \, dP_z$$

$$x^{n1} y^{n2} z^{n3} P_x^{n4} P_y^{n5} P_z^{n6} f(x, y, z, P_x, P_y, P_z, t).$$

The following combination of second moments of distribution function is called the root-mean-square beam emittance:

$$\epsilon_{rms} = \sqrt{<x^2><x'^2> - <x\,x'>^2}$$

and the normalized root-mean-square beam emittance is given by

$$\varepsilon_{rms} = \frac{1}{mc}\sqrt{<x^2><P_x{}^2> - <xP_x>^2}$$

By the reasons discussed below, beam emittance is adopted as the value, four times large than rms emittance

$$\epsilon = 4\sqrt{<x^2><x'^2> - <x\,x'>^2}$$

Consider the rms beam emittance concept in more detail. The density of particles in phase space, normalized by the total number of particles *N,* is described by a distribution function $\rho_x\,(x, x')$, which is an integral of the beam distribution function over the remaining variables:

$$\rho_x\,(x, x') = \frac{1}{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\,(x, x', y, y', z, z')\; dy\; dy\; dz\; dz'$$

It is convenient to consider distributions in phase space with elliptical symmetry:

$$\rho_x\,(x, x') = \rho_x\,(\gamma_x\, x^2 + 2\,\alpha_x x\, x' + \beta_x\, x'^2)$$

Such distributions have particle densities, $\rho_x\,(x, x')$, that are constant along concentric ellipses

$$r_x^2 = \gamma_x\, x^2 + 2\,\alpha_x x\, x' + \beta_x\, x'^2$$

but are different from ellipse to ellipse, so one can write $\rho_x\,(x, x') = \rho_x\,(r_x^2)$. Namely, equation this describes a family of similar ellipses, which differ from each other by their areas. Using transformation

$$\sigma = \sqrt{\beta} \qquad\qquad \sigma' = -\frac{\alpha}{\sqrt{\beta}}$$

the ellipse equation can be rewritten as

$$r_x^2 = (x\sigma_x' - x'\sigma_x)^2 + \left(\frac{x}{\sigma_x}\right)^2$$

Let us calculate rms beam parameters and rms beam emittance for an arbitrary function $\rho_x\ (x,\ x')$. We begin by changing variables:

$$\begin{cases} \dfrac{x}{\sigma_x} = r_x\ cos\varphi \\ x\sigma_x' - x'\sigma_x = r_x\ sin\varphi \end{cases}$$

Now we rewrite it as

$$\begin{cases} x = r_x \sigma_x cos\varphi \\ x' = r_x \sigma_x' cos\varphi - \dfrac{r_x}{\sigma_x}\ sin\varphi \end{cases}$$

The absolute value of the Jacobian of transformation gives us the volume transformation factor of the phase space element:

$$dx\ dx' = (abs \begin{vmatrix} \dfrac{\partial x}{\partial r_x} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial x'}{\partial r_x} & \dfrac{\partial x'}{\partial \varphi} \end{vmatrix})\ dr_x\ d\varphi = r_x\ dr_x\ d\varphi$$

Then, the rms values are:

$$<x^2> = \int_o^{2\pi} \int_o^{\infty} (r_x \sigma_x \cos\varphi)^2 \rho_x(r_x^2)\, r_x\, dr_x\, d\varphi$$

$$<x'^2> = \int_o^{2\pi} \int_o^{\infty} (r_x \sigma_x' \cos\varphi - \frac{r_x}{\sigma_x} \sin\varphi)^2 \rho_x(r_x^2)\, r_x\, dr_x\, d\varphi$$

$$<xx'> = \int_o^{2\pi} \int_o^{\infty} r_x \sigma_x \cos\varphi\, (r_x \sigma_x' \cos\varphi - \frac{r_x}{\sigma_x} \sin\varphi) \rho_x(r_x^2)\, r_x\, dr_x d\varphi$$

Let us take into account previously introduced expressions:

$$\sigma = \sqrt{\beta}$$

$$\sigma' = - \frac{\alpha}{\sqrt{\beta}}$$

$$\beta\gamma - \alpha^2 = 1$$

Calculation of integrals over φ gives:

$$<x^2> = \pi\ \beta_x \int_o^{\infty} r_x^3\ \rho_x(r_x^2)\, dr_x$$

$$<x'^2> = \pi \gamma_x \int_o^{\infty} r_x^3 \rho_x(r_x^2)\, dr_x$$

$$<x\ x'> = -\ \pi\ \alpha_x \int_o^{\infty} r_x^3\ \rho_x(r_x^2)\, dr_x$$

Therefore, beam emittance is given by

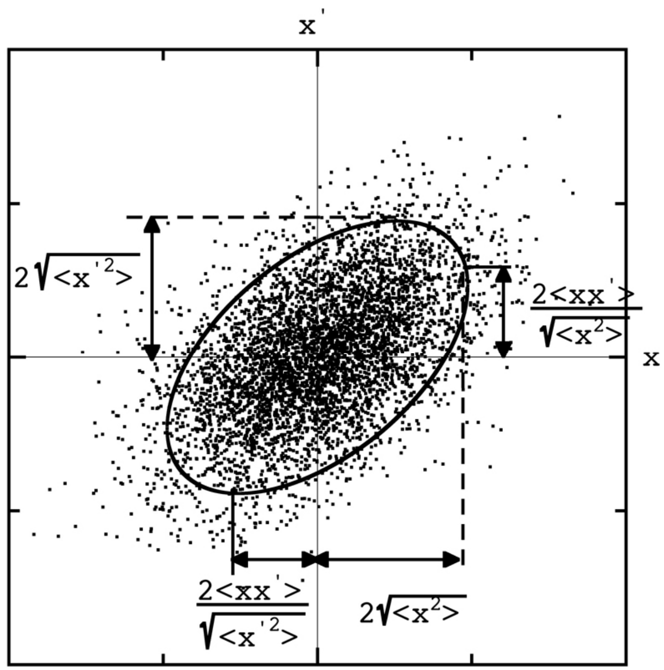
$$\text{э}_x = 4\pi \int_o^{\infty} r_x^3\ \rho_x(r_x^2)\, dr_x$$

Twiss parameters

$$\alpha_x = -4\,\frac{\langle x\,x'\rangle}{\epsilon_x} \qquad \beta_x = 4\,\frac{\langle x^2\rangle}{\epsilon_x} \qquad \gamma_x = 4\,\frac{\langle x'^2\rangle}{\epsilon_x}$$

Rms beam ellipse

$$\left(\frac{4\langle x'^2\rangle}{\epsilon_x}\right) x^2 - 2\left(\frac{4\langle x x'\rangle}{\epsilon_x}\right) x\,x' + \left(\frac{4\langle x^2\rangle}{\epsilon_x}\right) x'^2 = \epsilon_x$$



Beam distribution and rms ellipse.

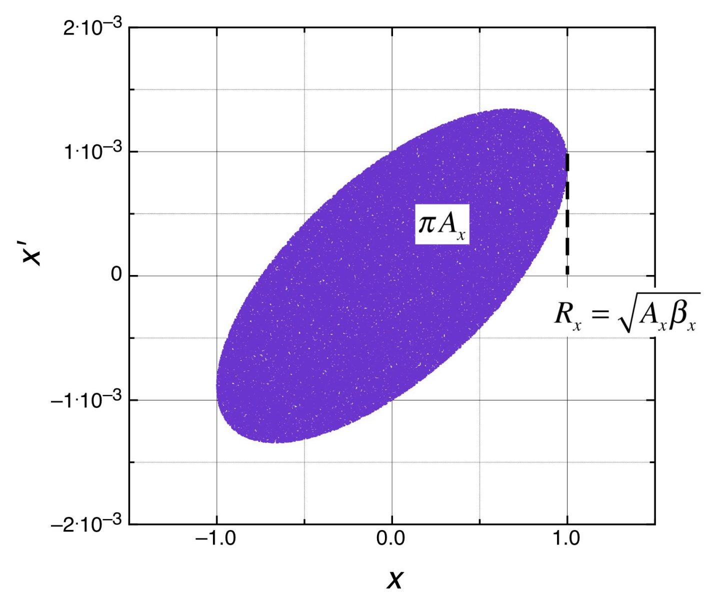
Example: Uniformly populated ellipse

Consider an example, where the beam ellipse has an area of πA_x, and is uniformly populated by particles. Particle density is constant inside the ellipse $r_x^2 = A_x$:

$$\rho_x(r_x^2) = \frac{1}{\pi A_x}$$

Calculation of the rms value, $<x^2>$, gives:

$$<x^2> = \pi\ \beta_x \int_o^{\sqrt{A_x}} r_x^3\ \rho_x(r_x^2)\, dr_x = \frac{A_x\ \beta_x}{4}$$



Uniformly populated ellipse at phase plane (x, x').

The beam boundary is given by

$$R_x = \sqrt{A_x \beta_x}$$

Radius of the beam represented as a uniformly populated ellipse is equal to twice the rms beam size:

$$R = 2\sqrt{<x^2>}$$

Rms beam emittance:

$$э_x = \frac{4}{A_x}\int_o^{\sqrt{A_x}} r_x^3 \, dr_x = A_x$$

Therefore, the area of an ellipse, uniformly populated by particles, coincides with the 4 x rms beam emittance. This explains the choice of the coefficient 4 in the definition of rms beam emittance.

Different Particle Distributions in Phase Space

Consider quadratic from of 4-dimensional phase space variables:

$$I = (\sigma_x x' - \sigma'_x x)^2 + (\frac{x}{\sigma_x})^2 + (\sigma_y y' - \sigma'_y y)^2 + (\frac{y}{\sigma_y})^2$$

Consider different distributions $f = f(I)$ in phase space which depend on quadratic form:

Water Bag:

$$f = \{ \begin{array}{ll} \frac{2}{\pi^2 F_o^2}, & I \le F_o \\ 0, & I > F_o \end{array}$$

Parabolic:

$$f = \frac{6}{\pi^2 F_o^2}(1 - \frac{I}{F_o})$$

Gaussian:

$$f = \frac{1}{\pi^2 F_o^2} exp(-\frac{I}{F_o})$$

Normalization: $\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f\, dx\, dx'\, dy\, dy' = 1$

Characteristics of Beam Distributions

Distribution	Definition $\rho(x,x',y,y')=\rho(I)$ $I=r_x^2+r_y^2$ $r_x^2=\gamma_x x^2+2\alpha_x xx'+\beta_x x'^2$ $r_y^2=\gamma_y y^2+2\alpha_y yy'+\beta_y y'^2$	Distribution in phase space $\rho(x,x')=\rho(r_x^2)$	Space charge density	Space charge field
KV	$\frac{1}{\pi^2 F_o}\delta(I-F_o)$	$\frac{1}{\pi\,\epsilon_x}$	$\frac{I}{\pi R^2\beta c}$	$\frac{I}{2\pi\varepsilon_o R^2\beta c}r$
Water Bag	$\frac{2}{\pi^2 F_o^2},\ I\le F_o$	$\frac{4}{3\pi\,\epsilon_x}(1-\frac{2}{3}\frac{r_x^2}{\epsilon_x})$	$\frac{4I}{3\pi R^2\beta c}(1-\frac{2}{3}\frac{r^2}{R^2})$	$\frac{2I}{3\pi\varepsilon_o\beta c}\frac{r}{R^2}(1-\frac{r^2}{3R^2})$
Parabolic	$\frac{6}{\pi^2 F_o^2}(1-\frac{I}{F_o})$	$\frac{3}{2\pi\,\epsilon_x}(1-\frac{r_x^2}{2\,\epsilon_x})^2$	$\frac{3I}{2\pi R^2\beta c}(1-\frac{r^2}{2R^2})^2$	$\frac{3I}{4\pi\varepsilon_o\beta c}\frac{r}{R^2}(1-\frac{r^2}{2R^2}+\frac{r^4}{12R^4})$
Gaussian	$\frac{1}{\pi^2 F_o^2}exp(-\frac{I}{F_o})$	$\frac{2}{\pi\,\epsilon_x}\exp(-2\frac{r_x^2}{\epsilon_x})$	$\frac{2I}{\pi R^2\beta c}\exp(-2\frac{r^2}{R^2})$	$\frac{I}{2\pi\varepsilon_o\beta cr}[1-\exp(-2\frac{r^2}{R^2})]$

Projection of distributions on phase plane

$$\rho_x(x, x') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, x', y, y')\, dy\, dy'$$

Let us change the variables (y, y') for new variables T, ψ

$$\sigma_y y' - \sigma_{y'} y = T \cos\psi$$

$$\frac{y}{\sigma_y} = T \sin\psi$$

Phase space element $dy\, dy'$ is transformed as

$$dy\, dy' = \begin{vmatrix} \dfrac{\partial y}{\partial T} & \dfrac{\partial y}{\partial \psi} \\ \dfrac{\partial y'}{\partial T} & \dfrac{\partial y'}{\partial \psi} \end{vmatrix} dT\, d\psi = T\, dT\, d\psi\,.$$

The quadratic form is $I = r_x^2 + T^2$.

where the following notation is used: $r_x^2 = (\sigma_x x' + \sigma_x' x)^2 + (\dfrac{x}{\sigma_x})^2$.

With new variables, the projection on phase space is

$$\rho_x(x, x') = \pi \int_o^{\infty} f(r_x^2 + T^2)\, dT^2.$$

Water Bag distribution

$$f = \begin{cases} \dfrac{2}{\pi^2 F_o^2}, & I = r_x^2 + T^2 \le F_o \\ 0, & I > F_o \end{cases}$$

is restricted by surface $\quad r_x^2 + T_1^2 = F_o\,, \quad T_1^2 = F_o - r_x^2$

Projection of *Water Bag* distribution on (x, x')

$$\rho_x(x, x') = \frac{2}{\pi F_o^2}\int_o^{T_1^2} dT^2 = \frac{2}{\pi F_o}\left(1 - \frac{r_x^2}{F_o}\right)$$

For *Parabolic* distribution, projection on *x, x′* plane is

$$\rho_x(x, x') = \frac{6}{\pi F_o^2}\int_0^{T_1^2}\left(1 - \frac{r_x^2 + T^2}{F_o}\right) dT^2 = \frac{3}{\pi F_o}\left(1 - \frac{r_x^2}{F_o}\right)^2$$

For *Gaussian* distribution projection on *x, x′* plane is

$$\rho_x(x, x') = \frac{1}{\pi F_o^2}\int_o^{\infty} exp\left(- \frac{r_x^2 + T^2}{F_o}\right) dT^2 = \frac{1}{\pi F_o}\, exp\left(- \frac{r_x^2}{F_o}\right).$$

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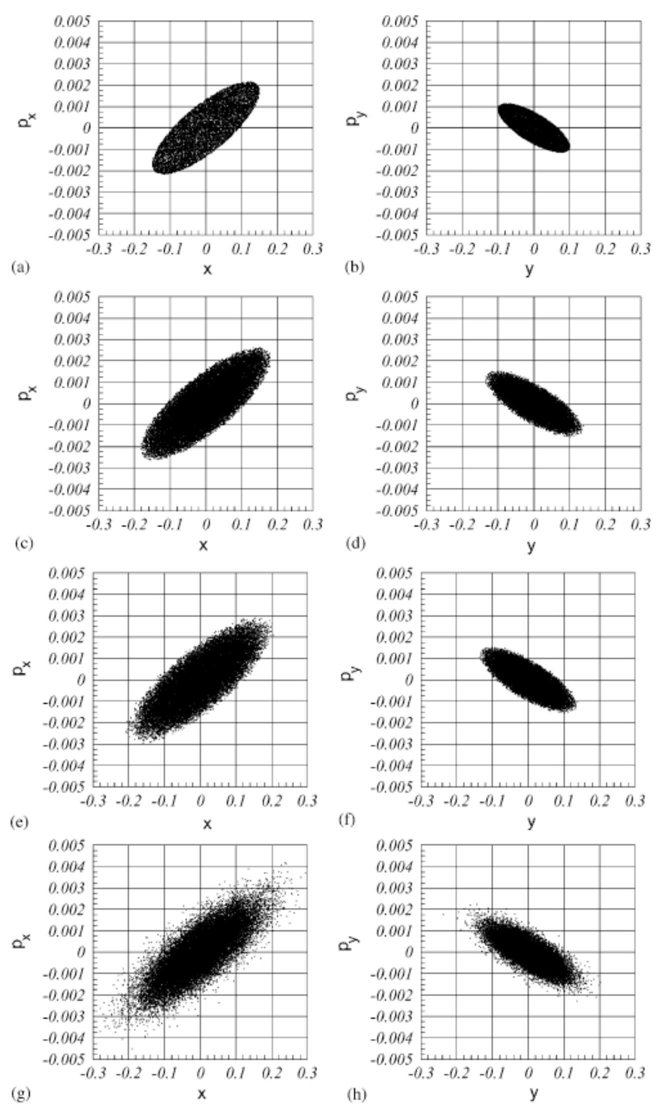


KV $\varepsilon_{\max} = 4\varepsilon_{rms}$

Water Bag $\varepsilon_{\max} = 6\varepsilon_{rms}$

Parabolic $\varepsilon_{\max} = 8\varepsilon_{rms}$

Gaussian $\varepsilon_{\max} = \infty$

Particle distributions with equal values of ε_{rms} .

Rms emittance of distributions with elliptical symmetry

Four rms beam emittance $$\backepsilon_x = 4\pi \int_o^{\infty} r_x^3 \, \rho_x (r_x^2) \, dr_x$$

Water bag distribution, is limited by the surface

$$r_x^2 + r_y^2 \le F_o, \qquad \text{or}$$

$$r_x^2 \le F_o - r_y^2$$

Maximum value of r_x^2 is achieved when $r_y^2 = 0$ and vise versa. Therefore, projection of water bag distribution, is limited by $r_{x,\ max}^2 = F_o$. Substituion of expressions for $\rho_x(r_x^2)$, and integration gives:

$$\backepsilon_x = \frac{8}{F_o} \int_o^{\sqrt{F_o}} r_x^3 \left(1 - \frac{r_x^2}{F_o}\right) dr_x = \frac{2}{3} F_o \,,$$

Analogously, for *parabolic* distribution

$$Э_x = \frac{12}{F_o}\int_o^{\sqrt{F_o}} r_x^3 \left(1 - \frac{r_x^2}{F_o}\right)^2 dr_x = \frac{F_o}{2},$$

For *Gaussian* distribution

$$Э_x = \frac{4}{F_o}\int_o^{\infty} r_x^3\, exp\left(- \frac{r_x^2}{F_o}\right) dr_x = 2\, F_o ,$$

Fraction of particles residing within a specific emittance

The distribution $\rho(r_x^2)$ is the particle density in the phase plane (x, x'), divided by the total number of particles, N. Fraction of particles

$$\eta = N(\text{э}_x)/N_o$$

within the emittance э_x is an integral of $\rho(r_x^2)$ over an ellipse area of э_x:

$$\eta = \frac{N(\text{э})}{N_o} = \int\int \rho_x(r_x^2)\ dx\, dx' = \int_o^{2\pi}\int_o^{\sqrt{\text{э}}} \rho_x(r_x^2)\, r_x\, dr_x\, d\varphi = \pi\int_o^{\text{э}} \rho_x(r_x^2)\ dr_x^2$$

Distributions on phase plane are:

$$\text{Water bag} \quad \rho_x(r_x^2) = \frac{4}{3\pi \epsilon_x}\left(1 - \frac{2}{3}\frac{r_x^2}{\epsilon_x}\right)$$

$$\text{Parabolic} \quad \rho_x(r_x^2) = \frac{3}{2\pi \epsilon_x}\left(1 - \frac{r_x^2}{2\,\epsilon_x}\right)^2$$

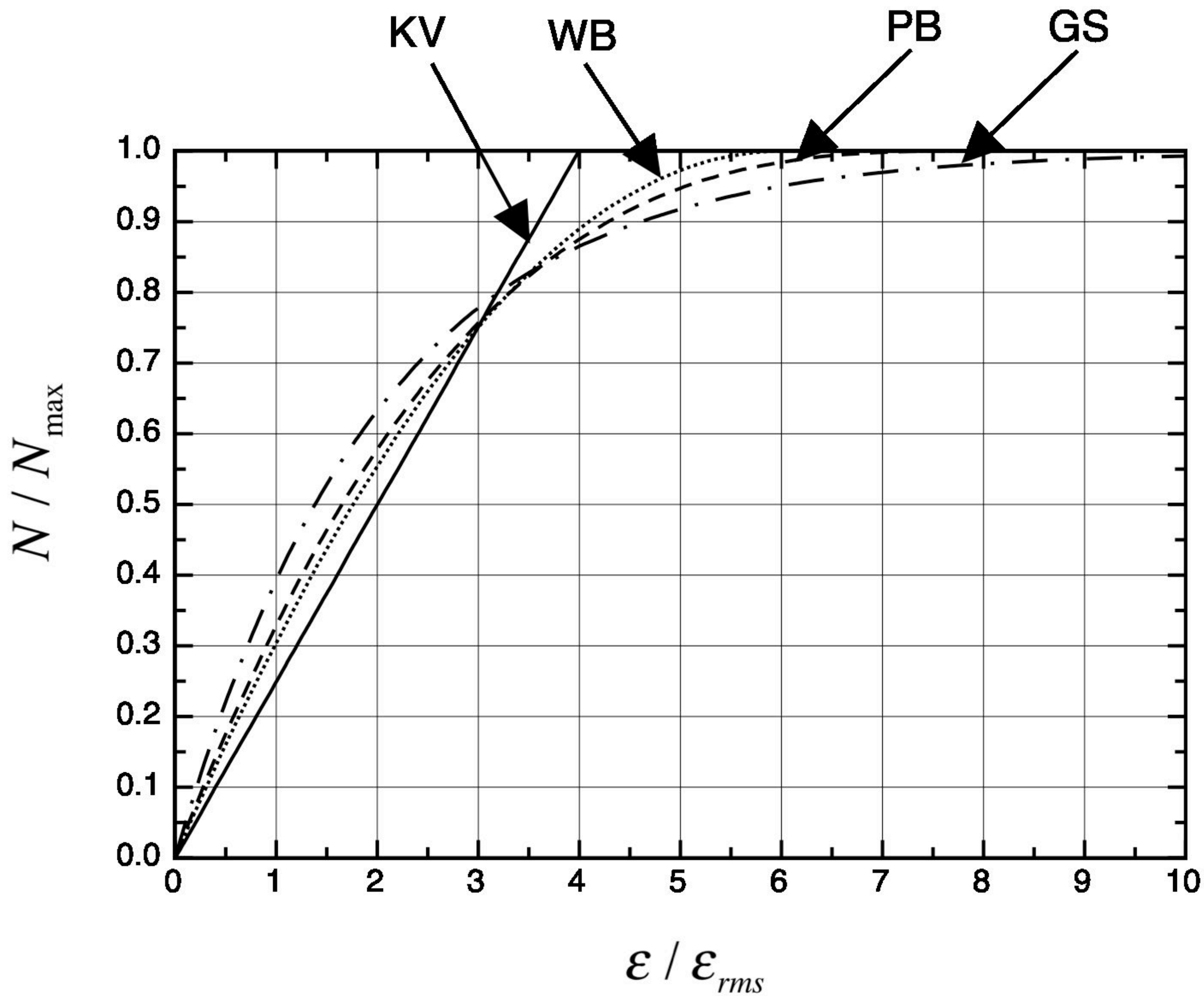
$$\text{Gaussian} \quad \rho_x(r_x^2) = \frac{2}{\pi \epsilon_x}\, exp\left(-2\frac{r_x^2}{\epsilon_x}\right)$$

Fraction of particles within phase space area is:

$$\text{Water bag} \quad \frac{N(\epsilon)}{N_o} = \frac{4}{3}\left(\frac{\epsilon}{\epsilon_x}\right)\left(1 - \frac{1}{3}\frac{\epsilon}{\epsilon_x}\right)$$

$$\text{Parabolic} \quad \frac{N(\epsilon)}{N_o} = \frac{3}{2}\left(\frac{\epsilon}{\epsilon_x}\right)\left[1 - \frac{1}{2}\frac{\epsilon}{\epsilon_x} + \frac{1}{12}\left(\frac{\epsilon}{\epsilon_x}\right)^2\right]$$

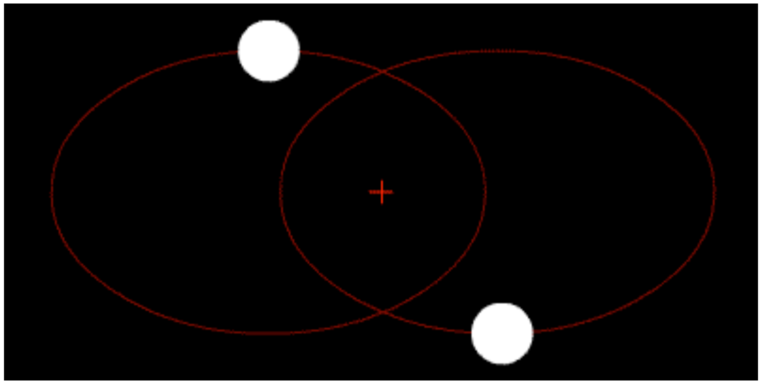
$$\text{Gaussian} \quad \frac{N(\epsilon)}{N_o} = 1 - exp\left(-2\frac{\epsilon}{\epsilon_x}\right)$$



Fraction of particles versus phase space area for different particle distributions.

Self-Consistent Particle Dynamics

Example of self-consistent dynamics: two - body problem



Every point mass attracts every single other point mass by a force pointing along the line intersecting both points. The force is directly proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:

$$F = G\frac{m_1 m_2}{r^2},$$

where:

- F is the magnitude of the gravitational force between the two point masses,
- G is the gravitational constant,
- m_1 is the mass of the first point mass,
- m_2 is the mass of the second point mass, and
- r is the distance between the two point masses.

In classical mechanics, the two-body problem is to determine the motion of two point particles that interact only with each other.

Self-Consistent Approach to N – Particle Dynamics

Solution to the equations of motion of the particles, together with the equations for the electromagnetic field which they create.

Evolution of charged particles interacting through long-range (Coulomb) forces is determined by *Vlasov's equation*

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{x}}\frac{d\vec{x}}{dt} + \frac{\partial f}{\partial \vec{P}}\frac{d\vec{P}}{dt} = 0$$

Solution of self-consistent problem: the phase space density, as a constant of motion, can be expressed as a function of other constants of motion $I_1, I_2, \ldots.$

$$f = f\,(I_1\,,\, I_2\,,\, \ldots.)$$

This equation automatically obeys Vlasov's equation

$$\frac{df}{dt} = \frac{\partial f}{\partial I_1}\frac{dI_1}{dt} + \frac{\partial f}{\partial I_2}\frac{dI_2}{dt} + \ldots. \quad = 0$$

because of vanishing derivatives, $dI_i\,/dt = 0$. Distribution function determined in this way, is then substituted to Maxwell's equation to find self-consistent field created by the beam together with the external electromagnetic field.

Field created by the beam is described by Maxwell's equations:

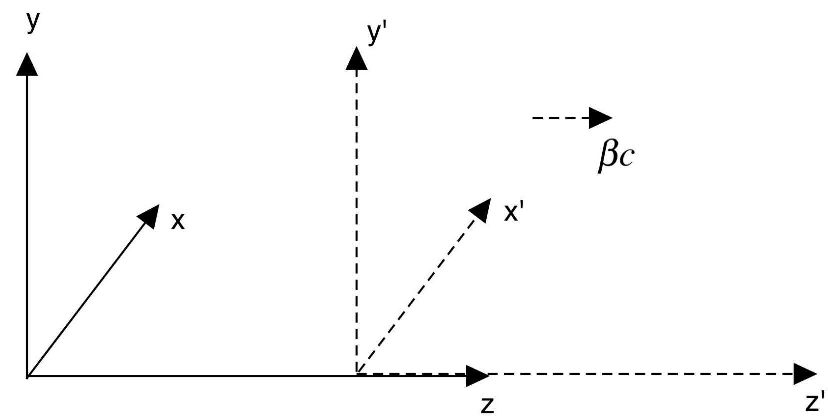
$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$
$\nabla \cdot \mathbf{B} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$

space charge density $$\rho = q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \, dP_x \, dP_y \, dP_z$$

beam current density $$\vec{j} = q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vec{v} f \, dP_x \, dP_y \, dP_z$$

$\varepsilon_o = 8.85 \times 10^{-12}$ F/m is the electric permittivity

$\mu_o = 4\pi \, 10^{-7}$ H/m is the magnetic permeability of free space



Consider system of coordinates, which moves with the average beam velocity β. We will denote all values in this frame by prime symbol. Potentials $U', \vec{A}'$ are connected with that in laboratory system, $U, \vec{A}$, by Lorentz transformation

$$A_z = \gamma(A_z' + \frac{\beta}{c}U')$$

$$U = \gamma(U' + \beta c A_z')$$

$$A_x = A_x', \quad A_y = A_y'$$

In the moving system of coordinates, particles are static, therefore, vector potential of the beam equals to zero, $\vec{A}_b' = 0$. According to Lorentz transformations, components of vector potential of the beam are converted into laboratory system of coordinates as follow

$$A_{xb} = 0\ , \qquad A_{yb} = 0\ , \qquad A_{zb} = \beta \frac{U_b}{c}$$

In a particle beam, the vector potential and the scalar potential are related via the expression $\vec{A}_b = \vec{v}_z \ /c^2 U_b$, therefore, it is sufficient to only solve the equation for the scalar potential. The unknown distribution function of the beam is then found by substituting equation for distribution function into the field equation and solving it. For example, for beam transport, equation for unknown space charge potential is

$$\Delta U_b = - \frac{q}{\varepsilon_o} \int_{-\infty}^{\infty} f(I_1, I_2, \ldots)\, d\vec{P}$$

Equation for unknown potential of the beam together with Vlasov's equation for beam distribution function constitute *self-consistent system of equations* describing beam evolution in the field created by the beam itself.

Applicability of Vlasov's equation to particle dynamics

Vlasov's equation describes behavior of non-interactive particles in given external field. Charged particles within the beam interact between themselves:

(i) interaction of large number of particles resulted in smoothed collective charge density and current density distribution

(ii) individual particle - particle collisions, when particles approach to each other at the distance, much smaller than the average distance between particles.

First type of interaction results in generation of smoothed electromagnetic field, which, being added to the field of external sources, act at the beam as an external field. The second type of interaction has a meaning of particle collisions resulting in appearance of additional fluctuating electromagnetic fields.

Using Vlasov's eqauiton, we *formally* expand it to dynamics of interacting charged particles, assuming that the total electromagnetic filed of the structure $(U, \vec{A})$

$$U = U_{ext} + U_b$$

$$\vec{A} = \vec{A}_{ext} + \vec{A}_b$$

U_{ext}, $\vec{A}_{ext}$, external field

U_b, $\vec{A}_b$ field created by the beam

and *neglecting* individual particle-particle interactions.

Vlasov's equation treats collisionless plasma, where individual particle-particle interactions are negligible in comparison with the collective space charge field

Quantative treatment of validity of collisionless approximation dynamics to particle dynamics:

n - particle density within the beam

$\bar{r}$ - the average distance between particles.

$$n\bar{r}^{3} = 1 \qquad \text{, or} \qquad \bar{r} = n^{-1/3}$$

Individual particle-particle collisions are neglected, when kinetic energy of thermal particle motion within the beam is much larger than potential energy of Coulomb particle-particle interaction:

$$\frac{mv_t^2}{2} >> \frac{q^2}{4\,\pi\,\varepsilon_o\,\bar{r}}$$

v_t is the root-mean square velocity of chaotic particle motion within the beam:

$$\frac{mv_t^2}{2} = \frac{kT}{2}$$

T is the “temperature” of chaotic particle motion

$k = 8.617342 \times 10^{-5}$ eV $K^{-1} = 1.3806504 \times 10^{-23}$ J K^{-1} is the Boltsman's constant.

Radius of Debye shielding in plasma: $$\lambda_D = \sqrt{\frac{\varepsilon_o kT}{q^2 n}}$$

Combining all equation one gets:

$$\bar{r} << \sqrt{2\pi}\,\lambda_D \quad \text{or} \quad N_D >> 1, \quad \text{or} \quad N_D = (2\pi)^{3/2} n \lambda_D^3$$

where N_D is the number of particles within Debye sphere.

<u>Individual particle-particle collisions can be neglected if number of particles within Debye sphere is much larger than unity (or average distance between particles is much smaller than λ_D).</u>

Particle density within uniformly charged cylindrical beam of radius R, with current I, propagating with longitudinal velocity βc, is

$$n = \frac{I}{\pi\, q\, \beta c\, R^2}$$