# United States Particles Accelerator School, FEL class notes, Wed. afternoon and Thursday

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LA-UR-14-24262

# Wed. Lecture, part 1: FEL Seeding Techniques

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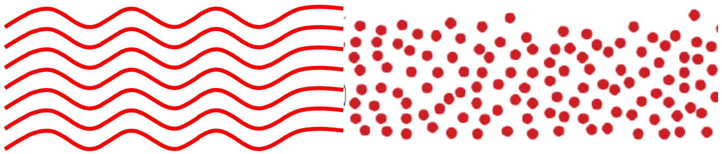
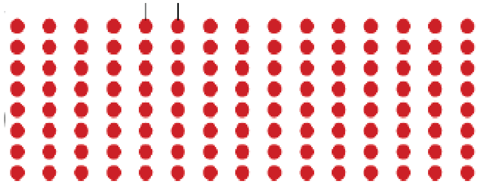
# Different Types of FEL Seeding

There are many different ways of starting the FEL process.  These can be divided into 3 general categories:

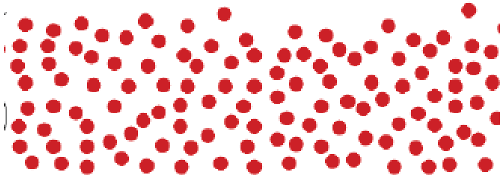
Seeding an FEL with electromagnetic radiation:



Seeding an FEL by pre-bunching the electron beam:



FEL starts through random noise (SASE):



# Seeding

Use eigenfunction expansion:

$$\tilde{E}_x(z) = \sum_j c_j \exp(\alpha_j z)$$

Take first and second derivative…

$$\tilde{E}_x^{'}(z) = \sum_j c_j \alpha_j \exp(\alpha_j z) \quad \text{and} \quad \tilde{E}_x^{''}(z) = \sum_j c_j \alpha_j^2 \exp(\alpha_j z)$$

In matrix form, this is written as:

$$\begin{pmatrix} \tilde{E}_x(z) \\ \tilde{E}_x^{'}(z) \\ \tilde{E}_x^{''}(z) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} c_1 \exp(\alpha_1 z) \\ c_2 \exp(\alpha_2 z) \\ c_3 \exp(\alpha_3 z) \end{pmatrix}$$

The **A** matrix is used to calculate the c's from the initial conditions.

# Seeding, 2

At z=0, we have:

$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x'(0) \\ \tilde{E}_x''(0) \end{pmatrix} = \mathbf{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

But we need to take the inverse in order to get c's from initial conditions:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x'(0) \\ \tilde{E}_x''(0) \end{pmatrix}$$

Next we plug in FEL solution to get values of $\mathbf{A}^{-1}$.

# Seeding, 3

The eigenvalue solutions for an FEL on resonance (η=0), zero energy spread, and negligible space charge ($k_p \approx 0$) is:

$$\alpha_1 = \left(i+\sqrt{3}\right)\Gamma/2 \qquad \alpha_2 = \left(i-\sqrt{3}\right)\Gamma/2 \qquad \alpha_3 = -i\,\Gamma$$

Insert values into **A** matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ \left(i+\sqrt{3}\right)\Gamma/2 & \left(i-\sqrt{3}\right)\Gamma/2 & -i\Gamma \\ \left(i+\sqrt{3}\right)^2\Gamma^2/4 & \left(i-\sqrt{3}\right)^2\Gamma^2/4 & -\Gamma^2 \end{pmatrix}$$

We need to take inverse of A so that we can solve for initial conditions:

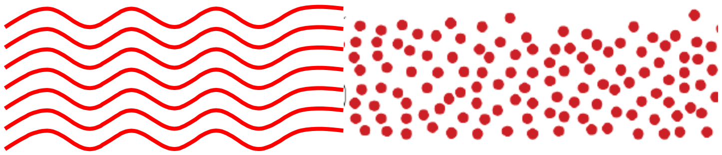
$$\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & \left(\sqrt{3}-i\right)/\left(2\Gamma\right) & \left(-i\sqrt{3}+1\right)/\left(2\Gamma^2\right) \\ 1 & \left(-\sqrt{3}-i\right)/\left(2\Gamma\right) & \left(i\sqrt{3}+1\right)/\left(2\Gamma^2\right) \\ 1 & i/\Gamma & -1/\Gamma^2 \end{pmatrix}$$

# Seeding with electromagnetic radiation

Simplest case: seed with electromagnetic radiation:

$$\tilde{E}_x(0) = E_{seed}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} E_{seed} \\ 0 \\ 0 \end{pmatrix}$$



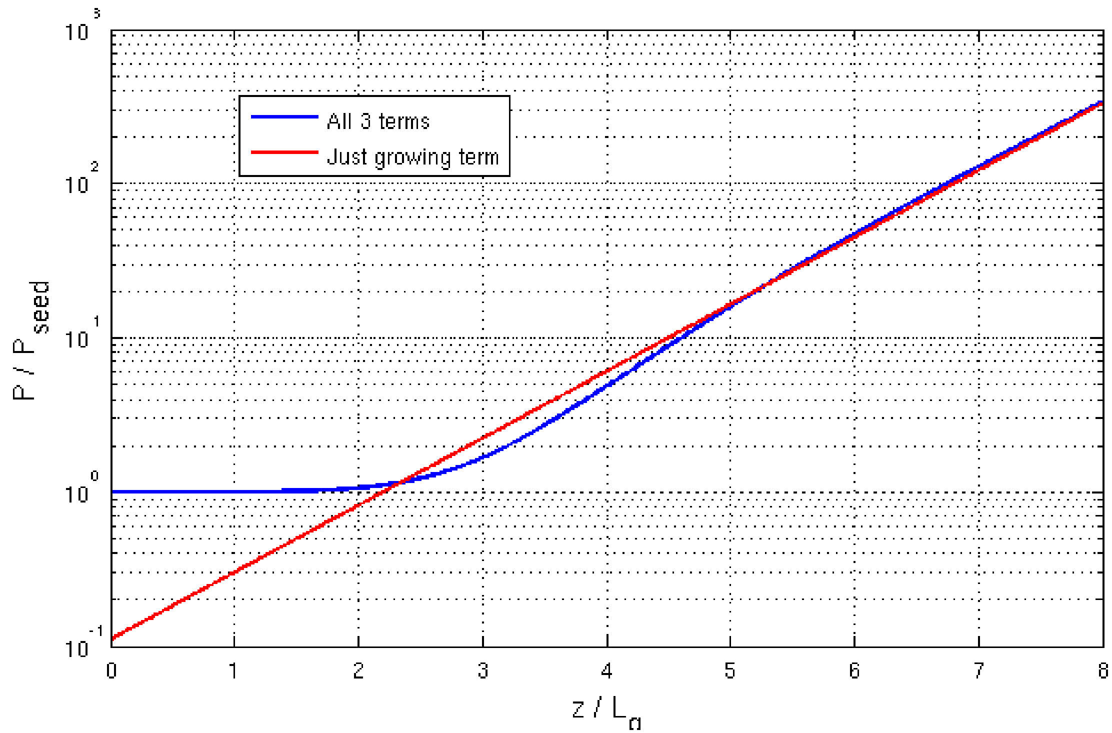
Insert value for **A$^{-1}$** on resonance and with zero energy spread (previous page):

$$c_1 = c_2 = c_3 = \frac{1}{3} E_{seed}$$

Evolution of electric field is:

$$E_x(z) = \frac{E_{seed}}{3} \exp\left[ z\left(i + \sqrt{3}\right)\Gamma / 2 \right] + \frac{E_{seed}}{3} \exp\left[ z\left(i - \sqrt{3}\right)\Gamma / 2 \right] + \frac{E_{seed}}{3} \exp\left(-i\Gamma z\right)$$

# Dominance of First Term




For large values of $z\Gamma$, the first term in equation for $E_x(z)$ dominates, because it is the only term with a positive, real component in the exponential. Then we have a very simple expression for $E_x(z)$:

$$\left|\tilde{E}_x(z)\right| = \frac{\left|E_{seed}\right|}{3}\exp\left[\frac{\sqrt{3}\,\Gamma z}{2}\right]$$

Then use the FEL gain length, and square to get in terms of power:

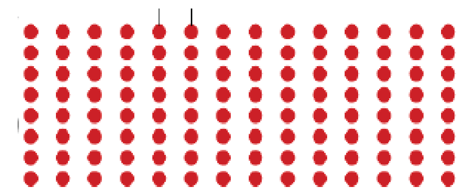
$$L_g = \frac{1}{\sqrt{3}\Gamma}$$

$$P(z) = \frac{P_{seed}}{9}\exp\left(\frac{z}{L_g}\right)$$

# Seeding with a pre-bunched electron beam

What are the initial conditions (ie,

$$\left[\tilde{E}_x(0), \tilde{E}_x^{'}(0), \tilde{E}_x^{''}(0)\right]$$



for an FEL with a pre-bunched electron beam?

From 1D FEL theory, we know that:

$$\frac{d\tilde{E}_x(0)}{dz} = -\frac{\mu_0 c\hat{K}}{4\gamma_r}\tilde{j}_1(0)$$

Here $\mu_0$ is the vacuum permeability, c is the speed of light, K is the undulator parameter, $\gamma_r$ is the average electron energy (in terms of the relativistic factor), and $j_1(0)$ is the oscillating current at the FEL wavelength:

$$\tilde{j}_z(z) = j_0 \frac{2}{N}\sum_{n=1}^{N}\exp\left[-i\Psi_n(z)\right]$$

Here $j_0$ is the DC current density and N is the # of electrons.

# Seeding with a pre-bunched electron beam, 2

We still need the second derivative of the E field, $\tilde{E}_x''(0)$

Equation for second derivative of E field:

$$\frac{d^2 \tilde{E}_x(0)}{dz^2} = -\frac{\mu_0 c \hat{K}}{4\gamma_r} \tilde{j}_1'(0)$$

Equation for derivative of harmonic current:

$$\frac{d\,\tilde{j}_z(z)}{d\,z} = j_0 \frac{-2i}{N} \sum_{n=1}^{N} \frac{d\Psi_n}{d\,z} \exp\left[-i\Psi_n(z)\right]$$

Next use the equation for the evolution of a single electron phase:

$$\frac{d\Psi_n}{d\,z} = 2k_u \eta_n$$

In our simple case, electrons are initially monoenergetic with energy η0. Then we have:

$$\frac{d\Psi_n\left(0\right)}{d\,z} = 2k_u \eta_0$$

All electrons have the same initial derivative with respect to phase.

# Seeding with a pre-bunched electron beam, 3

Now we can write a simple expression for the derivative of harmonic current at z=0:

$$\frac{d\,\tilde{j}_1(0)}{d\,z} = \left(-2ik_u\eta_0\right) j_0 \frac{2}{N}\sum_{n=1}^{N}\exp\left[-i\Psi_n(z)\right] = -2ik_u\eta_0\,\tilde{j}_1(0)$$

So now we have equations for both dE/dz and $d^2E/dz^2$.
We can write this in vector form:

$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x^{'}(0) \\ \tilde{E}_x^{''}(0) \end{pmatrix} = \frac{\mu_0 c\hat{K}}{4\gamma_r}\,\tilde{j}_1\left(0\right)\begin{pmatrix} 0 \\ -1 \\ i2k_u\eta \end{pmatrix}$$

# Seeding with a pre-bunched electron beam, 4

Now we want to calculate the starting coefficients (c's):

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \frac{\mu_0 c \hat{K} \tilde{j}_1(0)}{4\gamma_r} \mathbf{A}^{-1} \begin{pmatrix} 0 \\ -1 \\ 2ik_u\eta \end{pmatrix}$$

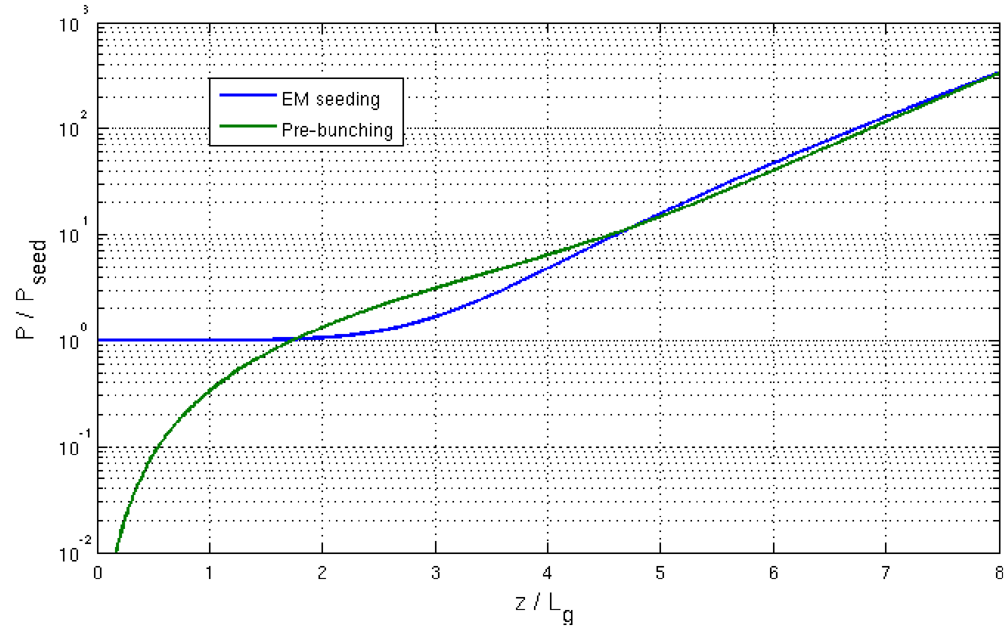
Let's just look at the $c_1$ term, which represents the only eigenvalue that grows exponentially:

$$c_1 = \frac{\mu_0 c \hat{K} \tilde{j}(0)}{12\gamma_r} \left[ \frac{-\left(\sqrt{3}-i\right)}{2\Gamma} + \left(2ik_u\eta\right) \frac{1-i\sqrt{3}}{2\Gamma^2} \right]$$

For on resonance FEL, η=0
Then, for after many gain lengths, we have:

$$\left|\tilde{E}_x(z)\right| \approx \frac{\mu_0 c \hat{K} \left|\tilde{j}_1(0)\right|}{12\gamma_r \Gamma} \exp\left[\frac{\sqrt{3}\Gamma z}{2}\right]$$

# Comparison of EM seeding and electron pre-bunching




At large z, the E field for a seeded FEL is:

$$\left|\tilde{E}_{seeded}\left(z\right)\right| = \frac{\left|E_{seed}\right|}{3}\exp\left[\frac{\sqrt{3}\,\Gamma z}{2}\right]$$

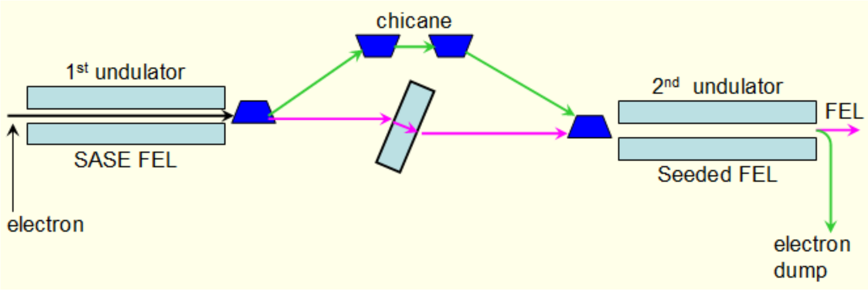
At large z, the E field for an FEL with a pre-bunched electron beam is:

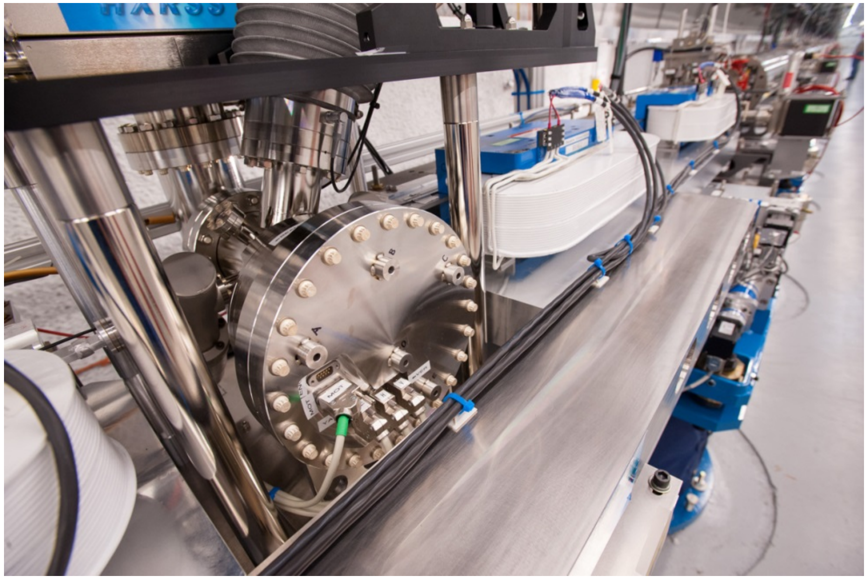
$$\left|\tilde{E}_{prebunched}\left(z\right)\right| \approx \frac{\mu_0 c\hat{K}\left|\tilde{j}_1(0)\right|}{12\gamma_r\Gamma}\exp\left[\frac{\sqrt{3}\Gamma z}{2}\right]$$

So the seeding strength that gives equivalent strength to a pre-bunched FEL is:

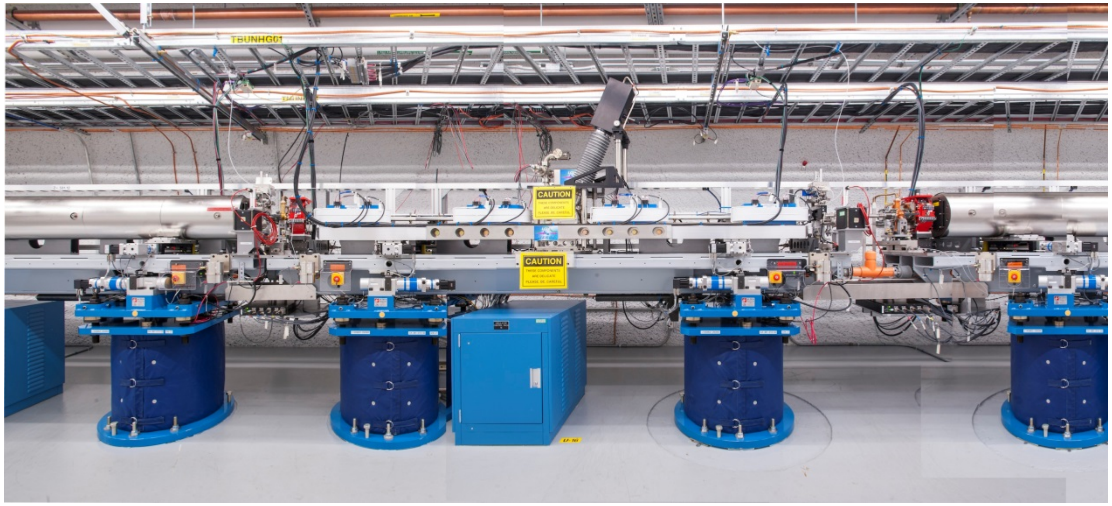
$$E_{equivalent} = \frac{\mu_0 c\hat{K}\left|j_1\right|}{4\gamma_r\Gamma}$$

# Example of self seeded FEL (LCLS)

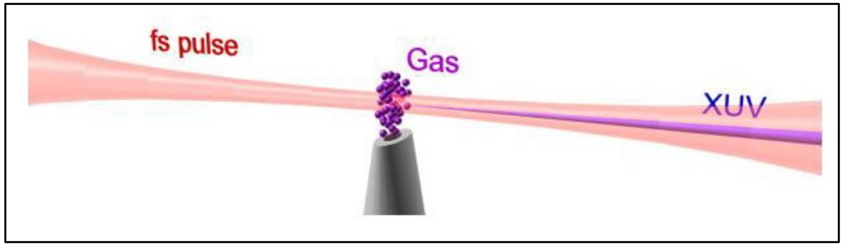


- SASE radiation is filtered through a crystal monochromator. The narrowband radiation then seeds the electron beam in a $2^{nd}$ undulator.
- This is a promising scheme which is operational at SLAC.

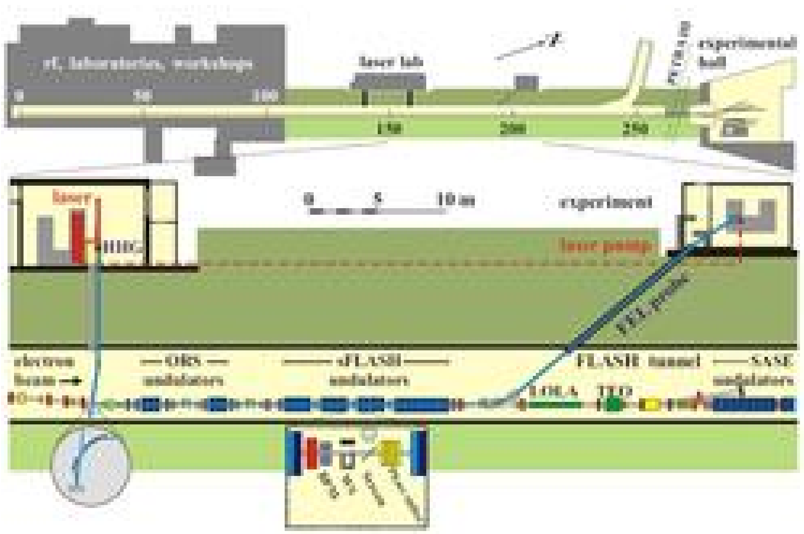


# Example of HHG FEL




**High Harmonic Generation**

- High Harmonic Generation uses a gas, plasma, or solid to generate high harmonics from an intense laser pulse.
- Experiments indicate that this method is currently impractical below 200 nm, because light becomes broadband with poor transverse coherence.
- In addition, extremely high powers of the laser fundamental are needed, which would limits the rep rate.




sFLASH is a planned experiment in Hamburg, Germany to seed an FEL directly with 38 nm radiation that has been generated through HHG.

# Wed. lecture part 2: SASE

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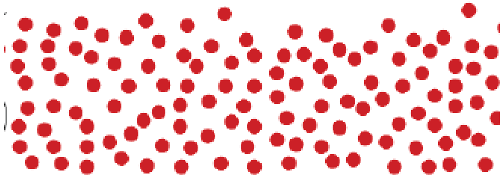


# Intro to SASE

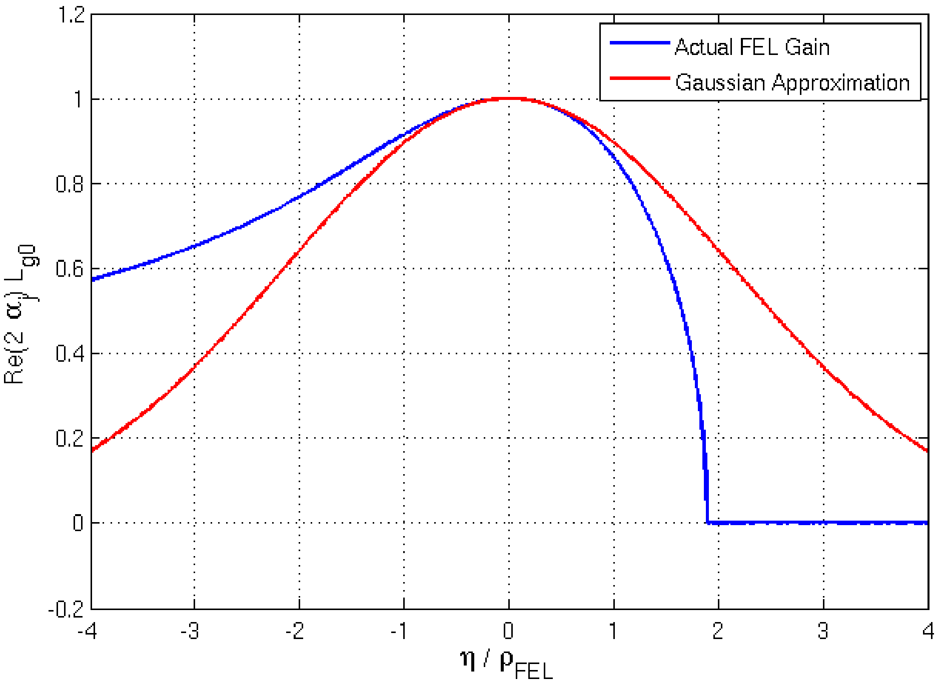
A SASE FEL is an FEL that is seeded from the random noise that comes from the discrete nature of electrons in an electron beam. The seed signal can either be thought of as broadband electron bunching, or as broadband synchrotron radiation.

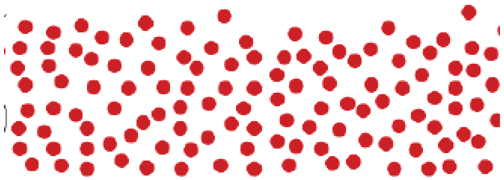
The FEL amplifies a narrowband portion of the signal. The amplified component is a complicated function of distance along the undulator and frequency:

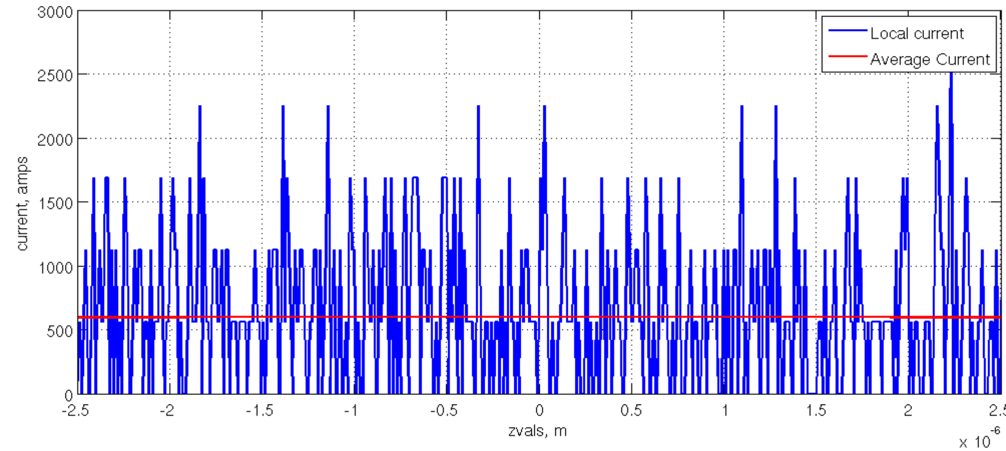
$$\left|\tilde{E}_x(\omega,z)\right| = \frac{\mu_0 c\hat{K}}{4\gamma_r}\sqrt{G(\omega,z)}\left|J(\omega)\right|$$

The gain function of an FEL is complicated (see p. 214 of textbook). We will use the simple Gaussian approximation of the gain function:

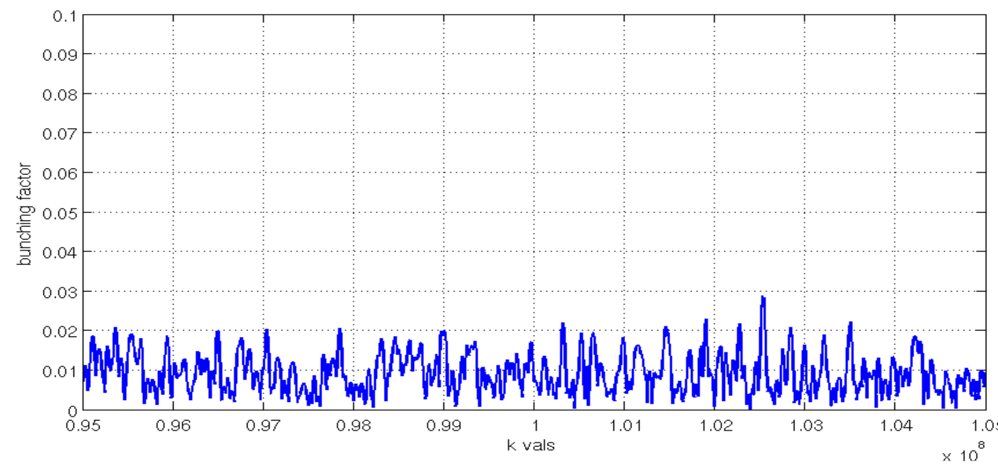
$$G(\omega,z) \propto \exp\left(\frac{z}{L_{g0}}\right)\exp\left(-\frac{\eta^2 z}{9\rho_{FEL}^2 L_{g0}}\right)$$



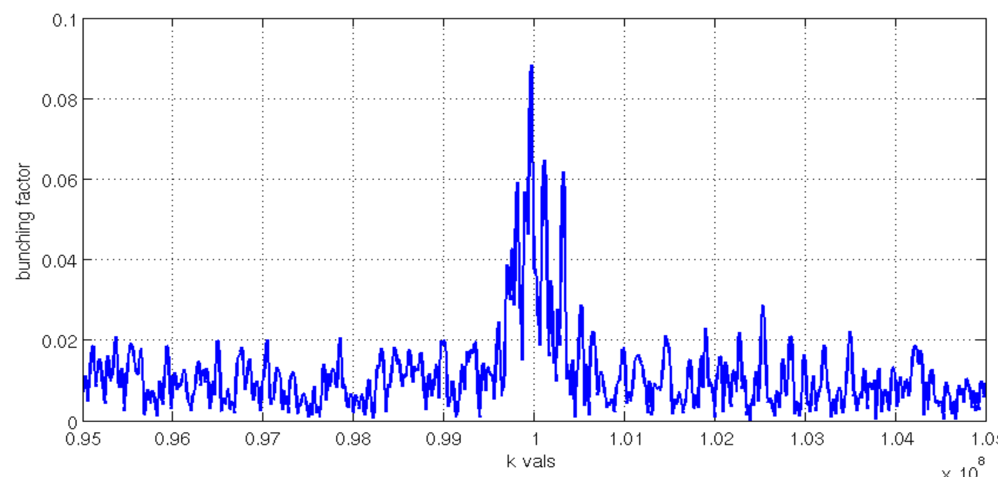





Discreteness of electrons leads to random fluctuations in current, as in the plot in top left.



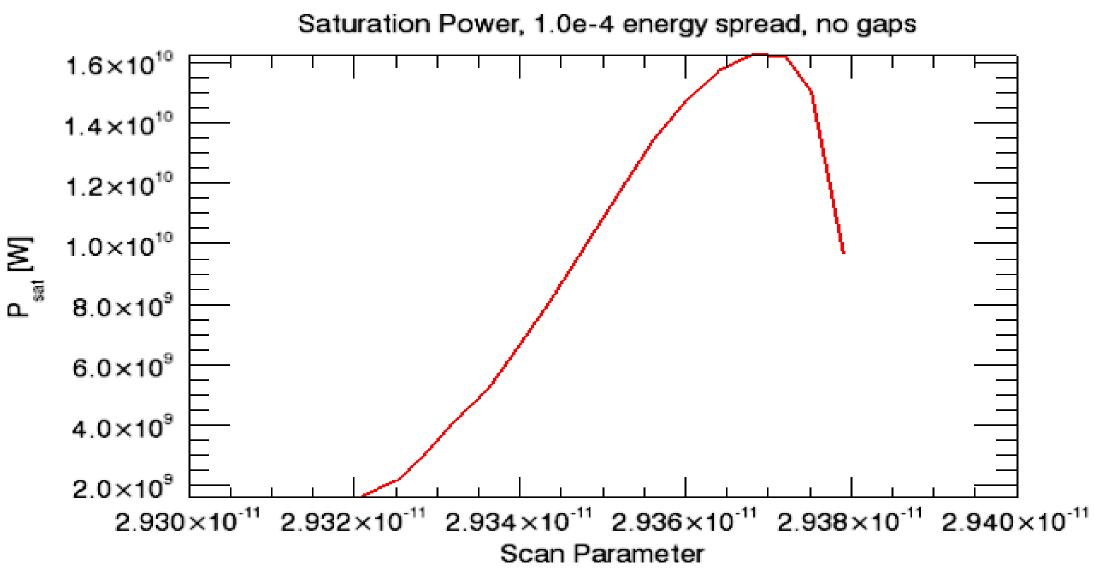

Taking FFT of current fluctuations, there is frequency dependent random bunching as a function of frequency.

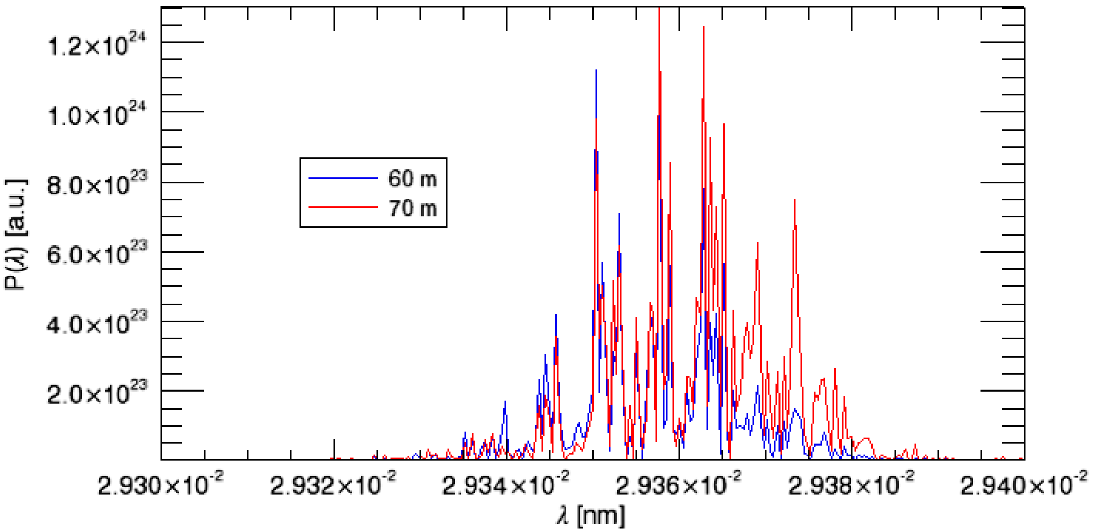



The FEL amplifies a narrow portion of the initial SASE seed. This portion grows up from noise. The randomness of the initial bunching is still apparent in the amplified FEL.

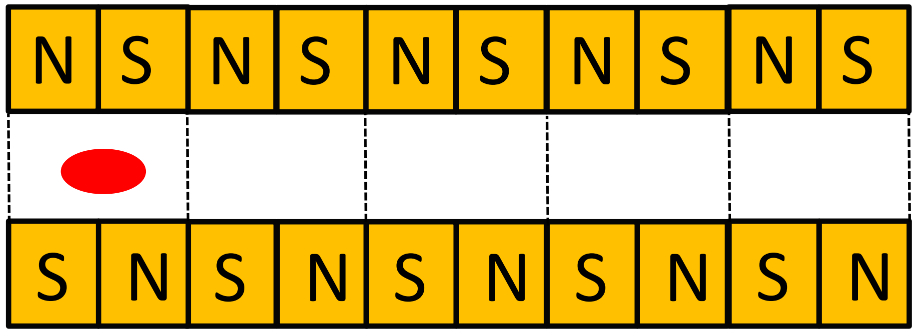
# SASE simulations of proposed LANL FEL

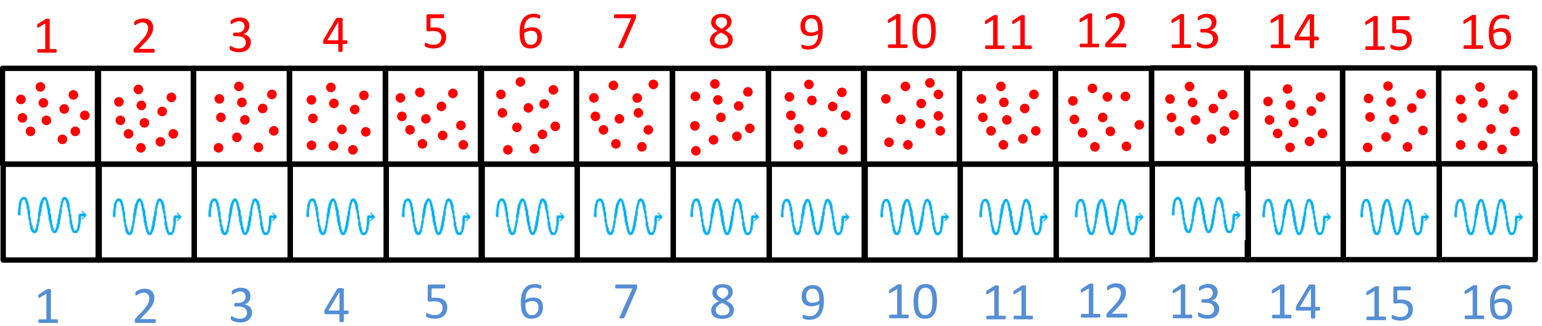
The top plot shows a scan of FEL saturation power vs. wavelength. This is a set of GENESIS simulations of the planned MaRIE XFEL at LANL. This result looks like the 1D FEL gain curve shown previously. The bottom shows the results of a time-dependent GENESIS simulation, with the same FEL parameters. The saturation power follows the gain curve, but there are random fluctuations in the power that reflect the initial random bunching of the SASE seed.



# Slippage illustration



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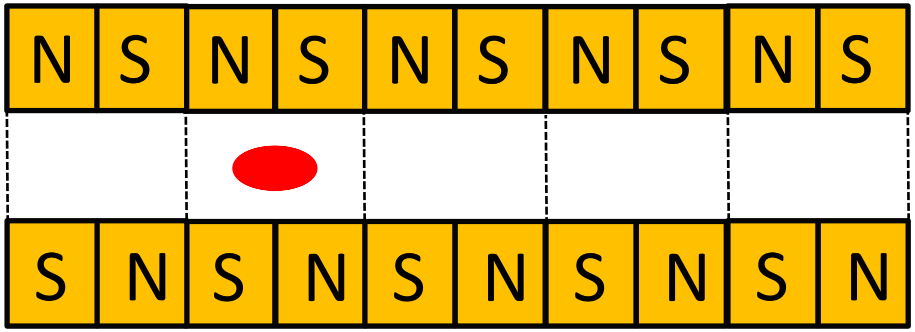


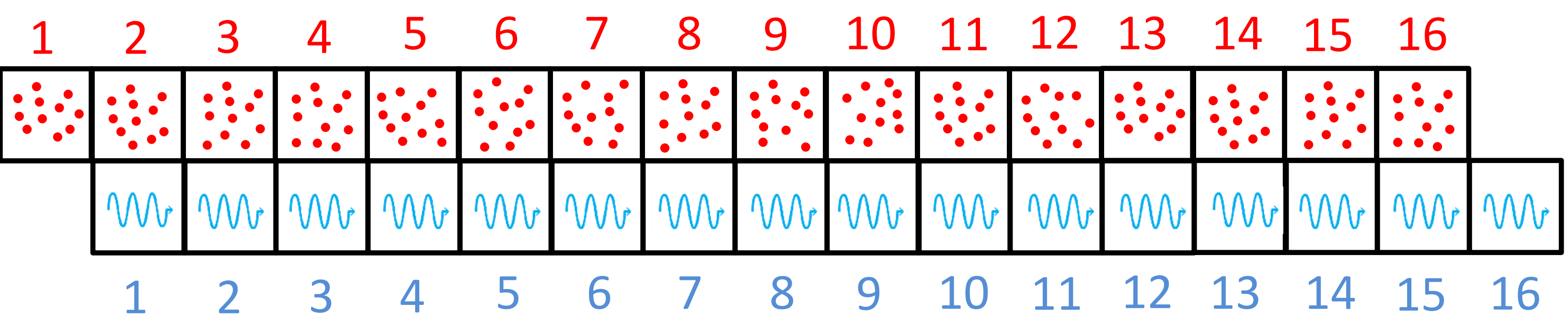

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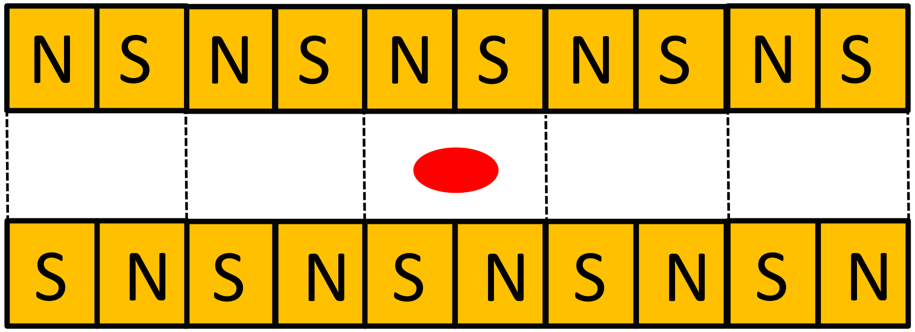


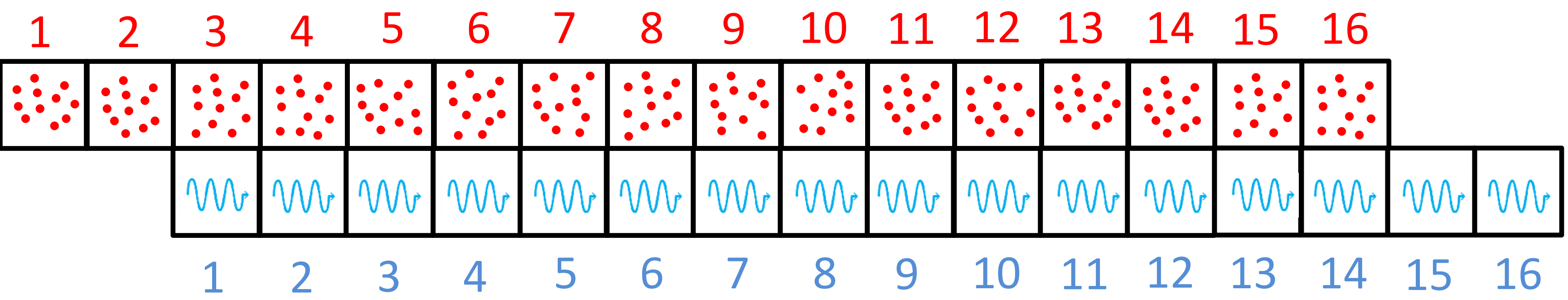

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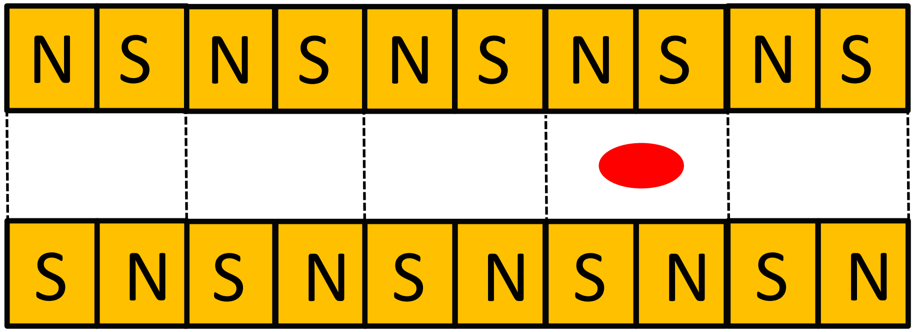


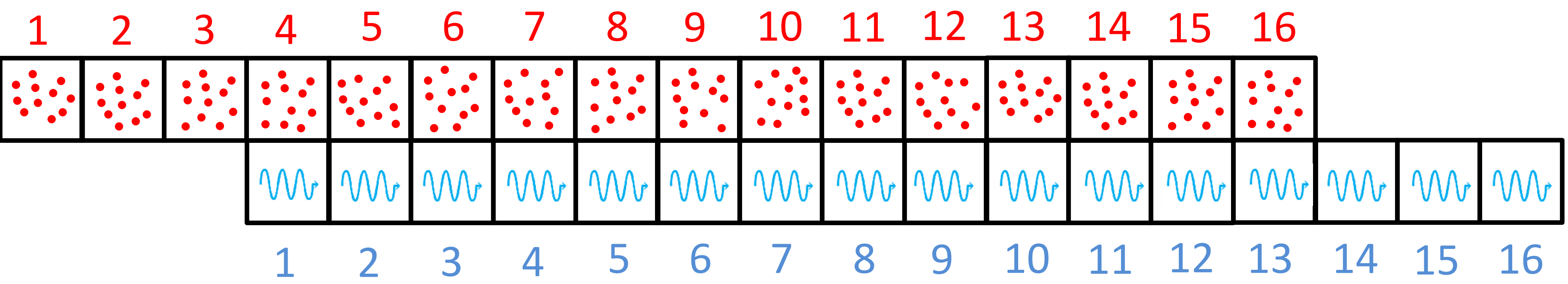

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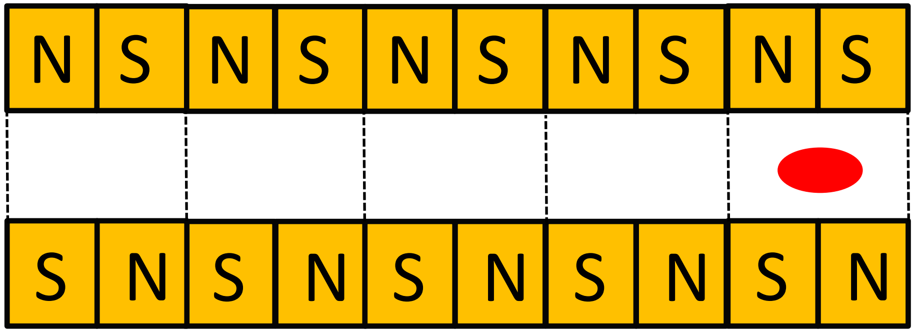
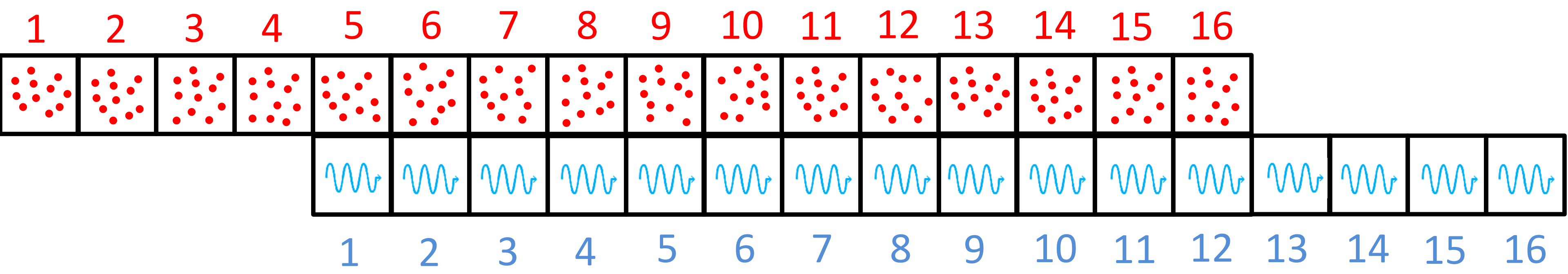
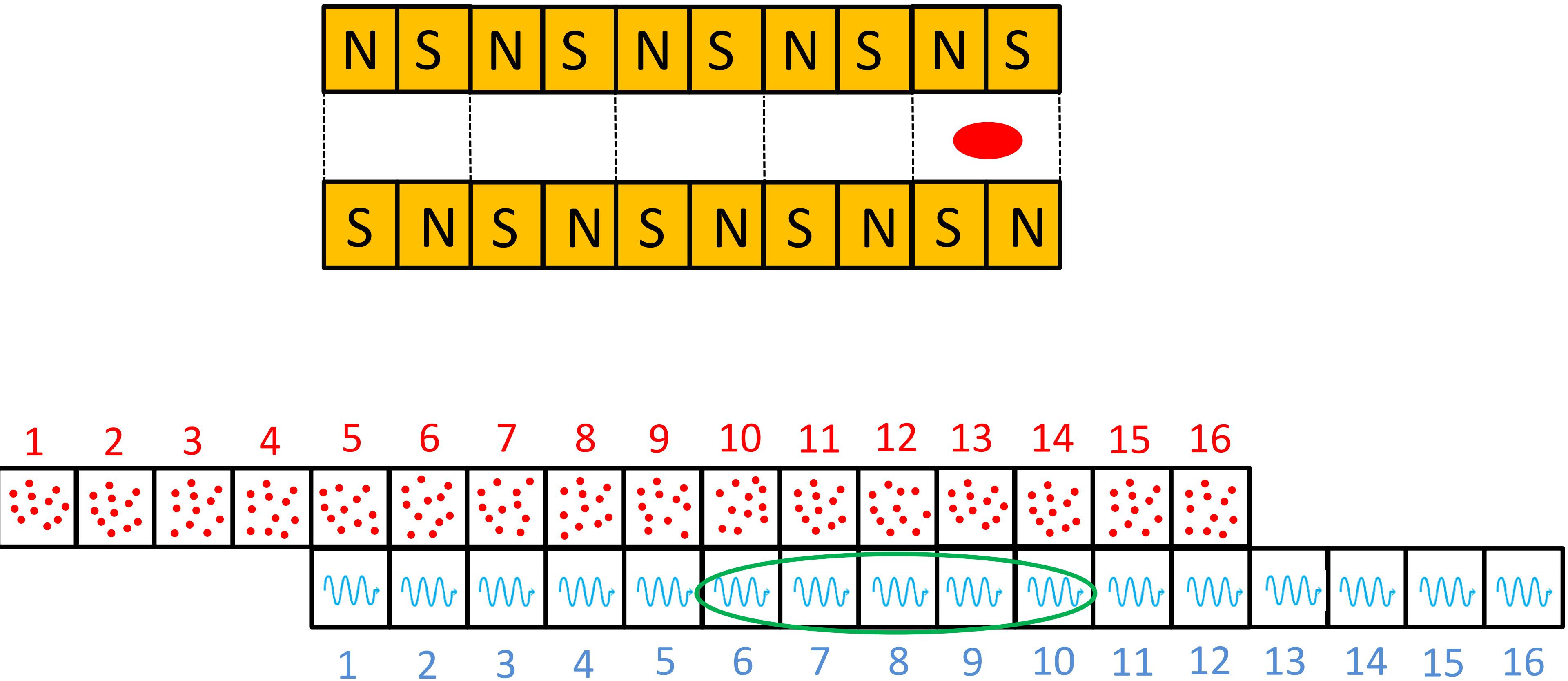


# Slippage illustration

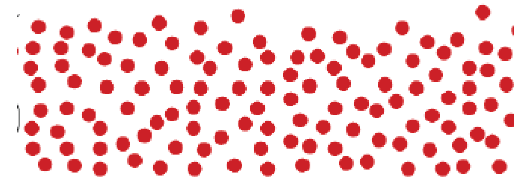



Slippage allows different slices of the photon beam to communicate with each other via the electron beam. This is what filters the wideband SASE input into the narrow FEL bandwidth.

# Calculation of SASE seed strength

Define Fourier transform of the electron current:

$$i_T(\omega) = \int_{-T/2}^{+T/2} I(t)\exp(i\omega t)\,dt$$



Because I(t) is a real function, we have $\left|\tilde{I}(\omega)\right| = \left|\tilde{I}(-\omega)\right|$

Then the total power in the signal can be written:

$$P = \frac{1}{2\pi T}\int_{-\infty}^{+\infty}\left|i_T(\omega)\right|^2 d\omega = \frac{1}{\pi T}\int_{0}^{+\infty}\left|i_T(\omega)\right|^2 d\omega$$

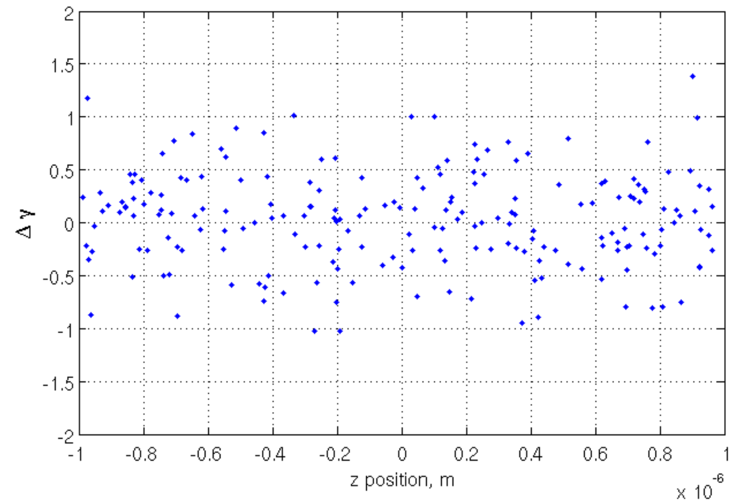
Then let: $S(\omega) = \frac{1}{\pi}\left\langle\left|\tilde{I}(\omega)\right|^2\right\rangle$

where < > denotes an average over a large number of shots with similar experimental conditions. The value S(ω)dω is the <u>average</u> amount of AC current within the frequency range [ω, ω + dω].

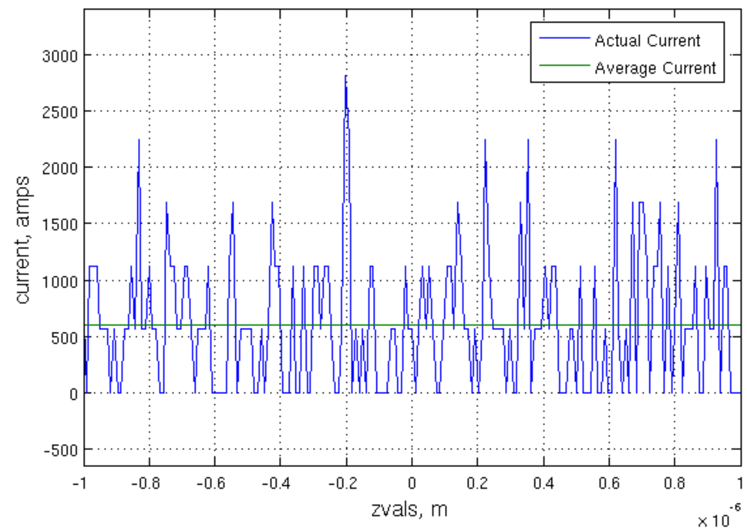
But what is S(ω) for a SASE FEL? To answer this, we need to look at shot noise, which is the AC current that exists because of discrete locations of electrons.

# Calculation of SASE seed strength, 2

We have been describing electron current as a smooth function I(t), where I(t) is something like a Gaussian. A more accurate description of current, which accounts for the discrete location of electrons, is a sum of Dirac delta functions:




$$I(t) = e\sum_{j=1}^{N}\delta\left(t - t_j\right)$$




Take the FT of I(t):

$$i_T(\omega) = \int_{-\infty}^{+\infty}\left[e\sum_{j=1}^{N}\delta\left(t - t_j\right)\right]\exp\left(i\omega t\right)dt = e\sum_{j=1}^{N}\exp\left(i\omega t_j\right)$$

# Calculation of SASE seed strength, 3

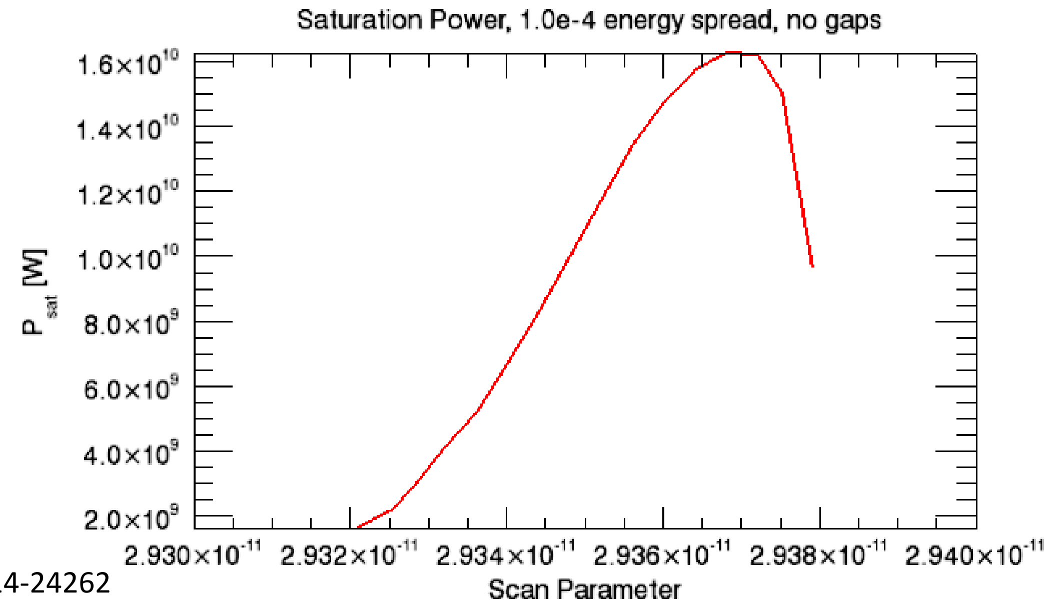
We want the spectral density function, S(ω):

$$S(\omega) = \frac{1}{\pi T}\left\langle \left| i_T(\omega) \right|^2 \right\rangle$$

$$= \frac{e^2}{\pi T}\left\langle \sum_{j=1}^{N} \exp\left(i\omega t_j - i\omega t_j\right) + \sum_{j=1}^{N}\sum_{k\neq j}^{N} \exp\left[ i\omega\left(t_j - t_k\right)\right]\right\rangle$$

$$= \frac{e^2 N}{\pi T} = \frac{e I_0}{\pi}$$

An FEL has a sharply defined gain curve, Δω, where the power at saturation drops off very rapidly outside $\omega_r \pm \Delta\omega / 2$. (Here $\omega_r$ is the resonant angular frequency). We can determine the initial SASE current by calculating the total current from random noise in this FEL bandwidth.



# Calculation of SASE seed strength, 4

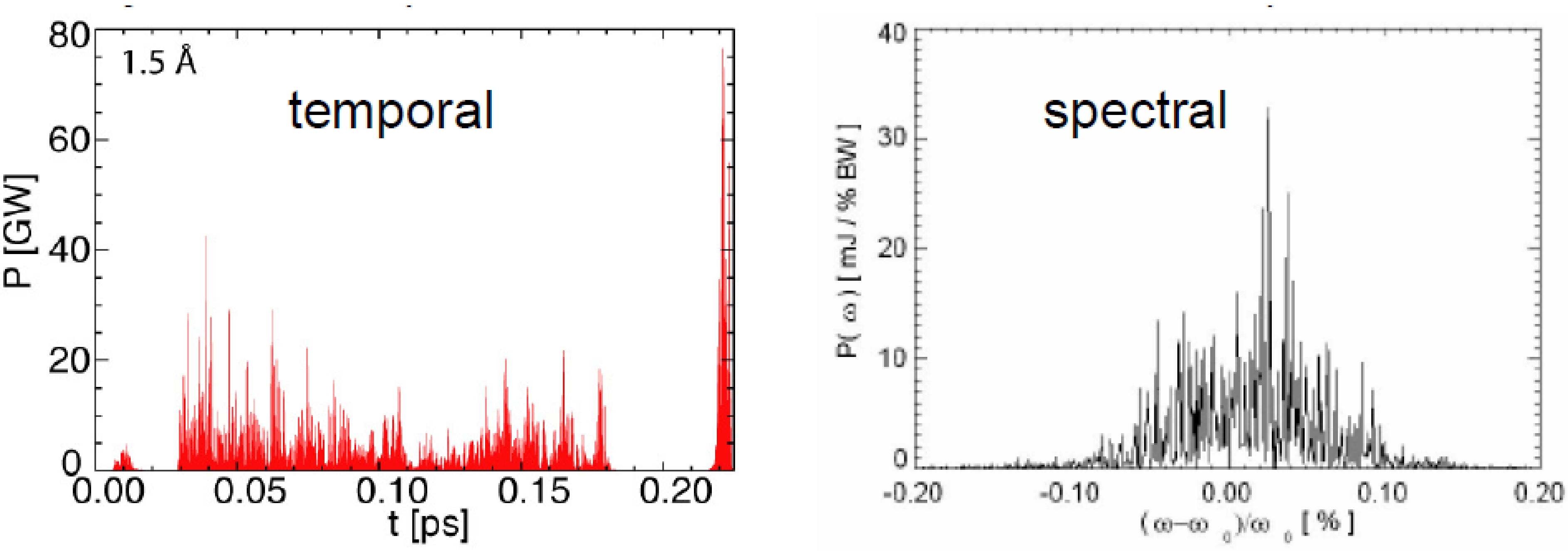
The total AC current from shot noise that is in the FEL resonant bandwidth is:

$$I_{rms}^2 = S(\omega)\Delta\omega = \frac{eI_0}{\pi}\Delta\omega$$

The FEL equations are in terms of the current density, $j_1$. The total starting current density for a SASE FEL is:

$$\left|\tilde{j}_1\right| = \frac{\sqrt{I_{rms}^2}}{A_b} = \frac{1}{A_b}\sqrt{\frac{eI_0\Delta\omega}{\pi}}$$

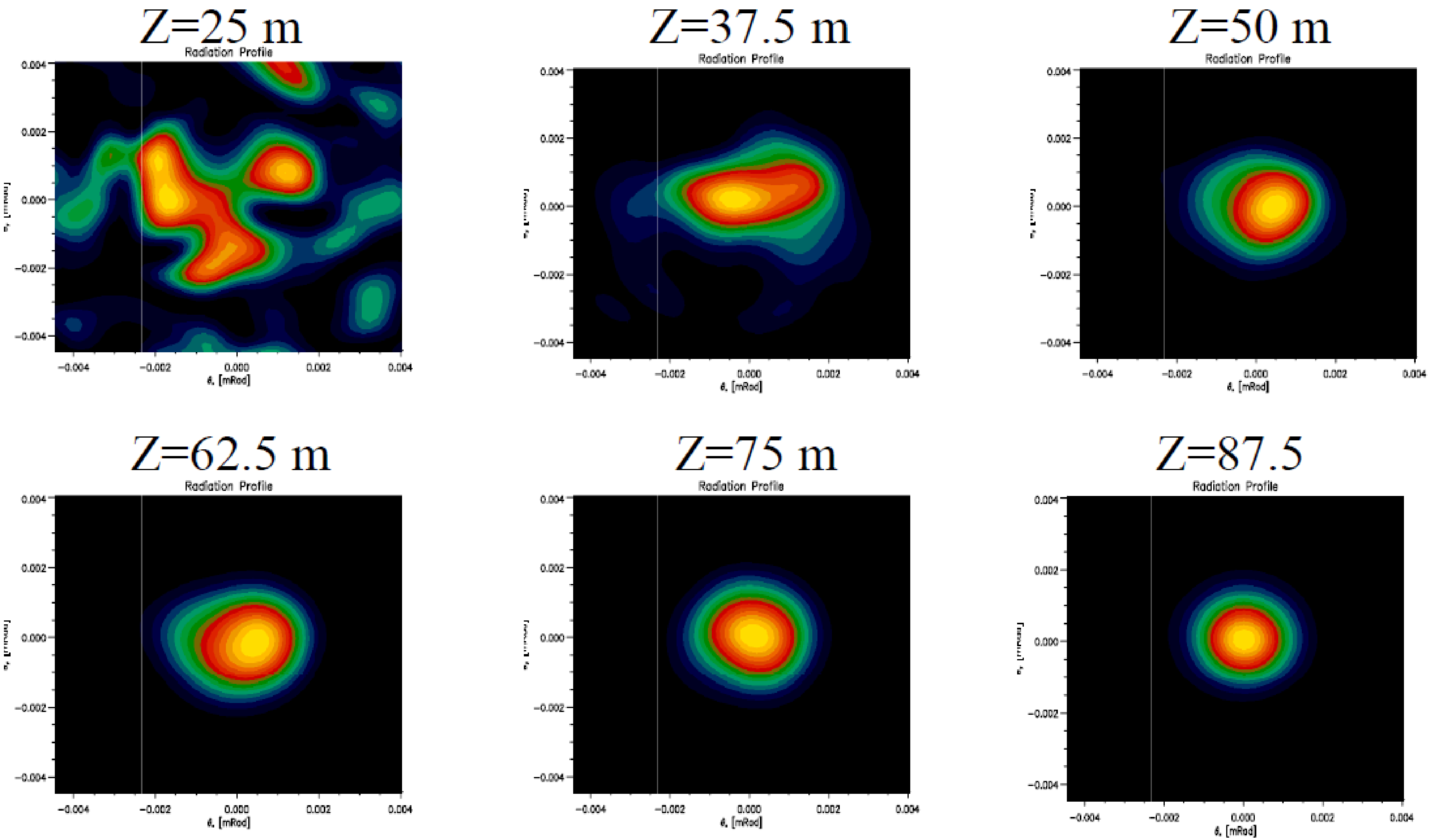
# Temporal coherence in the LCLS SASE FEL




The LCLS output consists of several spikes in both temporal and spectral profiles. The full width of the spectral profile is the Fourier transform of individual temporal "spikes," or coherence lengths. The width of each spectral "spikes" is the Fourier transform of the entire pulse length.

LCLS recently demonstrated that self-seeding reduces the FEL spectral linewidth

# Evolution of Transverse Coherence in the LCLS SASE FEL

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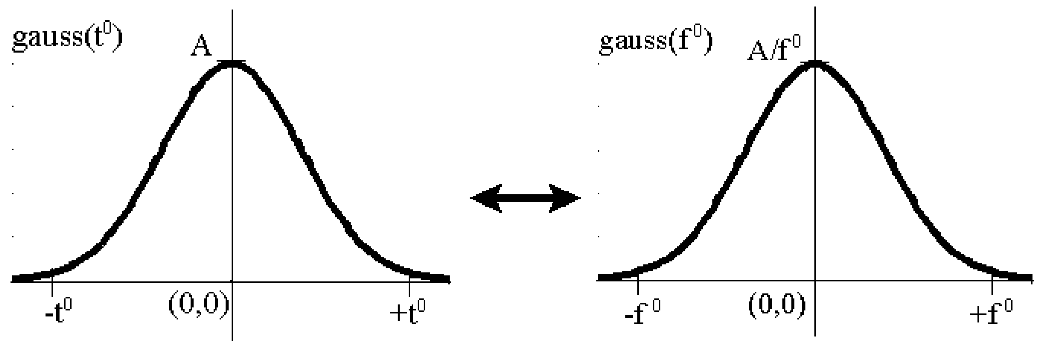
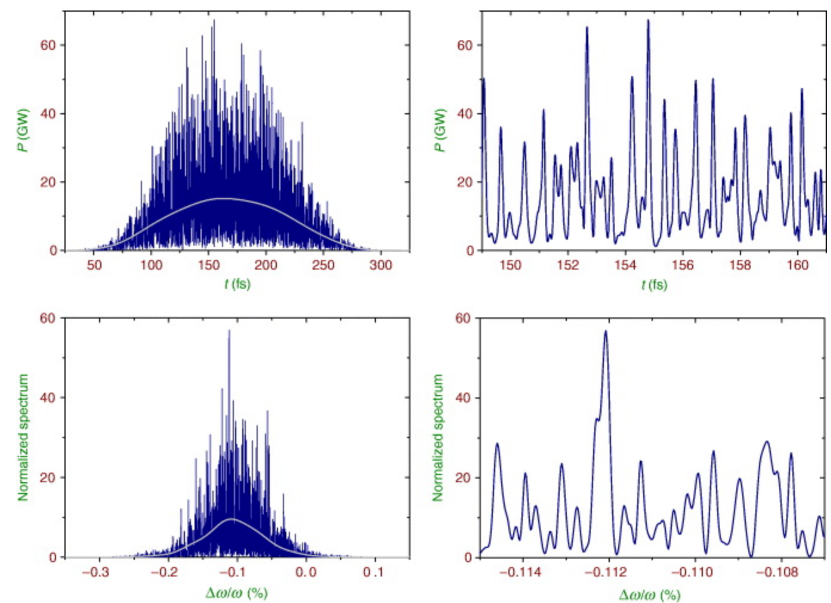
# Fluctuations in SASE Energy

A SASE FEL has a bandwidth $\Delta\omega$ that is determined by the FEL gain, and a corresponding coherence time $\tau_c$, with $\tau_c \sim 1 / \Delta\omega$.
We will estimate the random fluctuations in the SASE energy for 2 cases:

1. Electron beam is shorter than coherence time, $\tau_b < \tau_c$. Then SASE radiation will manifest itself as a single Gaussian pulse.




2. Electron beam is longer than coherence time, $\tau_b < \tau_c$. Then there will be many spikes of SASE power.

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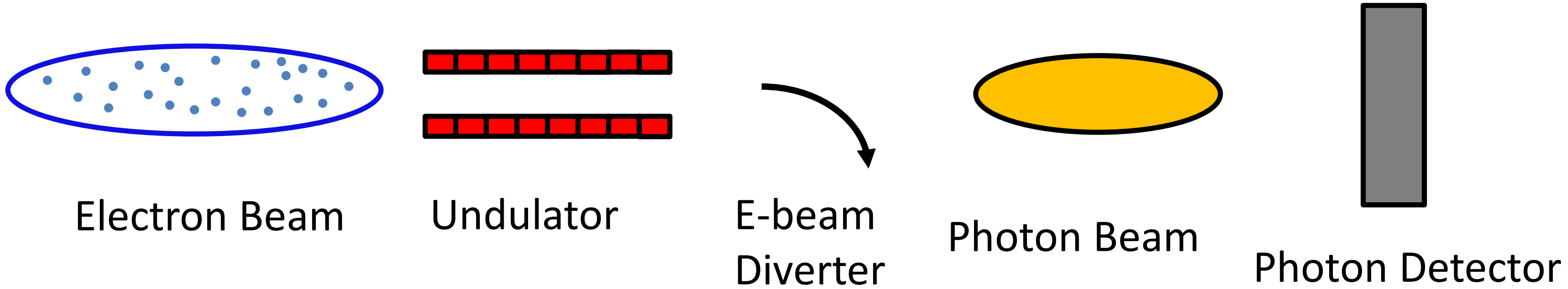
# Fluctuations in SASE Energy, 2

In a real FEL, you measure the total energy from each shot, and you can measure the average energy over a large number of shots.

We want to calculate the statistics that determine how the photon energy fluctuates from shot to shot.



# Fluctuations in SASE Energy, case #1

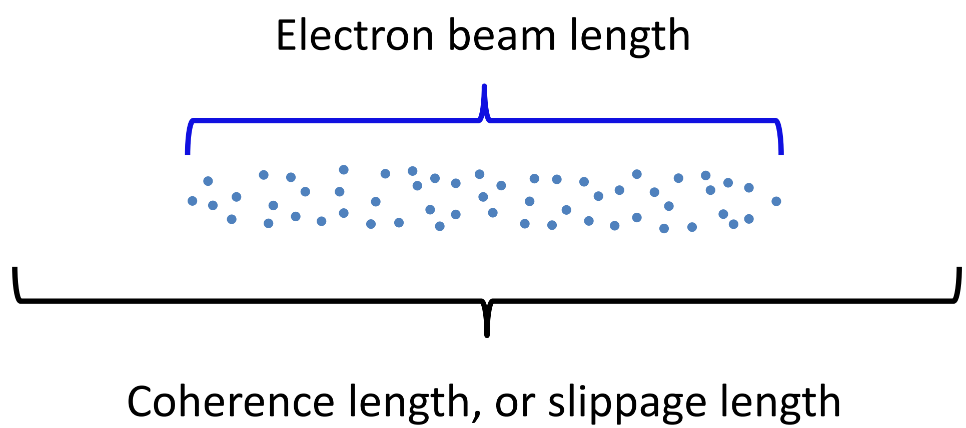
Let's look at case #1, with $\tau_b < \tau_c$. In this case, all of the electrons communicate with each other through slippage, so the total SASE seeding source is the signal that exists when all phases are added up.

1 electron: $$E_j(t) = E_0 \exp(-i\omega_l t)\exp(i\phi_j)$$

many electrons: $$E_j(t) = E_0 \exp(-i\omega_l t)\exp(i\phi_j)$$

We care about seeding energy:

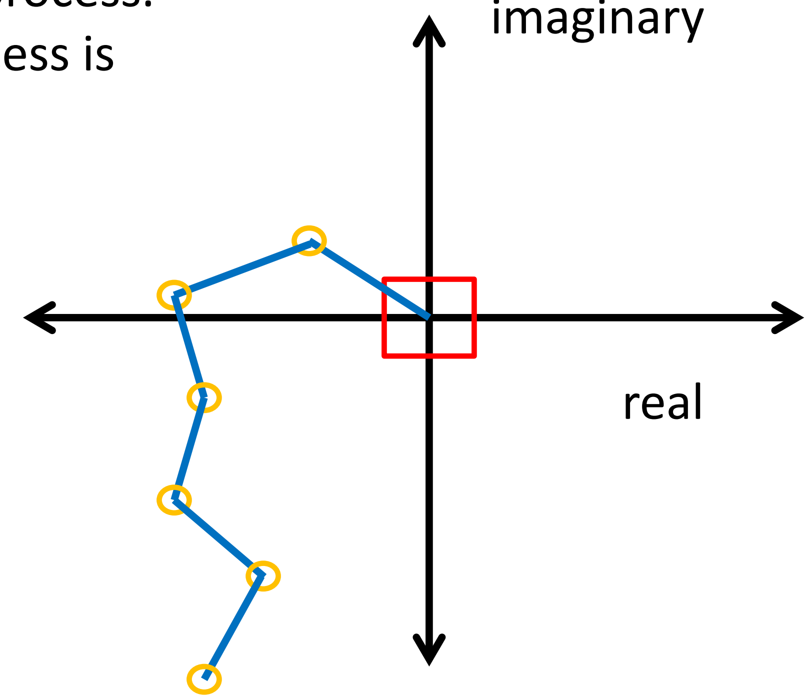
$$U \propto E_0^2 \left| \sum_j \exp(i\phi_j) \right|^2$$



# Random Walk

The addition of phases is a random walk process. The probability distribution of such a process is given by:

$$p(x,y)dxdy = \frac{1}{\pi N}\exp\left[\frac{-(x^2+y^2)}{N}\right]$$




Or, in cylindrical coordinates:

$$p(r)dr = \frac{2r}{N}\exp\left(\frac{-r^2}{N}\right)dr$$

The RMS value of this distribution is given by:

$$\langle r^2 \rangle = \int_0^{\infty} r^2 p(r)dr = N$$

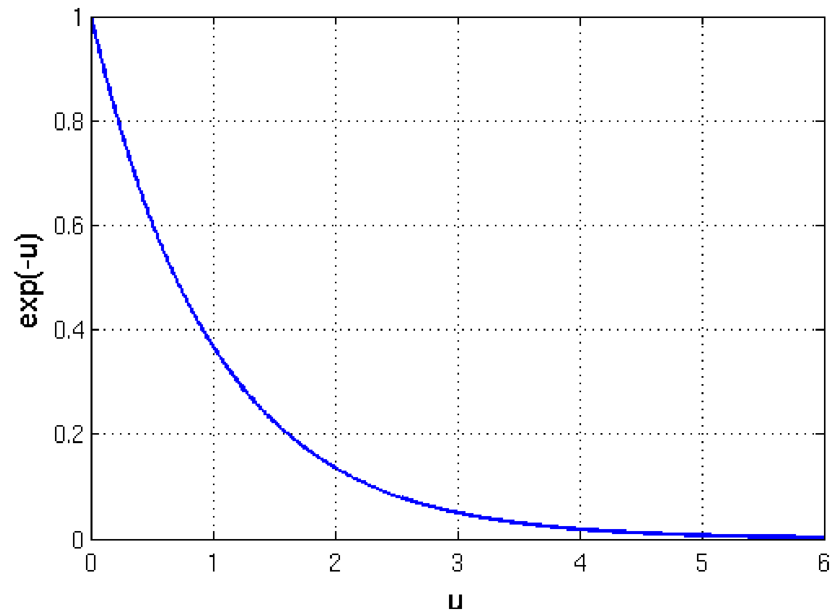
# Convert to dimensionless energy

Now we want to relate r from the random walk process to the electric field E and the pulse energy U. The electric field is proportional to r, while the pulse energy is proportional to $r^2$:

$$r \propto E, \quad U \propto r^2$$

Make U dimensionless by dividing by the average energy over a large number of fluctuations:

$$u = \frac{U}{\langle U \rangle}$$



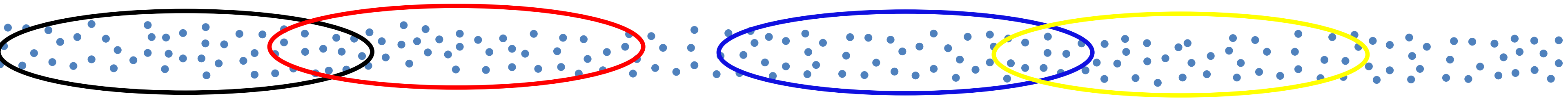

Transform: $u = \dfrac{r^2}{\langle r^2 \rangle} \qquad du = \dfrac{2r\,dr}{N}$

Get: $$p(u)du = \exp(-u)du$$

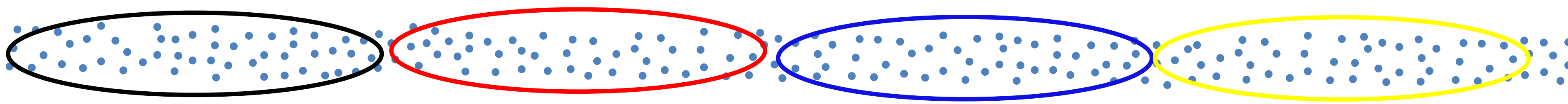
A SASE FEL with a short time duration will have energy fluctuations that follow these statistics.

# Fluctuations in SASE Energy, case #2

Let's look at case #2, which is a SASE FEL with $\tau_b >> \tau_c$. In this case, electrons in one part of the electron beam do not communicate with another part through slippage, so the output SASE signal consists of many independent coherent bunches.



Many SASE spikes will form in this case. Some of the spikes will overlap. For simplicity, let's assume the spikes do not overlap:



Now there are M statistically independent SASE spikes in the FEL. Each spike is coherent with itself, but is not coherent with the other SASE spikes. We have:

$$M = \frac{\tau_b}{\tau_c}$$

# Fluctuations in SASE Energy, case #2

Let: $\tilde{u} = \dfrac{U_{tot}}{\langle U_1 \rangle} = uM$ ← Average power of 1 longitudinal mode.

$$u = \frac{U_{tot}}{\langle U_{tot} \rangle}$$

We know that: $p_1(\tilde{u})d\tilde{u} = \exp(-\tilde{u})d\tilde{u}$

Correct solution for fluctuations in power must satisfy:

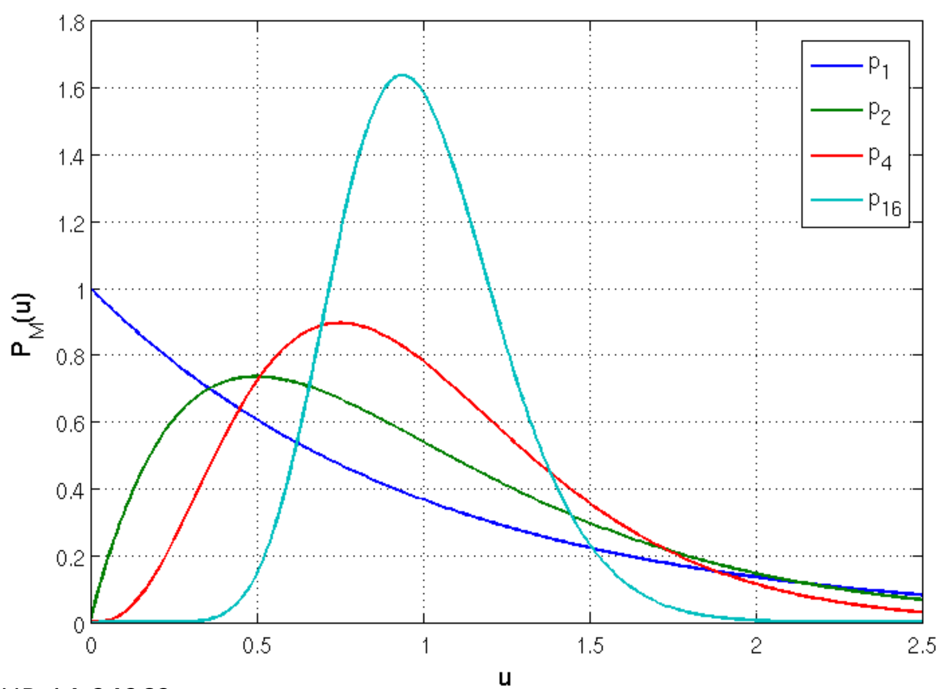
$$p_{M+1}(\tilde{u}) = \int_0^{\tilde{u}} p_M(\tilde{v}) * p_1(\tilde{u}-\tilde{v})d\tilde{v}$$

This will be satisfied for the following function:

$$P_M(\tilde{u})d\tilde{u} = \frac{\tilde{u}^{M-1}}{\Gamma(M)}\exp(-\tilde{u})d\tilde{u}$$

Or, convert back to u:

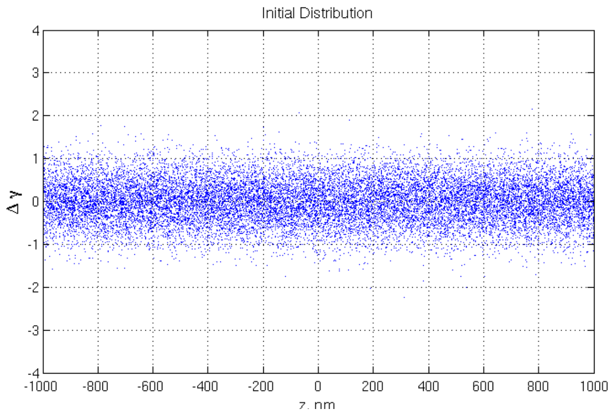
$$P_M(u)du = \frac{M^M u^{M-1}}{\Gamma(M)}\exp(-uM)du$$

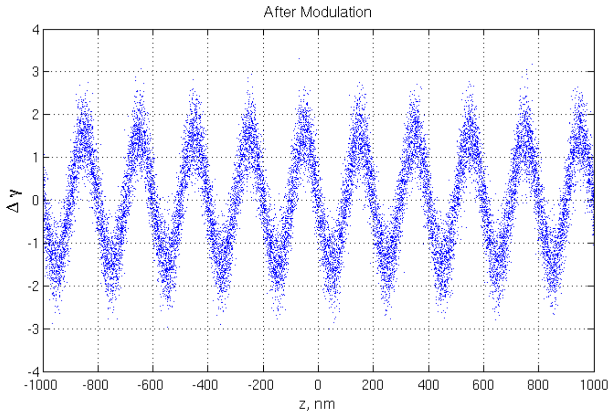


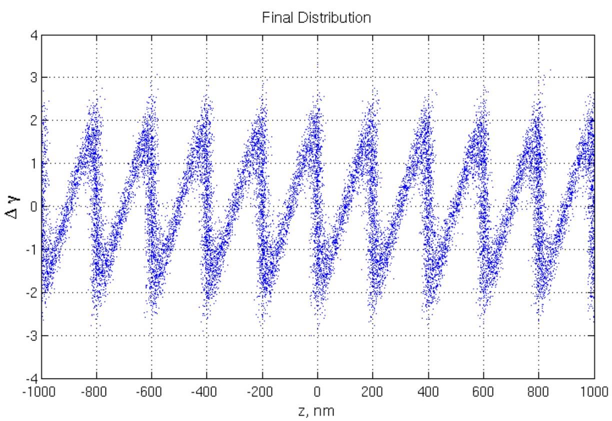
# Thursday lecture part 1: HGHG

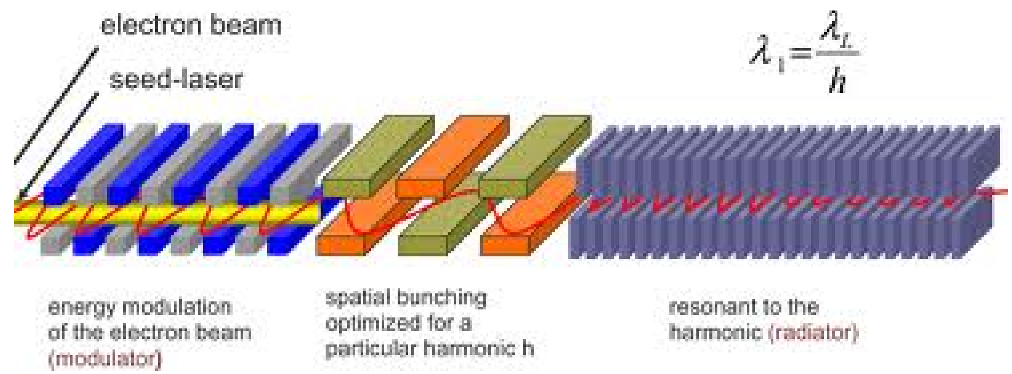
LA-UR-14-24262

# Introduction to HGHG




- High Gain Harmonic Generation uses a single modulator and a single chicane to generate harmonics in the electron current profile.
- The "bunching factor" at higher harmonics is severely limited by the electron beam energy spread. We will derive the formula for the induced harmonic content as a function of the harmonic number, the random energy spread, the induced modulation, and the strength of the chicane.

# Bunching Factor from HGHG, 1

We will use dimensionless variables to simplify this calculation.
To describe longitudinal position, we will use ξ, with:

$$\xi = \frac{2\pi s}{\lambda}$$

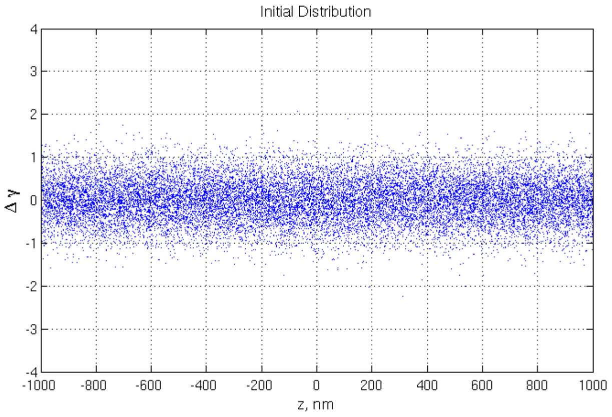
Here *s* is the longitudinal position in meters, and λ is the wavelength of the fundamental (i.e., the laser used to modulate the beam).

The energy of the electrons are described using *p*, with:
Here $\sigma_\gamma$ is the RMS energy spread in the electron beam,
Before the beam is modulated.

$$p = \frac{\gamma}{\sigma_\gamma}$$

We will assume that the initial electron distribution is Gaussian in energy, and is independent of longitudinal direction. In our dimensionless units, the initial distribution is:

$$f_0(p) = \frac{N_0}{\sqrt{2\pi}} e^{-p^2/2}$$



# Bunching Factor from HGHG, 2

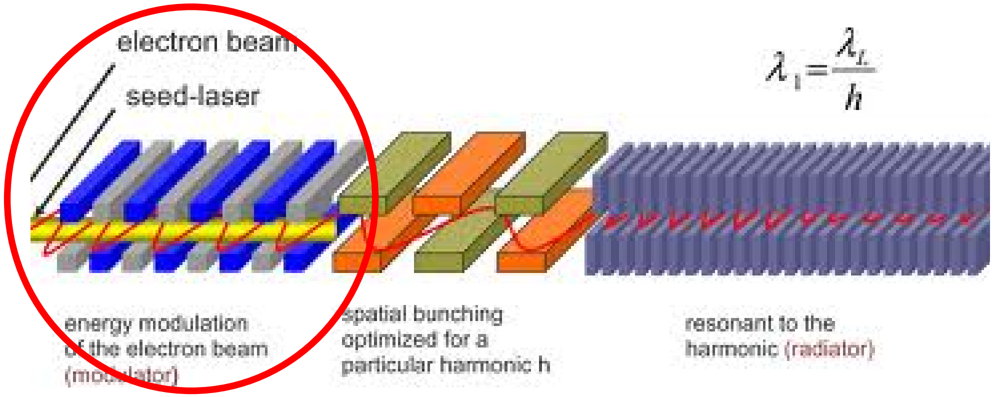
The electron beam passes through a wiggler and interacts with a laser. This gives the electrons a modulation in phase space:

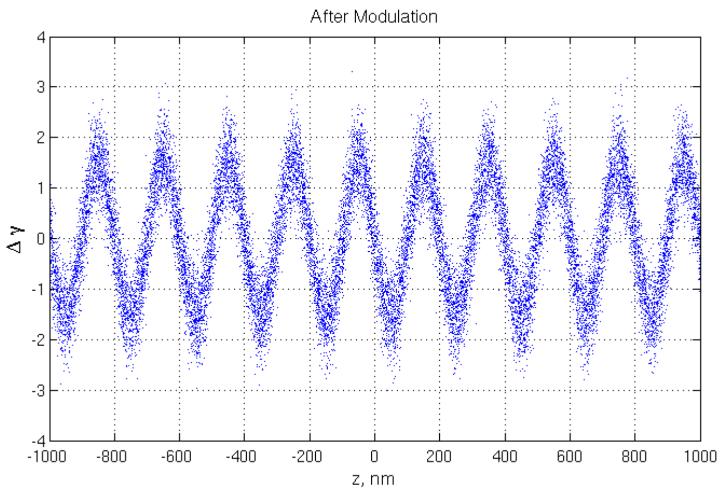
$$p' = p + A\sin(\xi)$$

The prime denotes the phase space after modulation, while the unprimed coordinates are the phase space before modulation. The dimensionless parameter *A* represents the strength of modulation, and is given by:

$$A = \frac{\Delta\gamma}{\sigma_\gamma}$$

The modulation strength $\Delta\gamma$ depends on the laser power, the laser transverse size, the wiggler length, and the wiggler strength. (More on this later).



# Bunching Factor from HGHG, 3

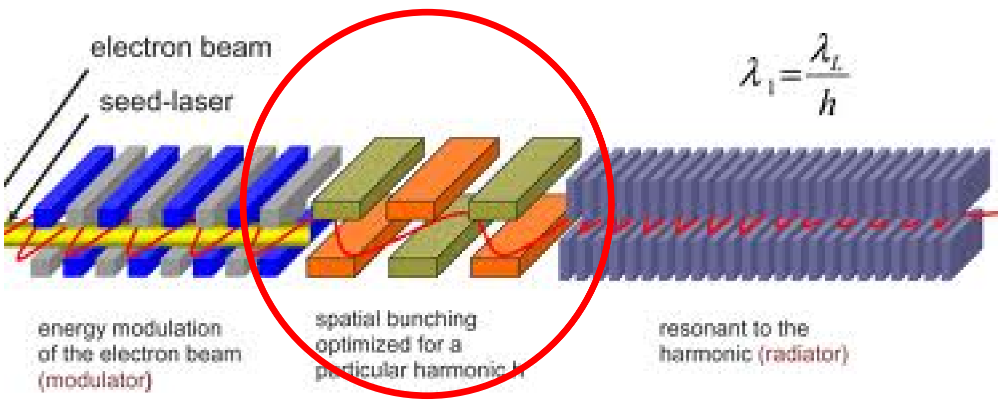
Next the electron beam passes through a chicane, which causes more energetic electrons to move forward with respect to less energetic electrons. The new phase space is:

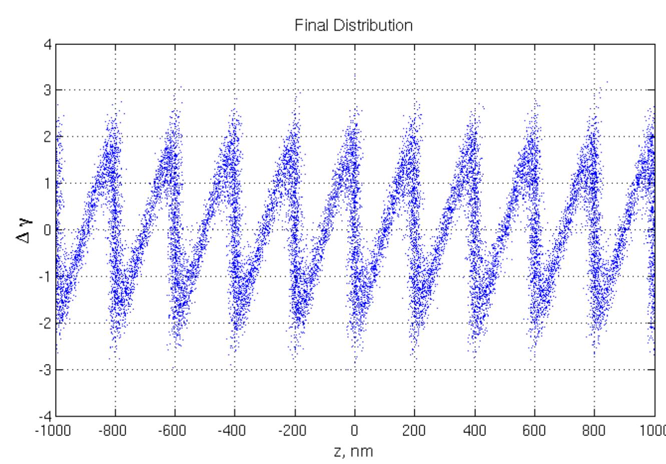
$$\xi' = \xi + B p'$$

$$\xi' = \xi + B\left(p + A\sin\xi\right)$$

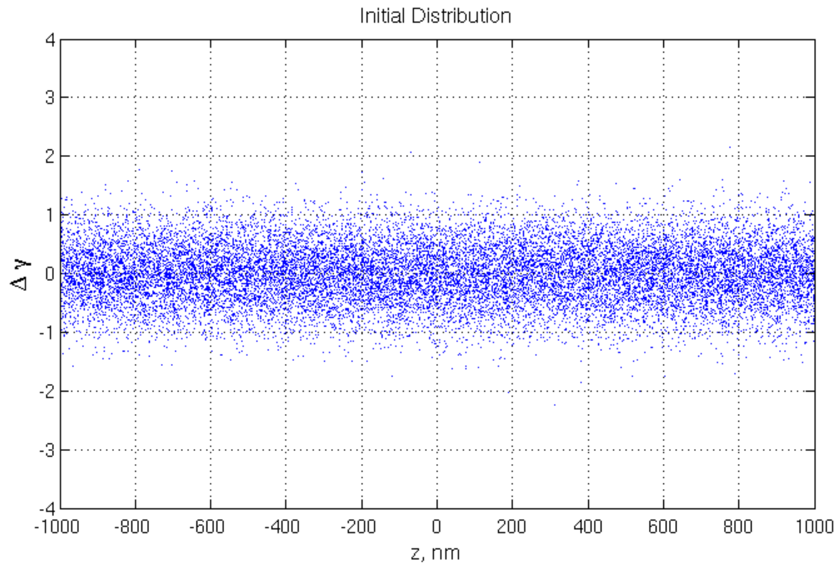
The dimensionless parameter *B* describes the strength of the chicane, and is given by:

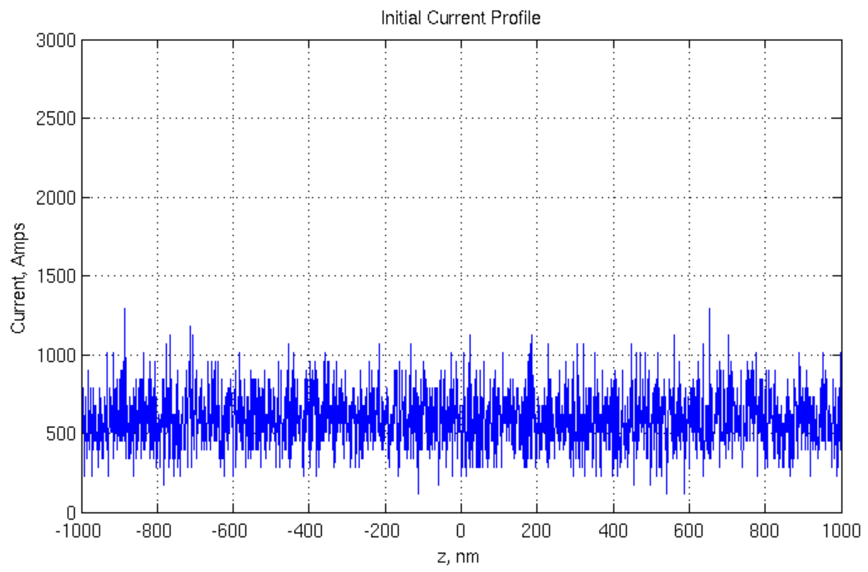
$$B = \frac{2\pi R_{56}\sigma_\gamma}{\lambda\gamma_0}$$

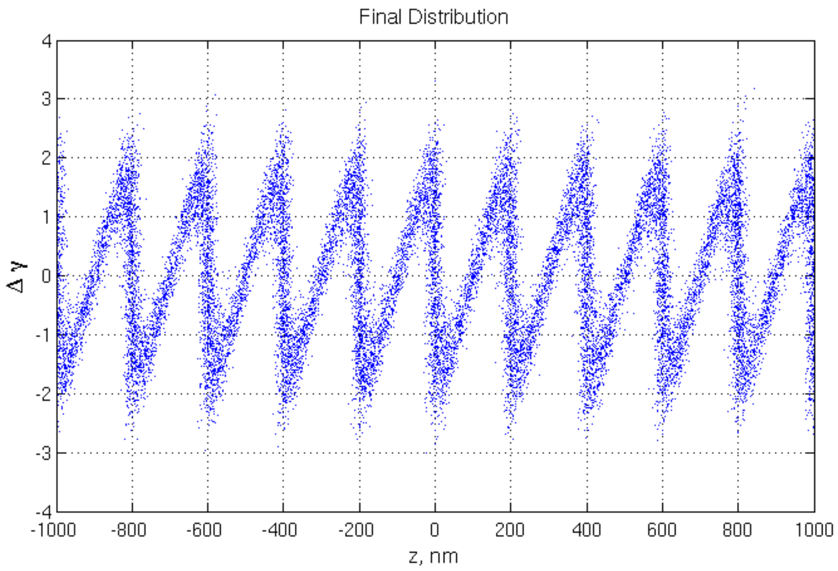


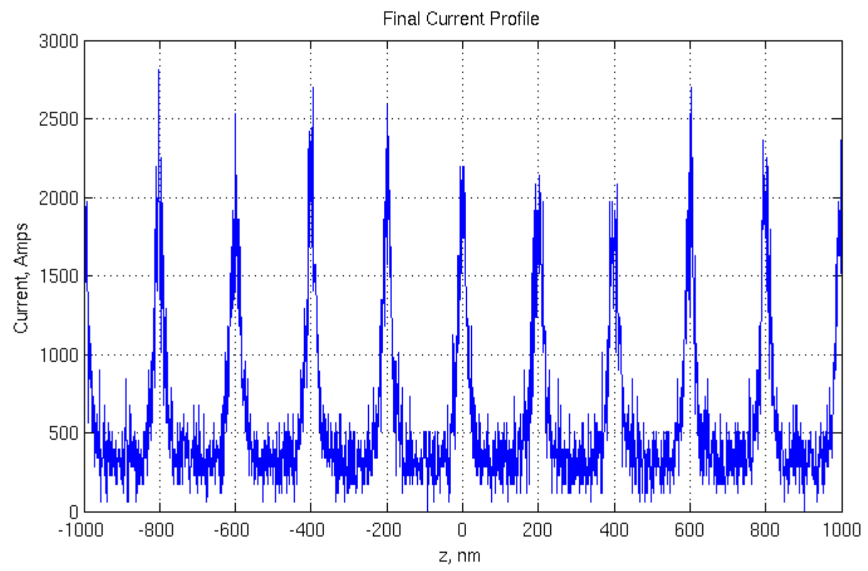



# Phase Space and Current Before and After HGHG

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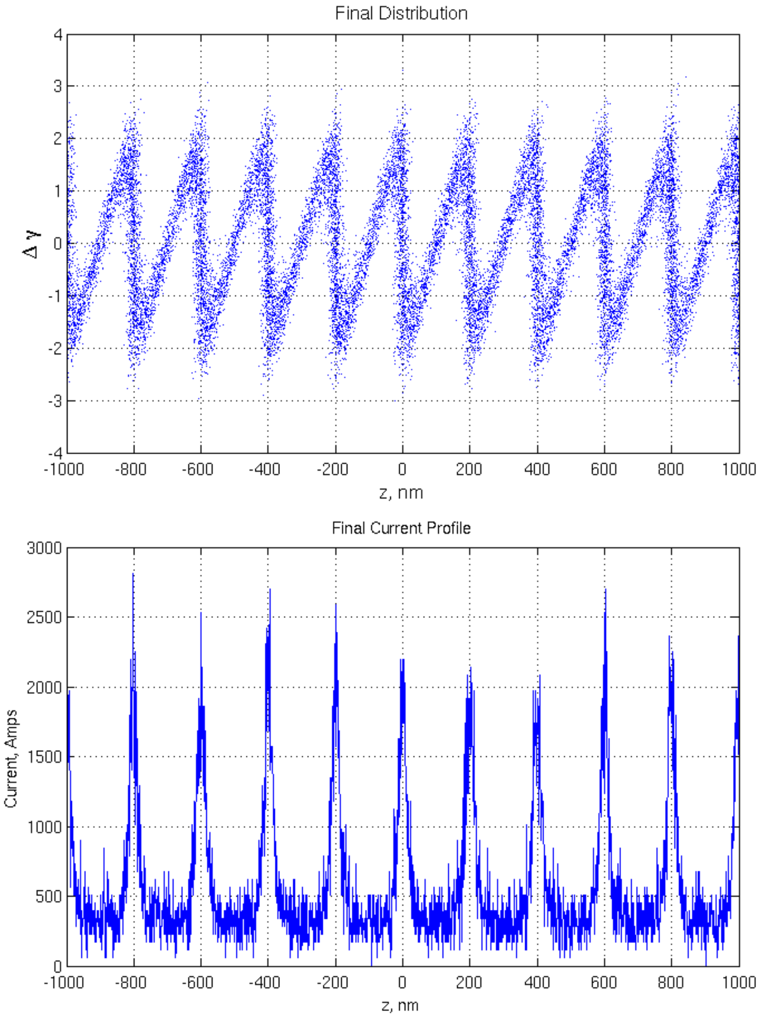
# Bunching Factor from HGHG, 4

We want to calculate the harmonic current that will be generated from the HGHG process. We can characterize the harmonic current by calculating the bunching factor at a certain harmonic n of the seeding laser:




$$b(n) = \frac{1}{I_0}\left|\left\langle e^{-in\xi'} I(\xi')\right\rangle\right|$$

The brackets ( < > ) denote averaging in ξ. We will switch from describing the electron current I to describing the electron line density N, with I = ecN. Then we have:

$$b(n) = \frac{1}{N_0}\left|\left\langle e^{-in\xi'} N(\xi')\right\rangle\right| \quad \text{with} \quad N(\xi') = \int_{-\infty}^{+\infty} f_f(\xi', p')\, dp'$$

# Bunching Factor from HGHG, 5

For simplicity, assume an infinitely long laser and electron beam (this will be a good approximation when $L_{beam} >> \lambda$). Then the averaging ( < > ) can be mathematically described as:

$$\langle ... \rangle = \lim_{L\to\infty} \frac{1}{L} \int_{-\infty}^{+\infty} ... d\xi$$

Then the calculation of bunching factor can be written:

$$b(n) = \lim_{L\to\infty} \frac{1}{N_0 L} \left| \int_{-L}^{L} \int_{-\infty}^{\infty} dp' d\xi' e^{-in\xi'} f_f(\xi', p') \right|$$

To simplify, use the fact that phase space is constant along particle trajectories. Mathematically, this means that:

$$f_f(\xi', p') = f_0(\xi, p) = \frac{N_0}{\sqrt{2\pi}} e^{-p^2/2}$$

Along trajectories, we also have $d\xi dp = d\xi' dp'$

# Bunching Factor from HGHG, 6

We can use this fact that phase space is conserved along particle trajectories to calculate the bunching factor in terms of the *initial* phase space distribution:

$$b(n) = \lim_{L\to\infty} \frac{1}{2N_0 L} \left| \int_{-L}^{L} \int_{-\infty}^{\infty} dp\, d\xi\, e^{-in\xi'(\xi,p)} f_0(p) \right|$$

Switch order of integration:

$$b(n) = \lim_{L\to\infty} \frac{1}{N_0} \left| \int_{-\infty}^{\infty} dp\, f_0(p) \left[ \frac{1}{2L} \int_{-L}^{L} d\xi e^{-in\xi'(\xi,p)} \right] \right|$$

Plugging in the value for ξ'(ξ,p), the integral over ξ becomes:

$$\left\langle \exp\left[-in\xi'(\xi,p)\right] \right\rangle = \lim_{L\to\infty} \frac{1}{L} \int_{-L}^{L} d\xi \exp\left\{-in\left[\xi + B\left(p + A\sin\xi\right)\right]\right\}$$

# Bessel Generating Function

Break up exponential into 2 parts:

$$e^{-in\left[\xi+B\left(p+A\sin\xi\right)\right]} = e^{-in\left(\xi+Bp\right)}\underbrace{e^{-inAB\sin\xi}}$$

To simplify the second exponential, we make use of the Bessel generating function (for more on this, see Arfken, p. 628):

$$e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)e^{i\theta n}$$

We will leave it to HW to modify this to be in a usable form.

The second exponential simplifies to:

$$e^{-inAB\sin\xi} = \sum_{-\infty}^{\infty} \mathrm{J}_m\left(-nAB\right)e^{i\xi m}$$

Then we get:

$$e^{-in\left[\xi+B\left(p+A\sin\xi\right)\right]} = e^{-in\left(\xi+Bp\right)}\sum_{-\infty}^{\infty} \mathrm{J}_m\left(-nAB\right)e^{i\xi m}$$

$$= \sum_{-\infty}^{\infty} \mathrm{J}_m\left(-nAB\right)e^{-inBp}\,e^{i\xi(m-n)}$$

# Formula for HGHG

Going back to the formula for bunching factor:

$$b(n) = \lim_{L\to\infty} \frac{1}{N_0} \left| \int_{-\infty}^{\infty} dp\, f_0(p) \left[ \frac{1}{2L} \int_{-L}^{L} d\xi\, e^{-in\xi'(\xi,p)} \right] \right|$$

$$= \lim_{L\to\infty} \frac{1}{N_0} \left| \int_{-\infty}^{\infty} dp\, f_0(p) \left[ \sum_{-\infty}^{\infty} \mathrm{J}_m(-nAB)\, e^{-inBp} \underbrace{\frac{1}{2L} \int_{-L}^{L} d\xi\, e^{i\xi(m-n)}} \right] \right|$$

This last integral will be zero (in the limit L → infinity) if m≠n, because the 1/L will dominate over the oscillating term in the integral. On the other hand, if m=n, the exponential becomes = 1, and the integral becomes 1/2L * (2L) = 1. Then we have:

$$b(n) = \frac{1}{N_0} \left| \int_{-\infty}^{\infty} dp\, f_0(p)\, \mathrm{J}_m(-nAB)\, e^{-inBp} \right| \qquad \text{if m=n}$$

$$= 0 \qquad \text{otherwise}$$

# Formula for HGHG, 2

Insert value for $f_0(p)$:

$$b(n)=\frac{1}{\sqrt{2\pi}}\left|J_n(-nAB)\int_{-\infty}^{\infty} dp\, e^{-\left(p^2/2+inBp\right)}\right|$$

You can look up this integral in an integral table:

$$\int_{-\infty}^{\infty} dp\, e^{-\left(p^2/2+inBp\right)}=\sqrt{2\pi}\, e^{-n^2B^2/2}$$

Plug in values for A and B:

$$A=\frac{\Delta\gamma}{\sigma_\gamma} \qquad B=\frac{2\pi R_{56}\sigma_\gamma}{\lambda\gamma_0}$$

Then you get Yu's formula for HGHG bunching:

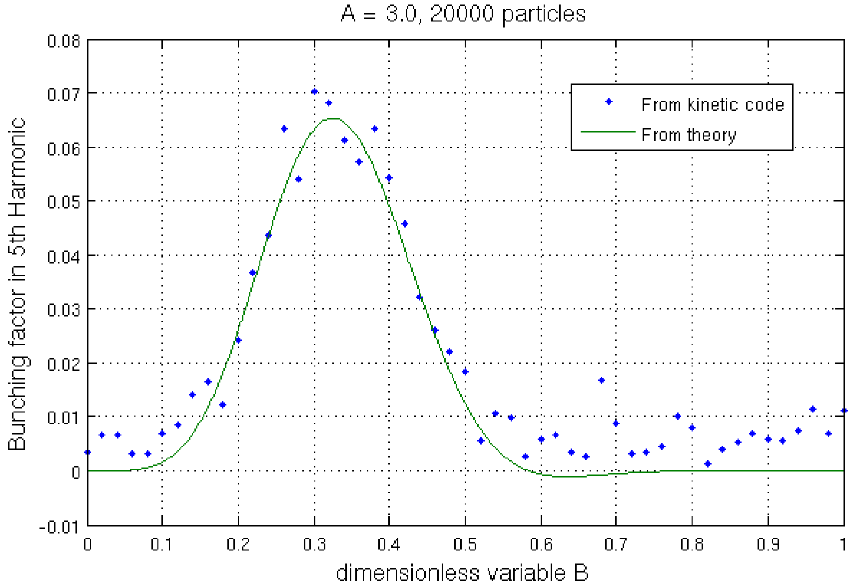
$$b(n)=J_n(nAB)\exp\left[-\frac{1}{2}n^2B^2\right]$$

Or:

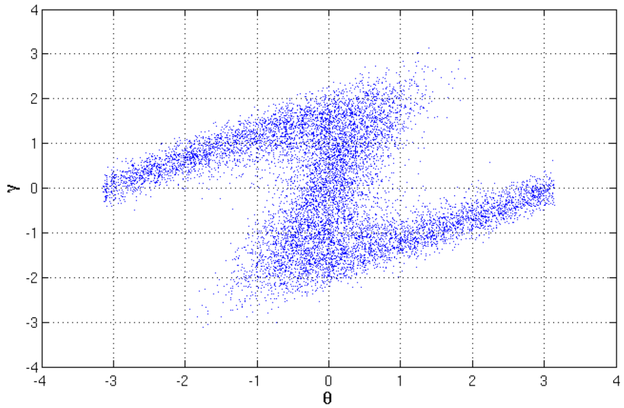
$$b(n)=J_n\left(n\Delta\gamma\frac{d\theta}{d\gamma}\right)\exp\left[-\frac{1}{2}n^2\sigma_E^2\left(\frac{d\theta}{d\gamma}\right)^2\right]$$

# Formula for HGHG, 3

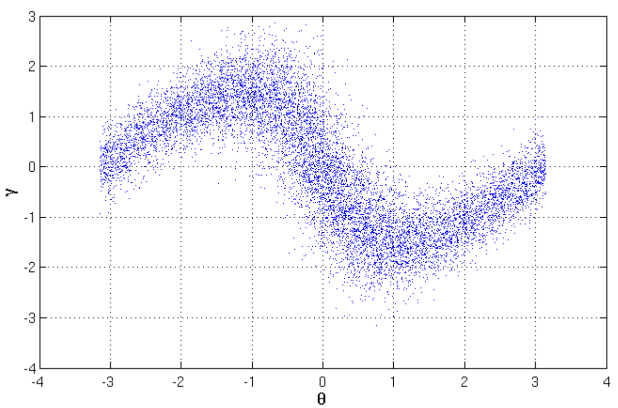
$$b(n) = \mathrm{J}_n\left(n\Delta\gamma \frac{d\theta}{d\gamma}\right)\exp\left[-\frac{1}{2}n^2\sigma_E^2\left(\frac{d\theta}{d\gamma}\right)^2\right]$$



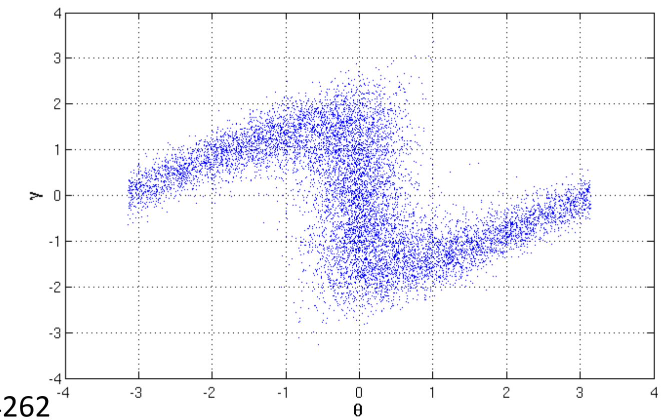

A=3, B=0.5




A=3, B=0.15




A=3, B=0.325



# Limits of HGHG

Bessel functions reach a peak at x > n+1, so in order to optimize the Bessel function, we have:

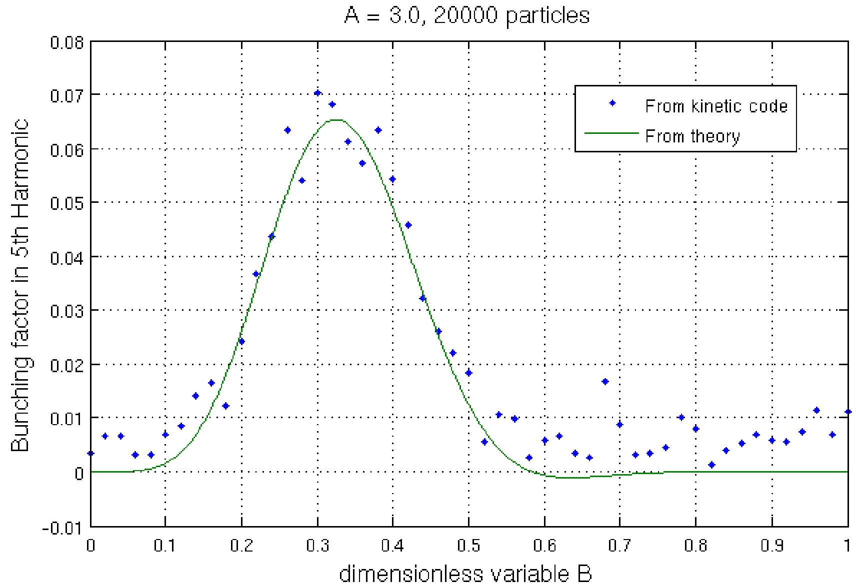
$$n\Delta\gamma \frac{d\theta}{d\gamma} > n$$

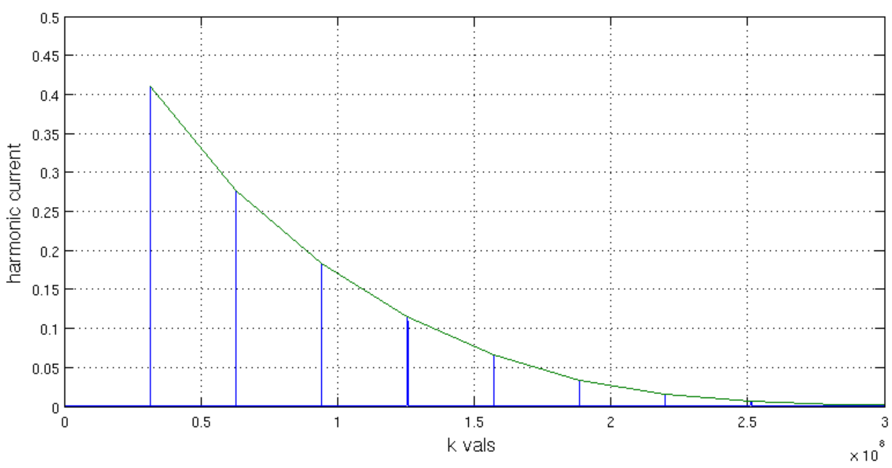
or:

$$\frac{d\theta}{d\gamma} > \frac{1}{\Delta\gamma}$$

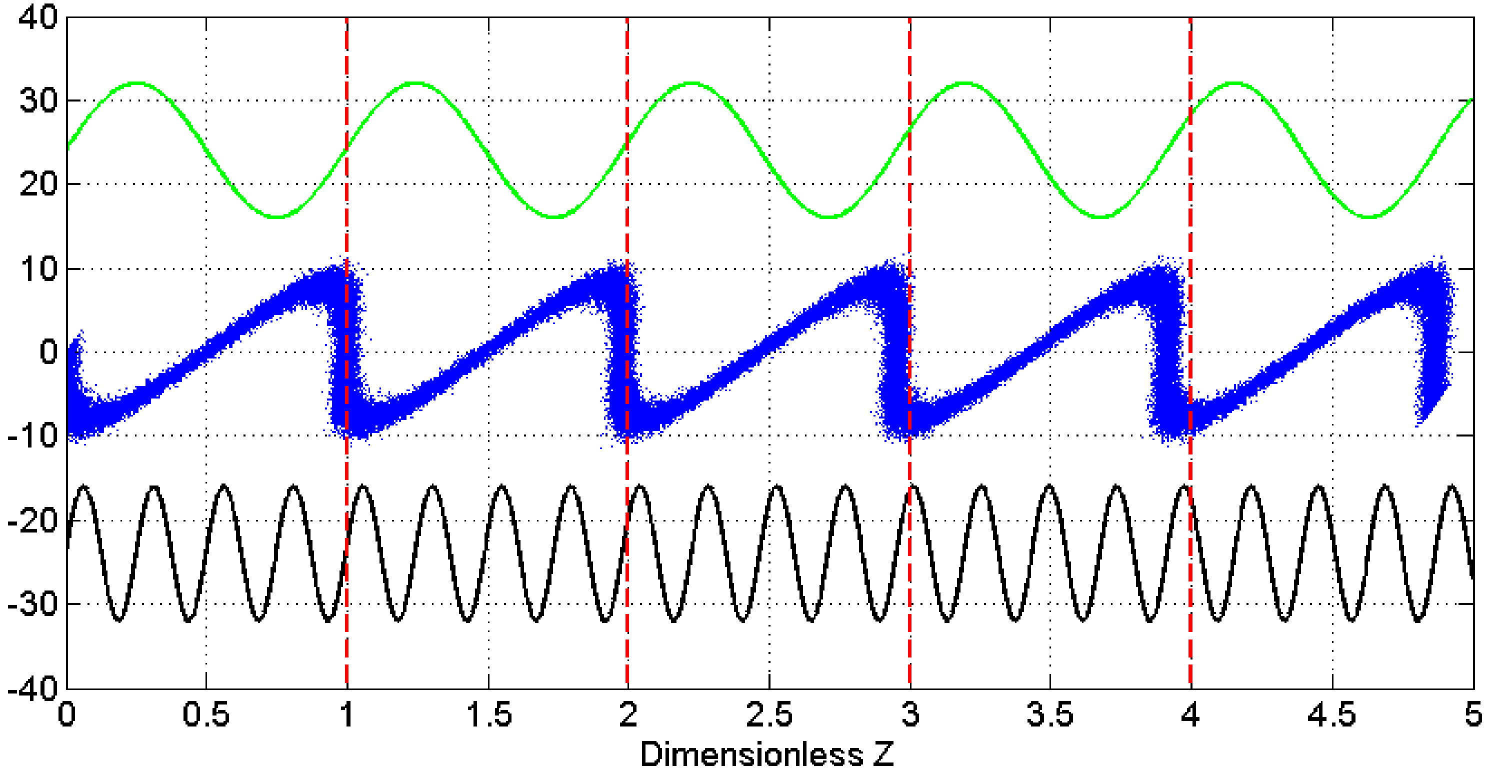
This gives an upper bound on the bunching vs. harmonic number:

$$b(n) < \exp\left[-\frac{1}{2}n^2\left(\frac{\sigma_E}{\Delta\gamma}\right)^2\right]$$



# Limits of HGHG – Effect of laser chirp




A small change in the phase at a low harmonic can transfer to a large change in the phase at a high harmonic.

# Effect of laser chirp – mathematical description

A Gaussian laser pulse with a frequency chirp can be described as:

$$E_{in}(\xi) = E_0 \exp\left[ i\left( \xi + \frac{\alpha\xi^2}{2\sigma_L^2} \right) - \frac{\xi^2}{2\sigma_L^2} \right]$$

Take Fourier transform:

$$\tilde{e}(\omega) = \frac{E_0}{2\pi} \int_{-\infty}^{\infty} d\xi \exp\left[ i\left( \frac{\alpha\xi^2}{2\sigma_L^2} + \omega\xi \right) - \frac{\xi^2}{2\sigma_L^2} \right]$$

$$= \frac{E_0\sigma_L}{\sqrt{2\pi}} \frac{1}{\left(1+\alpha^2\right)^{1/4}} \exp\left[ \frac{i}{2}\left( \tan^{-1}\alpha - \frac{\alpha\omega^2\tau_p^2}{1+\alpha^2} \right) \right] \underbrace{\exp\left[ -\frac{\omega^2\sigma_L^2}{2\left(1+\alpha^2\right)} \right]}$$

The Fourier transform is a Gaussian with $\sigma_\omega = \sqrt{\frac{1+\alpha^2}{\sigma_L^2}}$

If there is no chirp you get Fourier transform limit, $\sigma_\omega \sigma_L = 1$

If there is a chirp, you get: $\sigma_\omega \sigma_L = \sqrt{1+\alpha^2}$

# Effect of laser chirp – mathematical description 2

After harmonic generation, you get:

$$E_{out}(\xi) = \tilde{E}_0 \exp\left[ i\left( \xi + \frac{N\alpha\xi^2}{2\sigma_L^2} \right) - \frac{\xi^2}{2\sigma_L^2} \right]$$

Fourier transforming this gives a Gaussian with:

$$\sigma_\omega = \sqrt{\frac{1+N^2\alpha^2}{\sigma_L^2}}$$

or:

$$\sigma_\omega \sigma_L = \sqrt{1+N^2\alpha^2}$$

What is α for lasers used in HGHG seeding? Usually, a Ti-Sa laser is used, because it can produce the high powers needed at short wavelength (800 nm). Usually, laser quality is defined as:

$$M_0^2 = \sigma_\omega \sigma_L$$

For a TiSa laser,

$$M_0^2 - 1 \approx 0.01$$

# Example of HGHG FEL

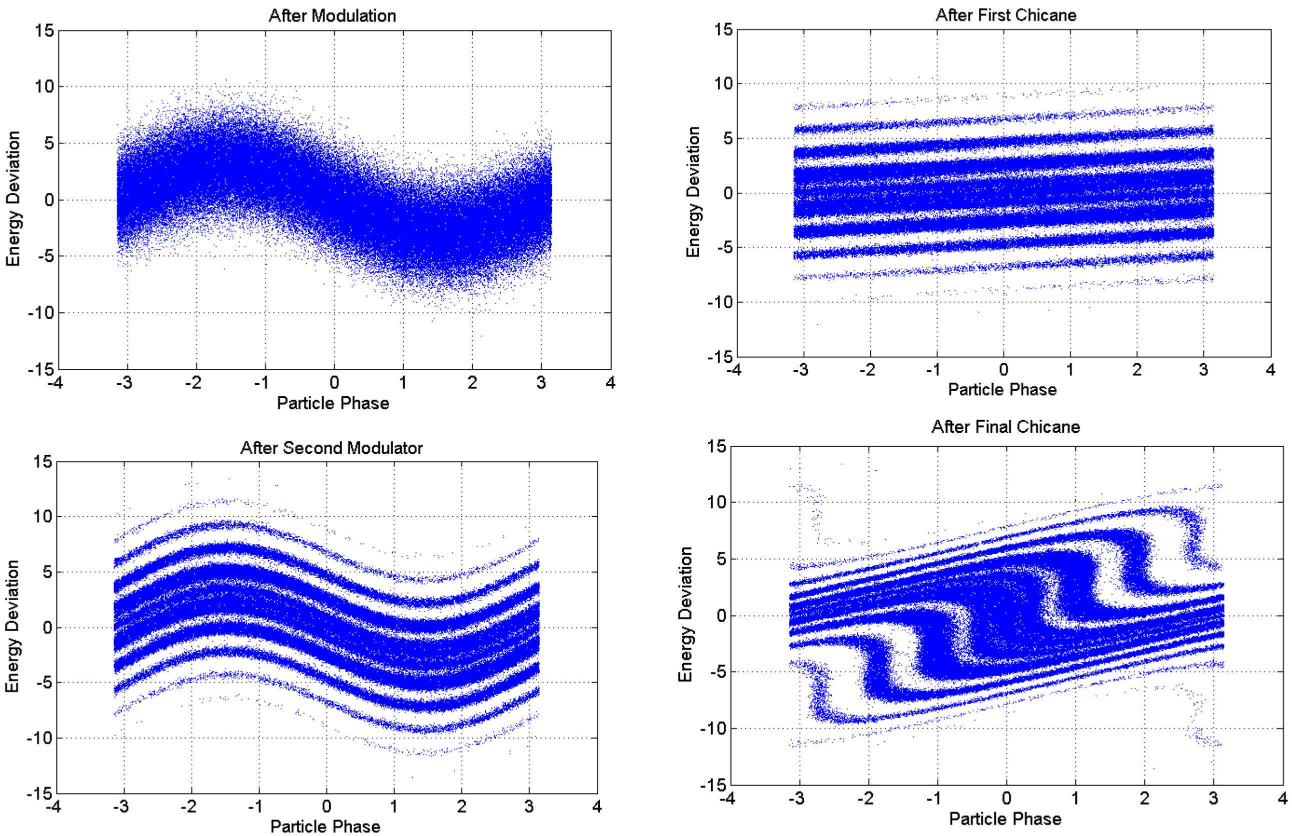
Fermi@Elettra is an FEL user facility in Trieste, Italy that uses the HGHG mechanism to produce coherent FEL pulses at wavelengths down to 43 nm. The electron beam is seeded with a 258 nm laser, and then a single HGHG stage produces radiation at the $6^{th}$ harmonic.  A delay line is used so that a "fresh" part of the electron bunch is seeded with the 43 nm radiation (more on this later).

For more information, see: www.elettra.trieste.it/FERMI/
Pictures of hardware taken from this page.

# Thursday lecture part 2: EEHG

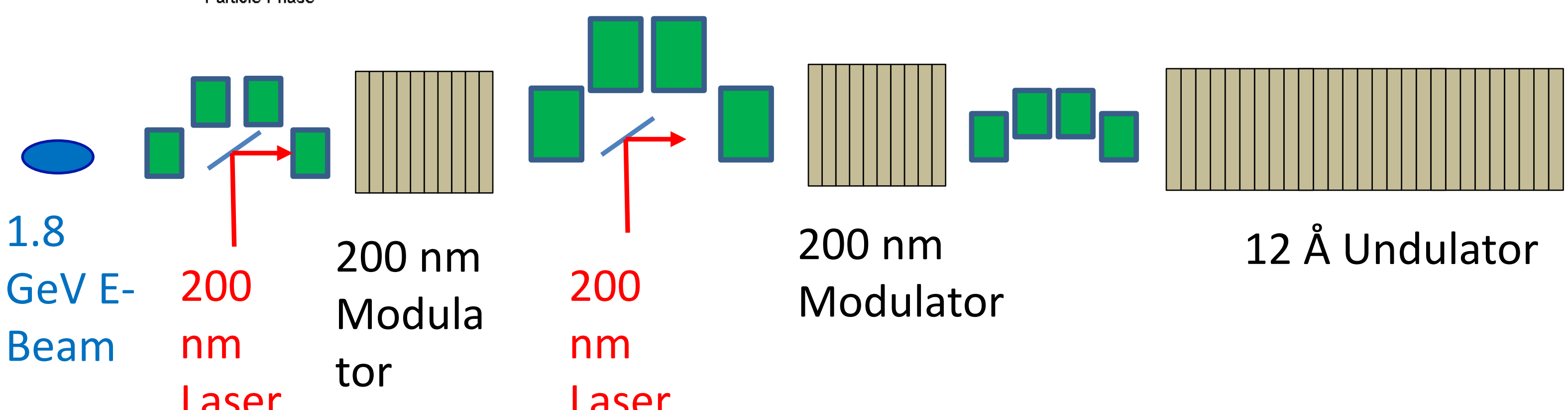
LA-UR-14-24262

# Introduction to EEHG




- Echo-Enabled Harmonic Generation[1] uses 2 modulators and 2 chicanes. The 1st chicane is large, and breaks the modulated beam into energy bands.
- Theory shows that EEHG has a very favorable scaling with harmonic number[1]:

$$b_n \approx \frac{0.39}{m^{1/3}}$$

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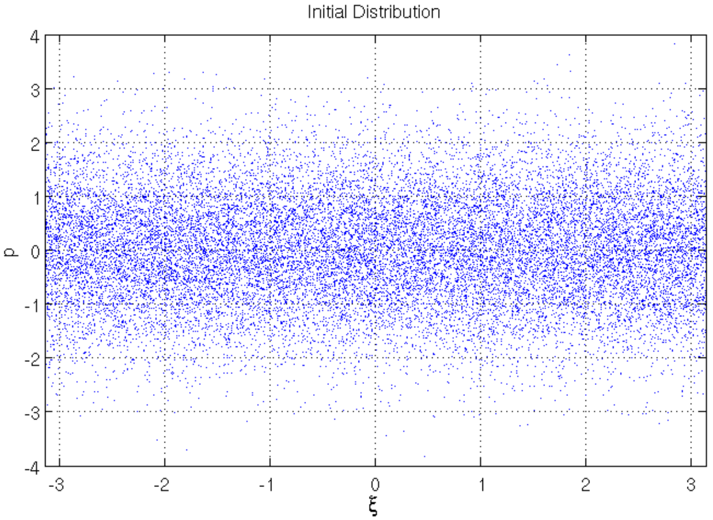
# Dimensionless variables for EEHG:

We will use the same dimensionless variables to do this calculation as the ones we used to calculate bunching factors from HGHG:

$$\xi = \frac{2\pi s}{\lambda} \quad \text{and} \quad p = \frac{\gamma}{\sigma_\gamma}$$

The initial phase space distribution will once again be a Gaussian, which, in our dimensionless variables is given by:

$$f_0(p) = \frac{N_0}{\sqrt{2\pi}} e^{-p^2/2}$$




We will also use the dimensionless variables A and B:

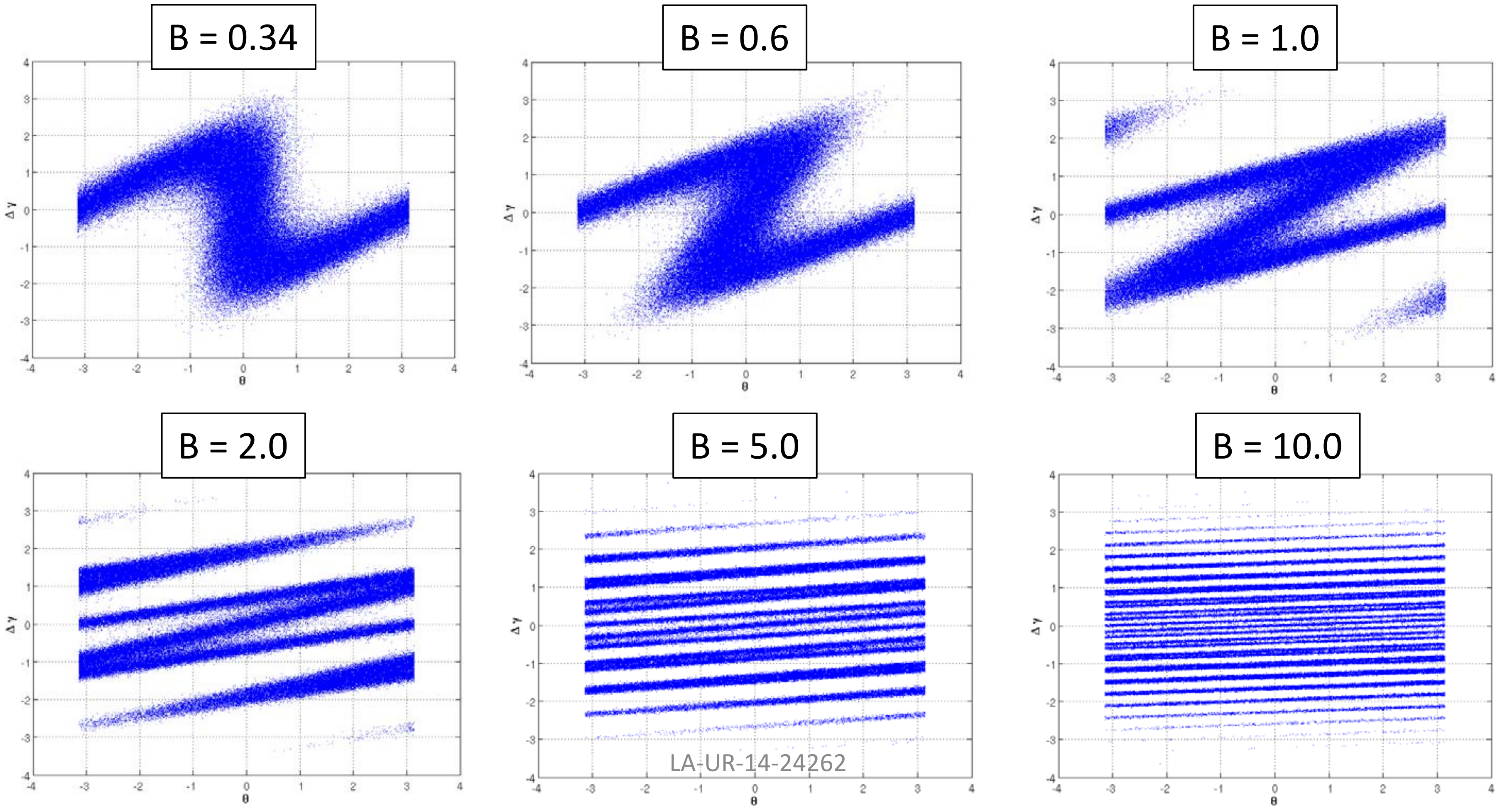
$$A = \frac{\Delta\gamma}{\sigma_\gamma} \qquad B = \frac{2\pi R_{56}\sigma_\gamma}{\lambda\gamma_0}$$

A for the strength of the modulation, and B for the strength of the chicane.

# Increasing B to get energy ladder:

The trick of EEHG is to use the first modulation and chicane combination to create an “energy ladder” in phase space, and then apply HGHG to the rungs of the energy ladder. Because each rung has a very small random energy spread, the limitations of HGHG are overcome.

Here is a picture showing how B can be increased to create this energy ladder:

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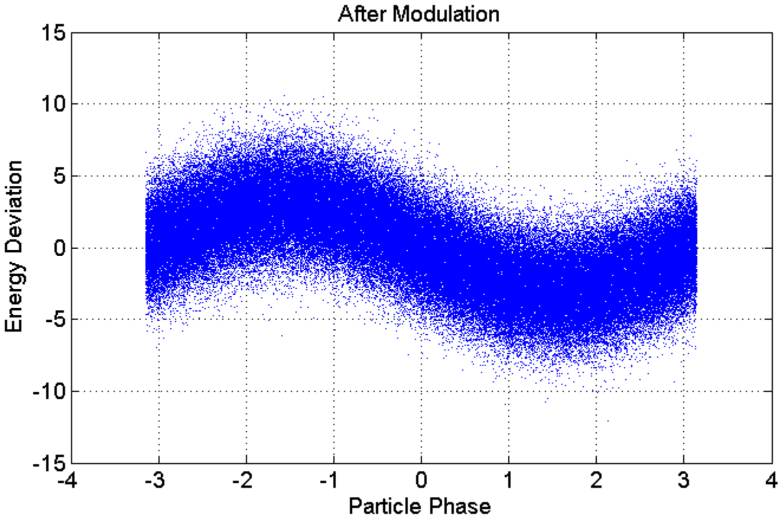
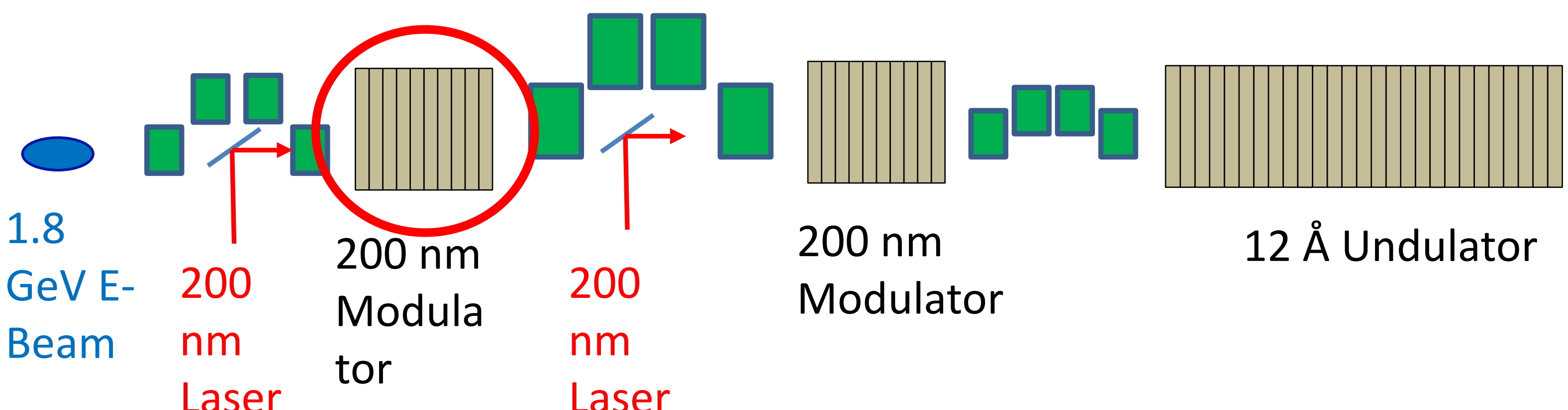
# Calculation of bunching in EEHG, 1




The first modulator imparts a sinusoidal oscillation on the particles, identical to what is done in HGHG. Typical values of the modulation are $A_1$~3.

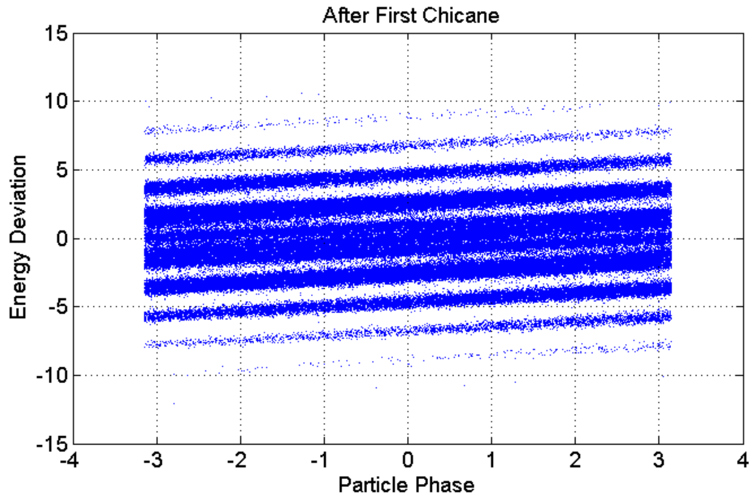
The first modulator modifies phase space according to:

$$p' = p + A_1 \sin(\xi)$$



# Calculation of bunching in EEHG, 2

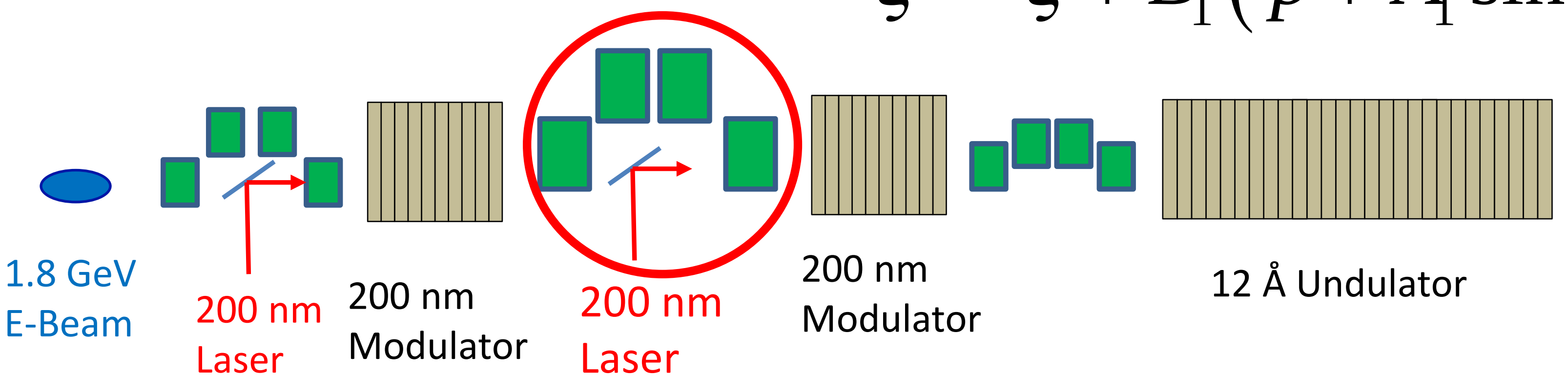
The first modulator-chicane combination produces the energy stripes. The larger the chicane, the thinner each energy stripe will be. This is ultimately how EEHG overcomes the exponential limit of HGHG.




The first chicane modifies phase space according to:

$$\xi' = \xi + B_1 p'$$

$$\xi' = \xi + B_1 \left( p + A_1 \sin \xi \right)$$




Note: the chicane is also used to displace the electron beam so that the 2nd laser can overlap with the electrons.

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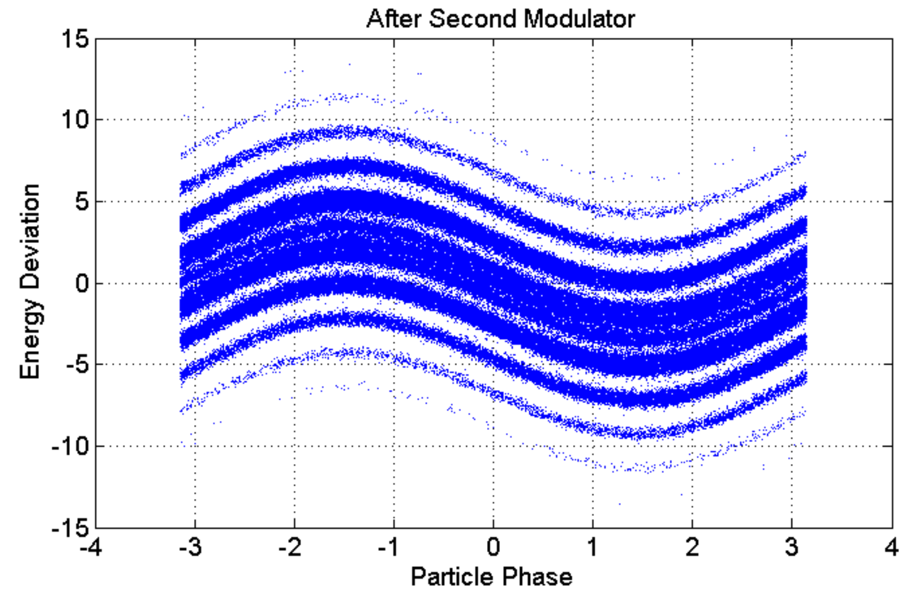
# Calculation of bunching in EEHG, 3

The second modulator gives each of the energy stripes a sinusoidal modulation.

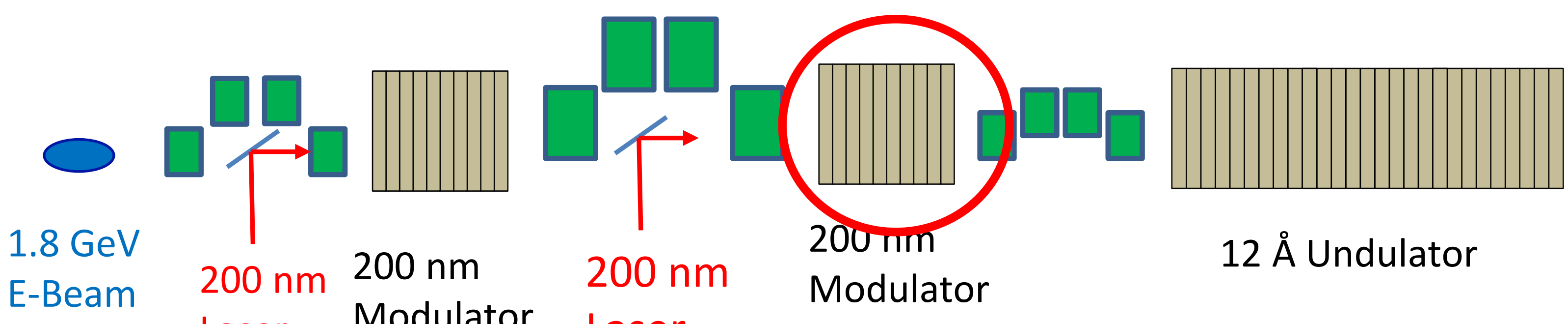



The second modulator modifies phase space according to:

$$p'' = p' + A_2 \sin\left(\kappa \xi'\right)$$

or:

$$p'' = p + A_1 \sin \xi + A_2 \sin\left\{\kappa\left[\xi + B_1\left(p + A_1 \sin \xi\right)\right]\right\}$$

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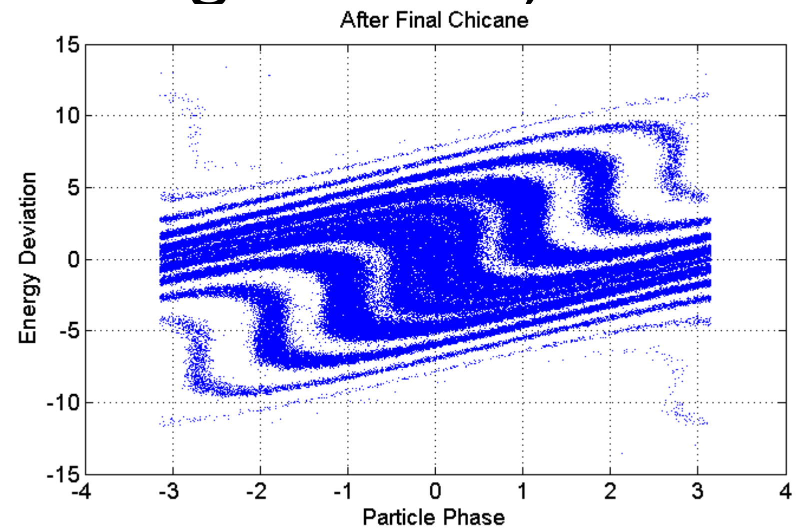
# Calculation of bunching in EEHG, 4

The final chicane rotates each of the stretched out energy bands in phase space, and produces bunching at a very high harmonic.

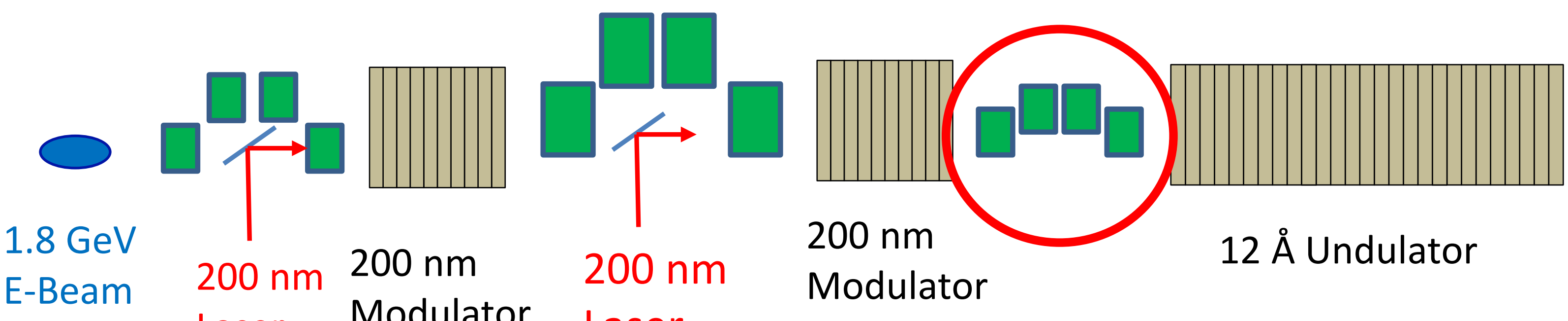



The second chicane modifies phase space according to:

$$\xi'' = \xi + B_1 \left( p + A_1 \sin \xi \right)$$

After doing some algebra, you get:

$$\xi'' = \xi + \left( B_1 + B_2 \right) p + A_1 \left( B_1 + B_2 \right) \sin \xi + A_2 B_2 \sin \left( \kappa \xi + \kappa B_1 p + \kappa A_1 B_1 \sin \xi + \phi \right)$$

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# Calculation of bunching in EEHG, 5

Use same trick we used before to calculate bunching for HGHG. The definition of bunching factor is:

$$b(n) = \lim_{L \to \infty} \frac{1}{N_0 L} \left| \int_{-L}^{L} \int_{-\infty}^{\infty} dp'' d\xi'' e^{-in\xi''} f_f(\xi'', p'') \right|$$

Use the fact that phase space is conserved, and do some algebra to get a formula for bunching factor that can be calculated:

$$b(n) = \lim_{L \to \infty} \frac{1}{N_0} \left| \int_{-\infty}^{\infty} dp\, f_0(p) \left[ \frac{1}{2L} \int_{-L}^{L} d\xi e^{-in\xi''(\xi, p)} \right] \right|$$

We will leave it as a HW assignment to derive the formula for the bunching factor in EEHG:

$$b_{n,m} = \left| e^{-(1/2)\left[nB_1 + (\kappa m + n) B_2\right]^2} \mathrm{J}_m\left[-(\kappa m + n) A_2 B_2\right] \mathrm{J}_n\left\{-A_1\left[nB_1 + (\kappa m + n) B_2\right]\right\} \right|$$

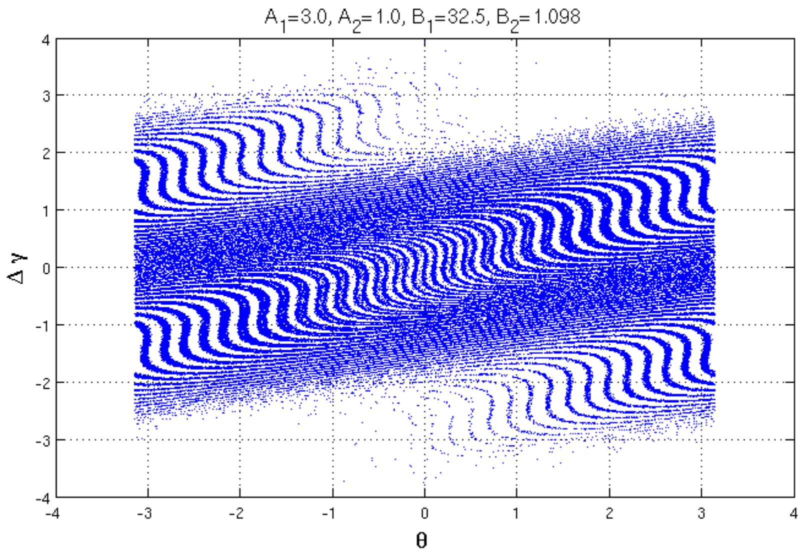
# Look at results of bunching factor for EEHG

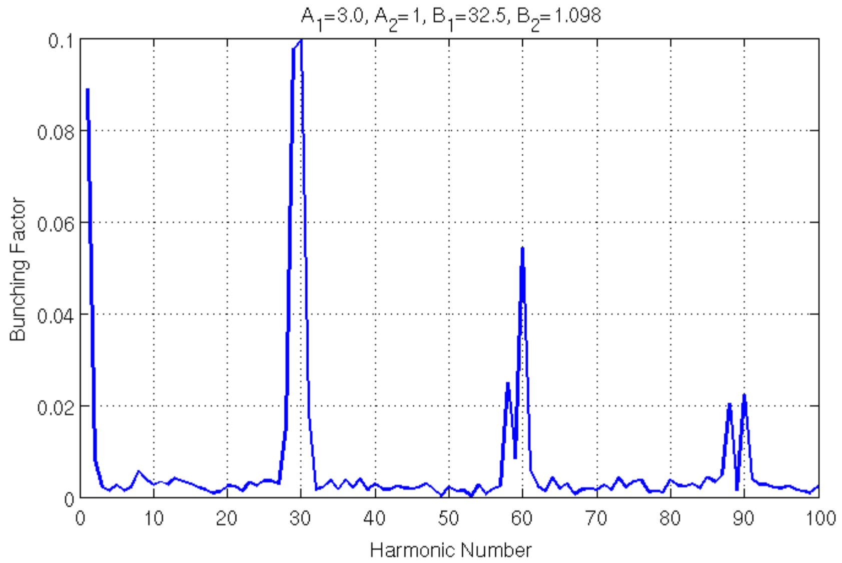
Write bunching formula again:

$$b_{n,m} = \left| e^{-(1/2)\left[nB_1 + (\kappa m + n) B_2\right]^2} \mathrm{J}_m \left[ -(\kappa m + n) A_2 B_2 \right] \mathrm{J}_n \left\{ -A_1 \left[ nB_1 + (\kappa m + n) B_2 \right] \right\} \right|$$

There are a lot of options. Let's look at n=-1, m>0:

$$b_{-1,m} = \left| \mathrm{J}_m \left[ (\kappa m - 1) A_2 B_2 \right] \mathrm{J}_1 \left\{ A_1 \left[ B_1 - (\kappa m - 1) B_2 \right] \right\} e^{-(1/2)\left[B_1 - (\kappa m - 1) B_2\right]^2} \right|$$




This plot shows the final phase space and the bunching factor at different harmonics when A1=3.0, A2=1.0, B1=32.5, and B2=1.098

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# Getting approximate values for EEHG

$$b_{-1,m} = \left| \underbrace{J_m\left[\left(\kappa m - 1\right)A_2 B_2\right]}_{\text{Term \#1}} J_1\left\{A_1 \underbrace{\left[B_1 - \left(\kappa m - 1\right)B_2\right]}_{\text{Term \#2}}\right\} e^{-(1/2)\overbrace{\left[B_1 - (\kappa m - 1)B_2\right]}^{\text{Term \#2}}{}^2} \right|$$

The Bessel function $J_m$ is maximum when x = m + 0.81m$^{1/3}$. So to maximize this term (term #1) we need to set:

$$\left(\kappa m - 1\right)A_2 B_2 = m + 0.81 m^{1/3}$$

To get some values of $A_1$, $A_2$, $B_1$, $B_2$ that are close to optimal, let k = 1, m >> 1, then we have:

$$m A_2 B_2 \approx m \quad \text{or} \quad A_2 \approx \frac{1}{B_2}$$

Term #2, which is in the J1 function and the exponential function, needs to be small, but slightly greater than zero. For this, we set:

$$B_1 - \left(\kappa m - 1\right)B_2 \approx {}^{1}\!/_{A_1}$$

Or, if k=1:

$$B_1 - m B_2 \approx {}^{1}\!/_{A_1}$$

# Other properties of EEHG

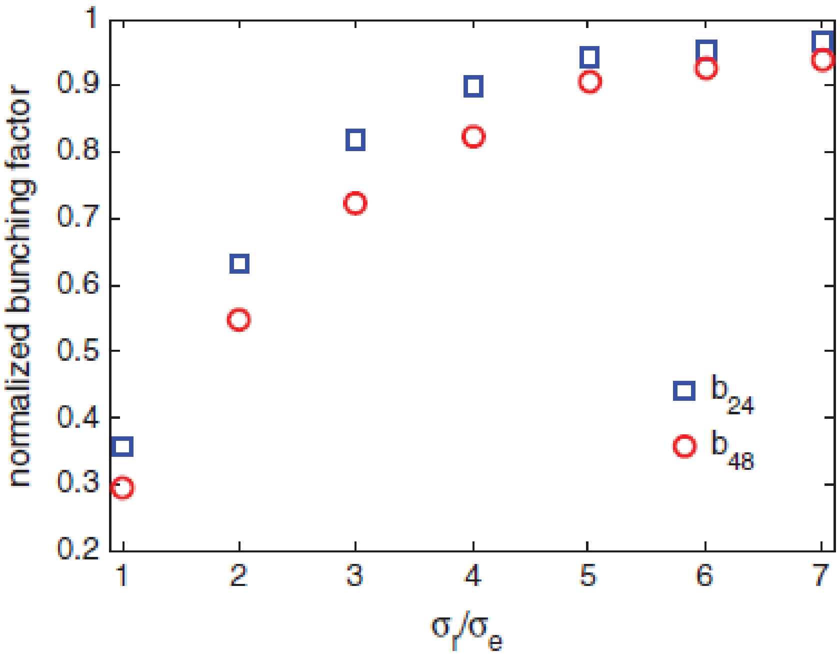


FIG. 8. (Color) Normalized bunching factors for $b_{24}$ (blue symbols) and $b_{48}$ (red circles) as functions of the ratio $\sigma_r/\sigma_e$.

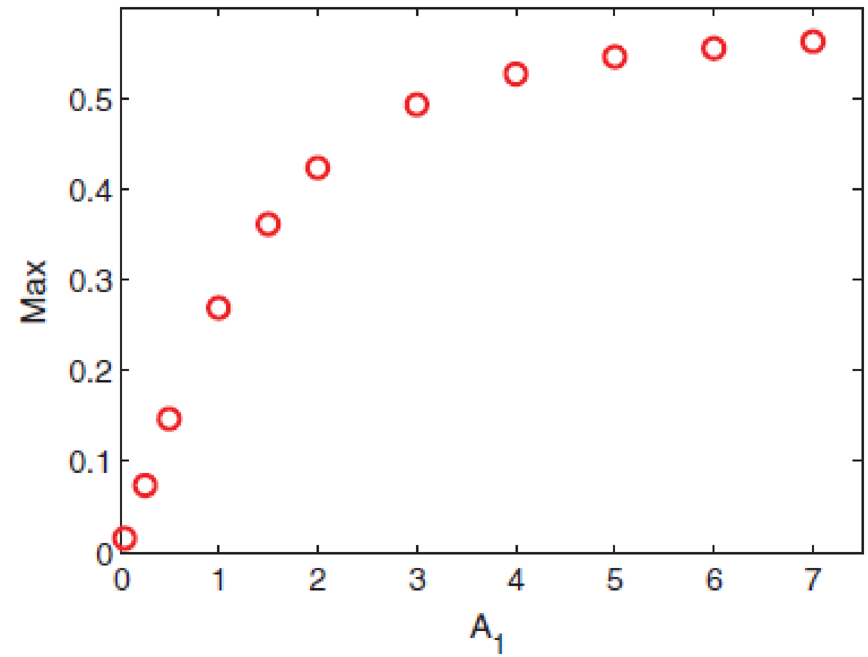


FIG. 2. (Color) Maximal value of $J_1(A_1\xi)e^{-\xi^2/2}$ as a function of parameter $A_1$.

These plots are from a paper by Xiang (Phys. Rev. STAB 12, 030702 (2009). The plot on the left shows how the bunching in EEHG is degraded if the laser transverse size is too small. The plot on the right shows how increasing the size of the initial modulation ($A_1$) increases the final bunching factor.

# Example of EEHG FEL

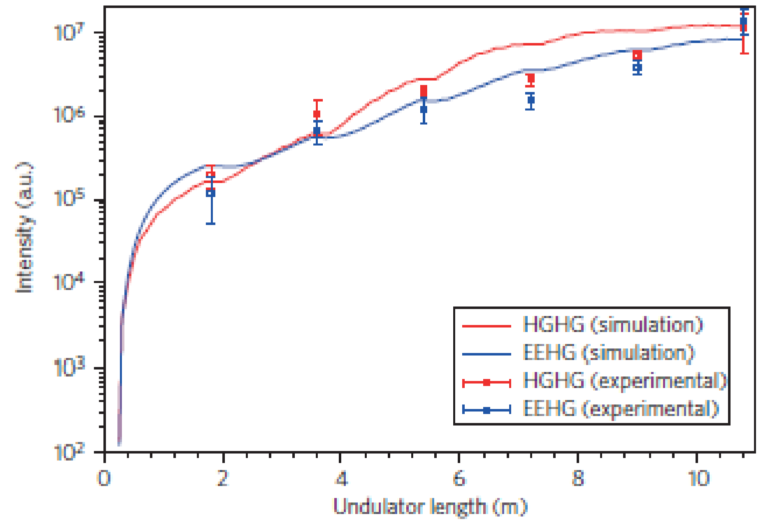


**Figure 5 | Gain curves of the EEHG and HGHG FEL at SDUV-FEL** Intensity is measured with a calibrated CCD at the end of the radiator (red open squares, HGHG; blue open squares, EEHG). Error bars correspond to the peak-to-peak intensity statistics of 100 measurements. Simulation results are shown as a red line (HGHG) and a blue line (EEHG).

Shanghai FEL has experiment that has produced an FEL seeded with EEHG and HGHG. There is also an experiment on EEHG at SLAC.