LA-UR-14-23792

High Intensity RF Linear Accelerators

2.5. Acceleration of Intense Beams in RF Linacs

Yuri Batygin

Los Alamos National Laboratory

U.S. Particle Accelerator School

Albuquerque, New Mexico, 23 – 27 June 2014

Content

Electromagnetic Wave Equations

To obtain the electromagnetic wave equation in a vacuum using the modern method, we begin with the modern 'Heaviside' form of Maxwell's equations. In a vacuum and charge free space, these equations are:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0\\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}\\ \nabla \cdot \mathbf{B} &= 0\\ \nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

Taking the curl of the curl equations gives:

$$\begin{aligned}\nabla \times \nabla \times \mathbf{E} &= -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}\\ \nabla \times \nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \times \mathbf{E} = -\mu_o \varepsilon_o \frac{\partial^2 \mathbf{B}}{\partial t^2}\end{aligned}$$

By using the vector identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$$

where $\mathbf{V}$ is any vector function of space, it turns into the wave equations:

$$\begin{aligned}\frac{\partial^2 \mathbf{E}}{\partial t^2} - {c_0}^2 \cdot \nabla^2 \mathbf{E} &= 0\\ \frac{\partial^2 \mathbf{B}}{\partial t^2} - {c_0}^2 \cdot \nabla^2 \mathbf{B} &= 0\end{aligned}$$

where $c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \times 10^8$ m/s is the speed of light in free space.

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

The solution is usually given in the form of a product of functions of one variable:

$$E_z\,(z,r,t) = Z(z)\,R(r)\,T(t)$$

Knowing E_z, one can compute the other electromagnetic field components: E_r with the divergence theorem

$$\operatorname{div}\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z}\ ,$$

giving

$$E_r(r) = -\frac{1}{r}\int_0^r \frac{\partial E_z}{\partial z} \mathrm{r}'\,\mathrm{dr}'\ ,$$

and B_ϑ via

$$\operatorname{rot}\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\ ,$$

giving

$$\frac{\partial B_\vartheta}{\partial z} = -\frac{1}{c^2}\frac{\partial E_r}{\partial t}\ .$$

Standing wave: $Z(z)T(t) = E_o \cos(k_z z)\cos(\omega t) = \frac{E_o}{2}[\cos(\omega t - k_z z) \quad + \quad \cos(\omega t + k_z z)]$

accelerating wave — opposite wave

Cyclic frequency of RF field $\omega = \frac{2\pi c}{\lambda} = 2\pi f_{RF}$

Wave number $k_z = \frac{2\pi}{L} = \frac{2\pi}{\beta\lambda}$

Equivalent traveling wave $E_z(z,\ r,\ t) = E\cos(\omega t - k_z z)R(r)$

Substitution into wave equation gives for radial field component:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial R}{\partial r}) - R(k_z^2 - \frac{\omega^2}{c^2}) = 0$$

$$k_z^2 - \frac{\omega^2}{c^2} = k_z^2(1 - \frac{\omega^2}{k_z^2 c^2}) = k_z^2\ (1-\beta^2) = \frac{k_z^2}{\gamma^2}$$

Solution for radial field component: $R(r) = I_o(\frac{k_z r}{\gamma})$

where $I_o(x)$ is the modified Bessel function

Finally, equivalent traveling wave is

$$E_z = E\, I_o(\frac{k_z r}{\gamma}) \cos(\omega t - k_z z), \tag{5.1}$$

$$E_r = -\gamma E I_1(\frac{k_z r}{\gamma}) \sin(\omega t - k_z z), \tag{5.2}$$

$$B_\theta = -\frac{1}{c} \beta\gamma\, E I_1(\frac{k_z r}{\gamma}) \sin(\omega t - k_z z) \tag{5.3}$$

Effective traveling wave can be represented in Hamiltonian by a potential function

$$U_a = \frac{E}{k_z} I_o(\frac{k_z r}{\gamma}) \sin(\omega t - k_z z). \tag{5.4}$$

Particle, which velocity coincides with the velocity of the accelerating wave, is called synchronous particle. Dynamics of the synchronous particle is described by the integration of equation for synchronous particle momentum, P_s, and position, z_s:

$$\frac{dP_s}{dt} = qE \cos \varphi_s$$

$$\frac{d z_s}{dt} = \frac{P_s}{m\gamma_s},$$

where $\varphi_s = \omega t - k_z z_s$ is the synchronous phase.

Hamiltonian of particle motion in RF field

Particle motion is governed by the single-particle Hamiltonian (Kapchinsky, "Theory of resonance linear accelerators", Harwood, 1985):

$$H = \frac{p_x^2 + p_y^2}{2\ m\ \gamma} + \frac{p_z^2}{2\ m\ \gamma^3} + q\ U_{ext} + q\ \frac{U_b}{\gamma^2}$$

$$U_{ext} = \frac{E}{k_z}\left[\ I_o\left(\frac{k_z r}{\gamma}\right) sin(\varphi_s - k_z\zeta) - sin\varphi_s + k_z\zeta\ cos\varphi_s\right] + G_t\ \frac{r^2}{2}$$

p_x, p_y transverse momentum

$p_z = P_z - P_s$ longitudinal momentum deviation from synchronous particle

$\zeta = z - z_s$ deviation from synchronous particle

φ_s synchronous phase

$k_z = \frac{2\pi}{\beta\lambda}$ wave number

U_{ext} potential of external field

U_b space charge potential of the beam

E amplitude of accelerating wave

G_t gradient of the focusing field

Hamiltonian of particle motion in RF field: Derivation

Consider Hamiltonian in a focusing channel with RF field:

$$K = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} + qU_a + qU_{el} + qU_b, \quad (5.9)$$

where U_{el} is the potential of electrostatic focusing lenses, and U_b is the scalar potential of field of the beam. For the further analysis, let us introduce new variables

$$p_z = P_z - P_s, \qquad \zeta = z - z_s, \qquad (5.10)$$

which define deviation from synchronous particle. Generating function of the transformation is

$$F_3(\zeta, P_z, t) = -(\zeta + z_s)(P_z - P_s), \qquad (5.11)$$

which can be easily verified by differentiation:

$$p_z = -\frac{\partial F_3}{\partial \zeta}, \qquad z = -\frac{\partial F_3}{\partial P_z}. \qquad (5.12)$$

New Hamiltonian is given by

$$T = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_s + p_z - qA_z)^2} + qU_a + qU_{el} + qU_b + \frac{\partial F_3}{\partial t}. \qquad (5.13)$$

Consider separately expression for square root in Hamiltonian:

$$s\,(p_x\,,p_y\,,p_\eta) = \sqrt{m^2c^2 + p_x^2 + p_y^2 + (P_s + p_\eta)^2}\,, \qquad (5.14)$$

where for simplification, the components of canonical momentum are substituted by that of mechanical momentum, $p_x = P_x - q\,A_x$, $p_y = P_y - q\,A_y$, and an additional variable is $p_\eta = p_z - q\,A_z$. Typically, momentum of the synchronous particle is much larger than transverse particle momentum and longitudinal momentum spread, $P_s >> p_x\,, p_y\,, p_\eta$. Let us expand expression for square root in the vicinity of $s(p_x,\,p_y,\,p_\eta)$ up to the order of p_x^2, p_y^2, p_η^2:

$$s = \sqrt{m^2c^2 + P_s^2} + \frac{\partial s}{\partial p_x}p_x + \frac{\partial s}{\partial p_y}p_y + \frac{\partial s}{\partial p_\eta}p_\eta + \frac{1}{2}\frac{\partial^2 s}{\partial p_x^2}p_x^2 + \frac{1}{2}\frac{\partial^2 s}{\partial p_y^2}p_y^2 +$$

$$+ \frac{1}{2}\frac{\partial^2 s}{\partial p_\eta^2}p_\eta^2 + \frac{1}{2}\frac{\partial^2 s}{\partial p_x \partial p_y}p_x p_y + \frac{1}{2}\frac{\partial^2 s}{\partial p_x \partial p_\eta}p_x p_\eta + \frac{1}{2}\frac{\partial^2 s}{\partial p_y \partial p_\eta}p_y p_\eta, \qquad (5.15)$$

where all derivatives are taken at $p_x = 0$, $p_y = 0$, $p_\eta = 0$. Calculations of expansion gives:

$$c\,\sqrt{m^2c^2 + p_x^2 + p_y^2 + (P_s + p_\eta)^2} = mc^2\gamma + \frac{p_x^2}{2m\gamma} + \frac{p_y^2}{2m\gamma} + \frac{P_s p_\eta}{m\gamma} + \frac{p_\eta^2}{2m\gamma^3}, \qquad (5.16)$$

where reduced energy is

$$\gamma = \sqrt{1 + \left(\frac{P_s}{mc}\right)^2}\,. \qquad (5.17)$$

Time derivative of the generating function, Eq. (5.11), is:

$$\frac{\partial F_3}{\partial t} = \zeta\dot{P}_s - \dot{z}_s P_z + \dot{z}_s P_s + z_s\dot{P}_s. \qquad (5.18)$$

where dot means derivative over time. Taking into account that the particle velocity is $\dot{z}_s = \frac{P_s}{m\gamma}$, the following expressions in time derivative, Eq. (5.11), are:

$$\dot{z}_s P_z = \frac{P_s}{m\gamma}(P_s + p_z), \qquad \dot{z}_s P_s = \frac{P_s^2}{m\gamma}, \qquad (5.19)$$

and the time derivative of the generating function is therefore

$$\frac{\partial F_3}{\partial t} = \zeta\dot{P}_s - \frac{P_s p_z}{m\gamma} + z_s\dot{P}_s. \qquad (5.20)$$

Substitution of expansions, Eqs. (5.16), (5.20), into Eq. (5.13) gives for the new Hamiltonian, $H = T - m^2c^2\gamma$:

$$H = \frac{(P_x - qA_x)^2}{2m\gamma} + \frac{(P_y - qA_y)^2}{2m\gamma} + \frac{(p_z - qA_z)^2}{2m\gamma^3} + qU_a + qU_{el} + qU_b - \frac{qP_sA_z}{m\gamma} + \dot{P}_s(z_s + \zeta). \quad (5.21)$$

The term $\dot{P}_s z_s$ can be excluded, because it does not depend on canonical variables and does not contribute to equations of particle motion. The acceleration of synchronous particle according to Eq. (5.7) is $\dot{P}_s = qE\, cos\varphi_s$. The term $\dot{P}_s\zeta$ can be combined with the accelerating potential:

$$qU_a + \dot{P}_s\zeta = q\,\frac{E}{k_z}\,[I_o(\frac{k_z r}{\gamma})\,sin\,(\varphi_s - k_z\zeta) + k_z\zeta cos\varphi_s]. \quad (5.22)$$

Finally, the new Hamiltonian is

$$H = \frac{(P_x - q\,A_x)^2}{2m\gamma} + \frac{(P_y - q\,A_y)^2}{2m\gamma} + \frac{(p_z - q\,A_z)^2}{2m\gamma^3} +$$

$$+ q\,\frac{E}{k_z}\,[I_o(\frac{k_z r}{\gamma})\,sin\,(\varphi_s - k_z\zeta) + k_z\zeta cos\varphi_s] + qU_{el} + qU_b - \frac{qP_sA_z}{m\gamma}. \quad (5.23)$$

Consider the following terms in the Hamiltonian:

$$\frac{(p_z - q\,A_z)^2}{2m\gamma^3} - \frac{qP_sA_z}{m\gamma} = \frac{p_z^2}{2m\gamma^3} - \frac{qP_sA_z}{m\gamma}\,(1 + \frac{p_z}{P_s\,\gamma^2} - \frac{qA_z}{2P_s\gamma^2}). \tag{5.24}$$

As soon as $p_z << P_s$, $qA_z << P_s$, the second and the third terms in parentheses in Eq. (5.24) can be omitted:

$$\frac{qP_sA_z}{m\gamma}\,(1 + \frac{p_z}{P_s\,\gamma^2} - \frac{q\,A_z}{2P_s\gamma^2}) \approx \frac{qP_sA_z}{m\gamma} = q\beta cA_z. \tag{5.25}$$

The vector - potential is $A_z = A_{z\ magn} + \frac{\beta}{c} U_b$. Therefore, in the adopted assumptions, the Hamiltonian becomes:

$$H = \frac{(P_x - q\,A_x)^2}{2m\gamma} + \frac{(P_y - q\,A_y)^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} +$$

$$+q\,\frac{E}{k_z}\,[I_o(\frac{k_z r}{\gamma})sin\,(\varphi_s - k_z\zeta) + k_z\zeta cos\varphi_s] + q(U_{el} - \beta cA_{z\ magn}) + q\,\frac{U_b}{\gamma^2}. \tag{5.26}$$

Consider separately structures with quadrupole focusing and with longitudinal magnetic focusing. In the absence of longitudinal magnetic field, transverse components of the vector potential ar e $A_x = 0$, $A_y = 0$, therefore, the transverse components of canonical momentum coincide with that of mechanical momentum: $p_x = P_x$, $p_y = P_y$. The term $U_{el} - \beta c A_{z,\,magn}$ is the focusing potential of the structure. Averaged potential of quadrupole structure is given by

$$U_{el} - \beta c A_{z\ magn} = G_t \frac{(x^2 + y^2)}{2}, \qquad (5.27)$$

where G_t is the gradient of averaged focusi ng potential. The Hamiltonian for particle motion in RF field with quadrupole focusing is

$$H = \frac{p_x^2}{2m\gamma} + \frac{p_y^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}\left[I_o\left(\frac{k_z r}{\gamma}\right)\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s\right] + qG_t\frac{(x^2+y^2)}{2} + q\frac{U_b}{\gamma^2}. \qquad (5.28)$$

In presence of longitudinal magnetic field, the Hamiltonian, Eq. (5.26), is

$$H = \frac{(P_x - qA_x)^2}{2m\gamma} + \frac{(P_y - qA_y)^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}\left[I_o\left(\frac{k_z r}{\gamma}\right)\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s\right] + q\frac{U_b}{\gamma^2} \qquad (5.29)$$

where transverse components of vector-potential are given by

$$A_{x\ magn} = -B\frac{y}{2} \qquad (5.30)$$

$$A_{y\ magn} = B\frac{x}{2} \qquad (5.31)$$

Transformation to Larmor system is given by

$$\hat{x} = x\cos\theta - y\sin\theta \,,$$

$$\hat{y} = x\sin\theta + y\cos\theta \,,$$

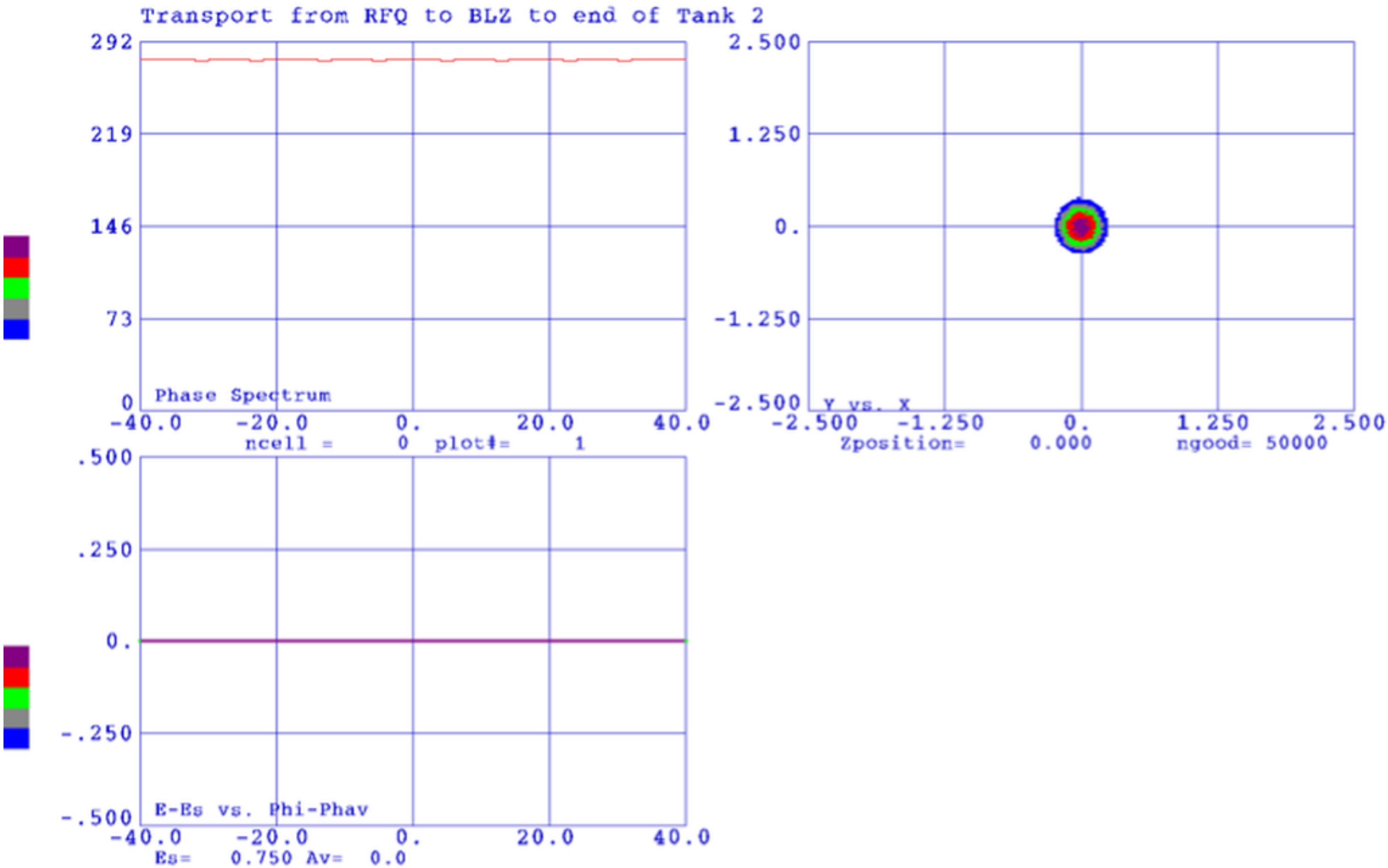
$$\hat{P}_x = P_x \cos\theta - P_y \sin\theta \,,$$

$$\hat{P}_y = P_y \cos + P_x \sin\theta \,.$$

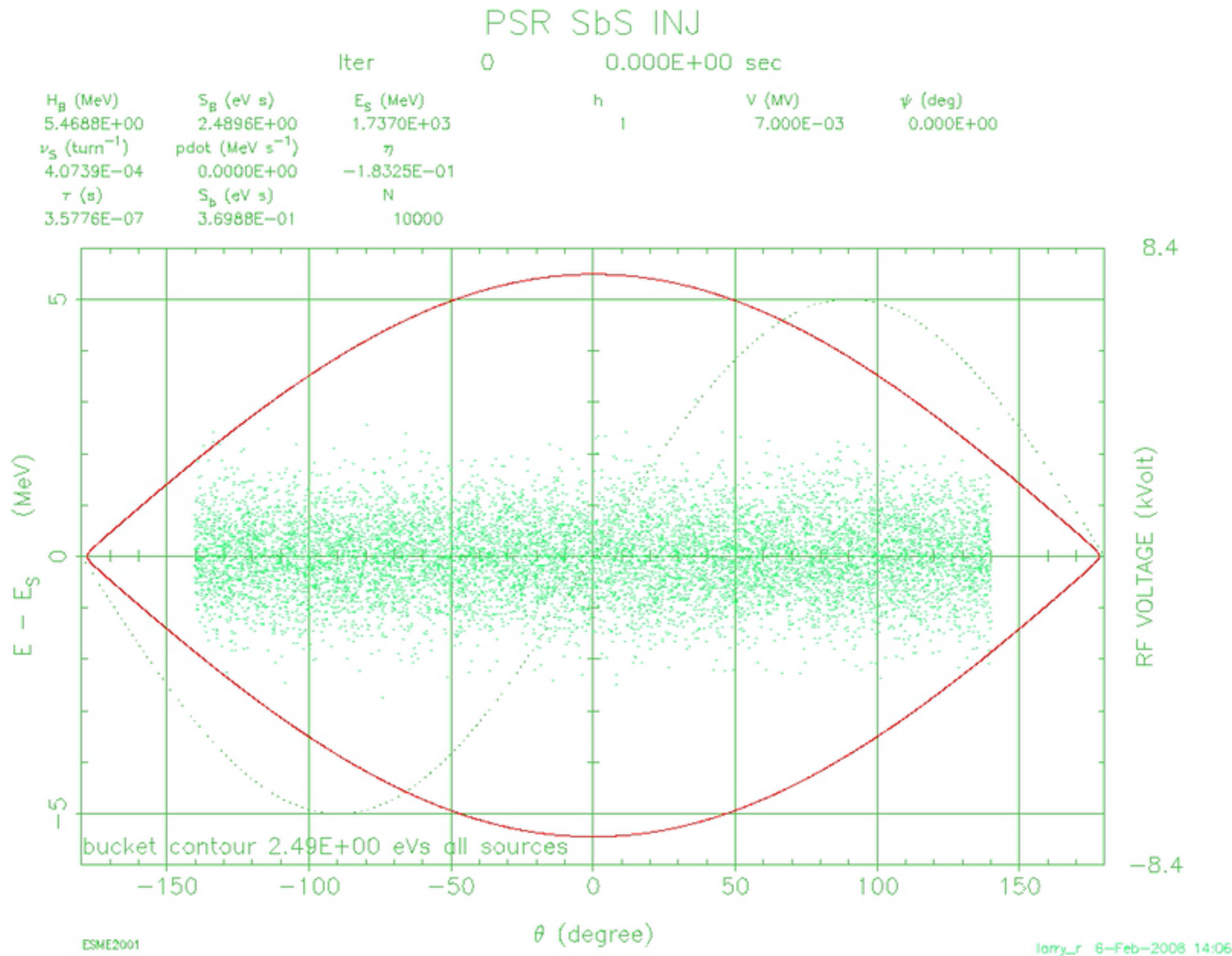
where angle $\theta(z) = \int_{z_o}^{z} \omega_L(z)\,dz$ and Larmor frequency $\omega_L = \dfrac{qB}{2m\gamma}$

Hamiltonian of particle motion in magnetic field :

$$H = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s] + m\gamma\omega_L^2\frac{r^2}{2} + q\frac{U_b}{\gamma^2}$$



Example of beam dynamics in accelerating structure. (Courtesy of Larry Rybarcyk.)



Longitudinal oscillations in RF field with φ_s = - 90°. (Courtesy of Larry Rybarcyk.)

Paraxial Approximation of Hamiltonian: Transverse Particle Motion in RF Field

Hamiltonian of particle motion in RF field:

$$H = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{p_z^2}{2m\gamma^3} + \frac{qE}{k_z}[I_o(\frac{k_z r}{\gamma})\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s] + m\gamma\Omega_r^2\frac{r^2}{2} + q\frac{U_b}{\gamma^2}$$

Near-axis approximation:

$$I_o(\frac{k_z r}{\gamma}) \approx 1 + \frac{1}{4}(\frac{k_z r}{\gamma})^2$$

Hamiltonian of *transverse* motion: $H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{qE}{4k_z}(\frac{k_z r}{\gamma})^2\sin(\varphi_s - k_z\zeta) + m\gamma\Omega_r^2\frac{r^2}{2} + q\frac{U_b}{\gamma^2}$

$$\frac{qE}{4k_z}(\frac{k_z r}{\gamma})^2 = \frac{qE\pi r^2}{2\beta\gamma^2\lambda}$$

Transverse Oscillation Frequency in RF Field

Expansion near synchronous particle: $$\sin(\varphi_s - k_z\zeta) \approx \sin\varphi_s - k_z\zeta\cos\varphi_s = \sin\varphi_s(1-\psi\,\mathrm{ctg}\,\varphi_s)$$

Phase deviation from synchronous particle $$\psi = k_z\zeta$$

Hamiltonian of near-axis, near synchronous particle motion, with $U_b = 0$:

$$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{qE\pi}{2\beta\gamma^2\lambda}\sin\varphi_s(1-\psi\,\mathrm{ctg}\,\varphi_s)r^2 + m\gamma\,\Omega_r^2\frac{r^2}{2}$$

Frequency of longitudinal oscillations: $$\Omega^2 = \frac{2\pi}{\lambda}\frac{qE}{m}\frac{|\sin\varphi_s|}{\beta\gamma^3}$$

Hamiltonian becomes: $$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} - \frac{m\gamma}{4}\Omega^2(1-\psi\,\mathrm{ctg}\,\varphi_s)r^2 + m\gamma\,\Omega_r^2\frac{r^2}{2}$$

$$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{m\gamma}{2}r^2[\Omega_r^2 - \frac{\Omega^2}{2}(1-\psi\,\mathrm{ctg}\,\varphi_s)]$$

Transverse oscillation frequency of synchronous particle in presence of RF field: $$\Omega_{rs}^2 = \Omega_r^2 - \frac{\Omega^2}{2}$$

Phase advance of transverse oscillations of synchronous particle in presence of RF field: $$\mu_{rs} = \sqrt{\mu_o^2 - \frac{1}{2}\Omega^2(\frac{L}{\beta_z c})^2}$$

Parametric Resonance in RF Field

Hamiltonian becomes:

$$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{m\gamma}{2} r^2 (\Omega_{rs}^2 + \frac{\Omega^2}{2} \psi \operatorname{ctg} \varphi_s)$$

Longitudinal particle oscillations with amplitude Φ and frequency Ω :

$$\psi = -\Phi \sin(\Omega t + \psi_o)$$

Finally, Hamiltonian is:

$$H_t = \frac{p_x^2 + p_y^2}{2m\gamma} + \frac{m\gamma}{2} r^2 [\Omega_{rs}^2 - \frac{\Omega^2}{2} \operatorname{ctg} \varphi_s \Phi \sin(\Omega t + \psi_o)]$$

Transversal equation of motion:

$$\frac{d^2 x}{dt^2} + x[\Omega_{rs}^2 - \frac{\Omega^2}{2} \operatorname{ctg} \varphi_s \Phi \sin(\Omega t + \psi_o)] = 0$$

Transverse particle oscillation frequency in RF field:

$$\boxed{\Omega_{r\ RF} = \sqrt{\Omega_{rs}^2 - \frac{\Omega^2}{2} \operatorname{ctg} \varphi_s \Phi \sin(\Omega t + \psi_o)}}$$

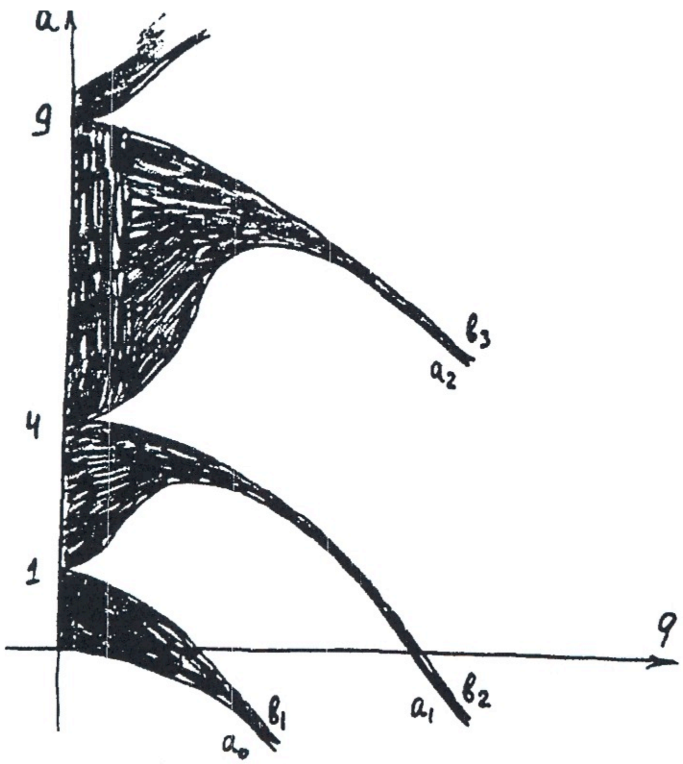
Parametric resonance occurs when

$$\Omega_{rs} = \frac{n}{2} \Omega, \quad n = 1, 2, 3$$

General form of Mathieu - Hill equation

Mathieu - Hill equation $$\frac{d^2x}{d\tau^2} + \pi^2(a - 2q\sin 2\pi\tau)x = 0$$

Unstable solutions are around $a = n^2$, or when average frequency of oscillator is close to half-integer value of that of driving force.



Shaded are stable regions of solutions of Mathieu-Hill equation.

First region of parametric instability $b_1 < a < a_1,$

where

$$b_1 = 1 - q - \frac{1}{8}q^2 + \frac{1}{64}q^3 - \cdots$$

$$a_1 = 1 + q - \frac{1}{8}q^2 - \frac{1}{64}q^3 - \cdots$$

The second region of parametric instability is

$$b_2 < a < a_2,$$

where

$$b_2 = 4 - \frac{1}{12}q^2 + \frac{5}{13824}q^4 - \cdots$$

$$a_2 = 4 + \frac{5}{12}q^2 - \frac{763}{13824}q^4 + \cdots$$

Regions of Parametric Resonance

Condition for parametric resonance

$$\Omega_{rs} = \frac{n}{2}\Omega, \qquad n = 1, 2, 3\ldots$$

The regions of parametric instability are

$$\frac{\sqrt{b_n}}{2} < \frac{\Omega_{rs}}{\Omega} < \frac{\sqrt{a_n}}{2}$$

where for the first two regions of instability, $n = 1, 2$, the parameters a_n, b_n are:

$$a_1 = 1 + q - \frac{q^2}{8} - \frac{q^3}{64}, \qquad b_1 = 1 - q - \frac{q^2}{8} + \frac{q^3}{64} \tag{10}$$

$$a_2 = 4 + \frac{5q^2}{12} - \frac{763q^4}{13824}, \qquad b_2 = 4 - \frac{q^2}{12} + \frac{5q^4}{13824} \tag{11}$$

and the parameter

$$q = \frac{\Phi}{|tg\varphi_s|} \approx \frac{\varphi_s}{tg\varphi_s}$$

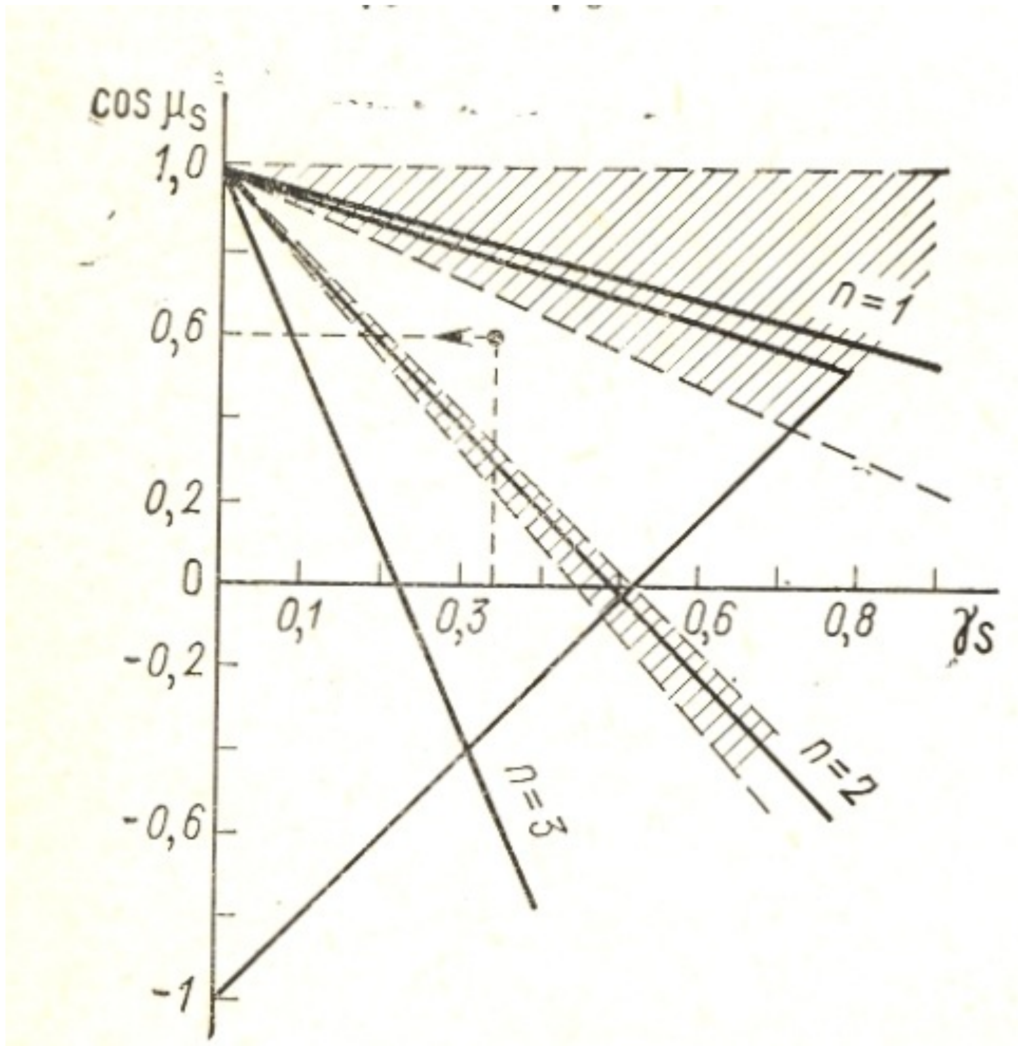
In linac, the transverse oscillation frequency is typically larger than the longitudinal oscillation frequency, and the first $n=1$ parametric resonance instability region is avoided. The potentially dangerous region in this case is the second parametric resonance bandwidth where $n = 2$. Instabilities of higher-order resonance regions are typically unimportant

Let us introduce phase advance for synchronous particle in RF field

$$\mu_s = \Omega_{rs} \frac{L}{\beta_z c}$$

and defocusing factor

$$\gamma_s = \frac{1}{4}\Omega^2 (\frac{L}{\beta_z c})^2$$



Parametric resonance regions.

Experimental Observation of Parametric Resonance (L.Groening et al, LINAC2010)

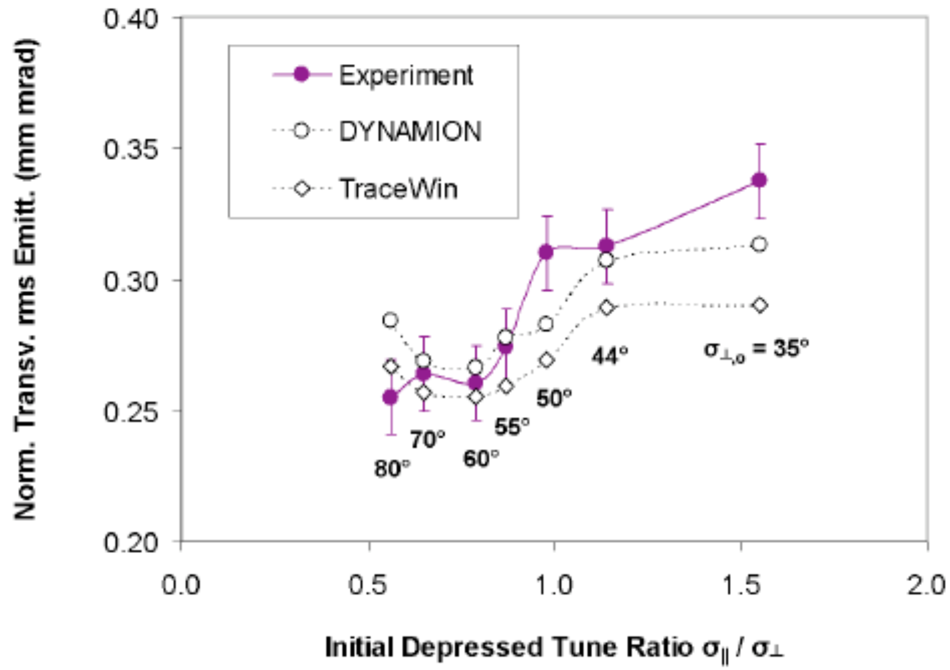
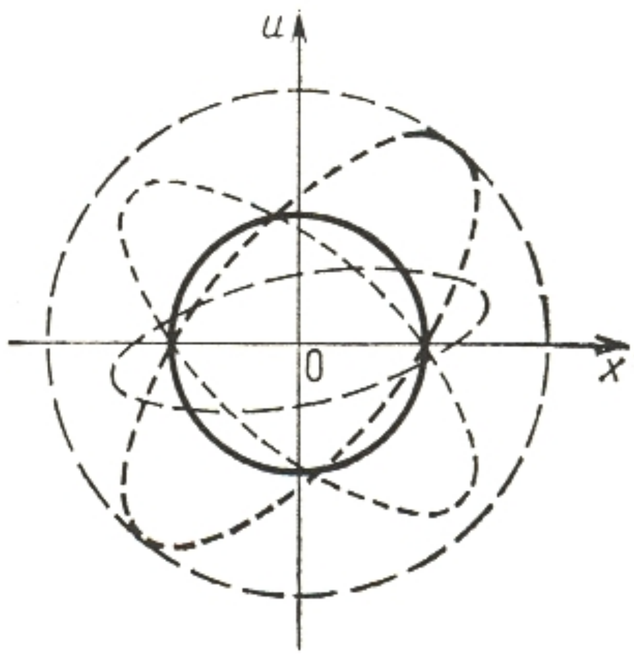


Figure 8: Mean of horizontal and vertical rms emittance at the DTL exit as a function of the initial ratio of depressed longitudinal and transverse phase advance.

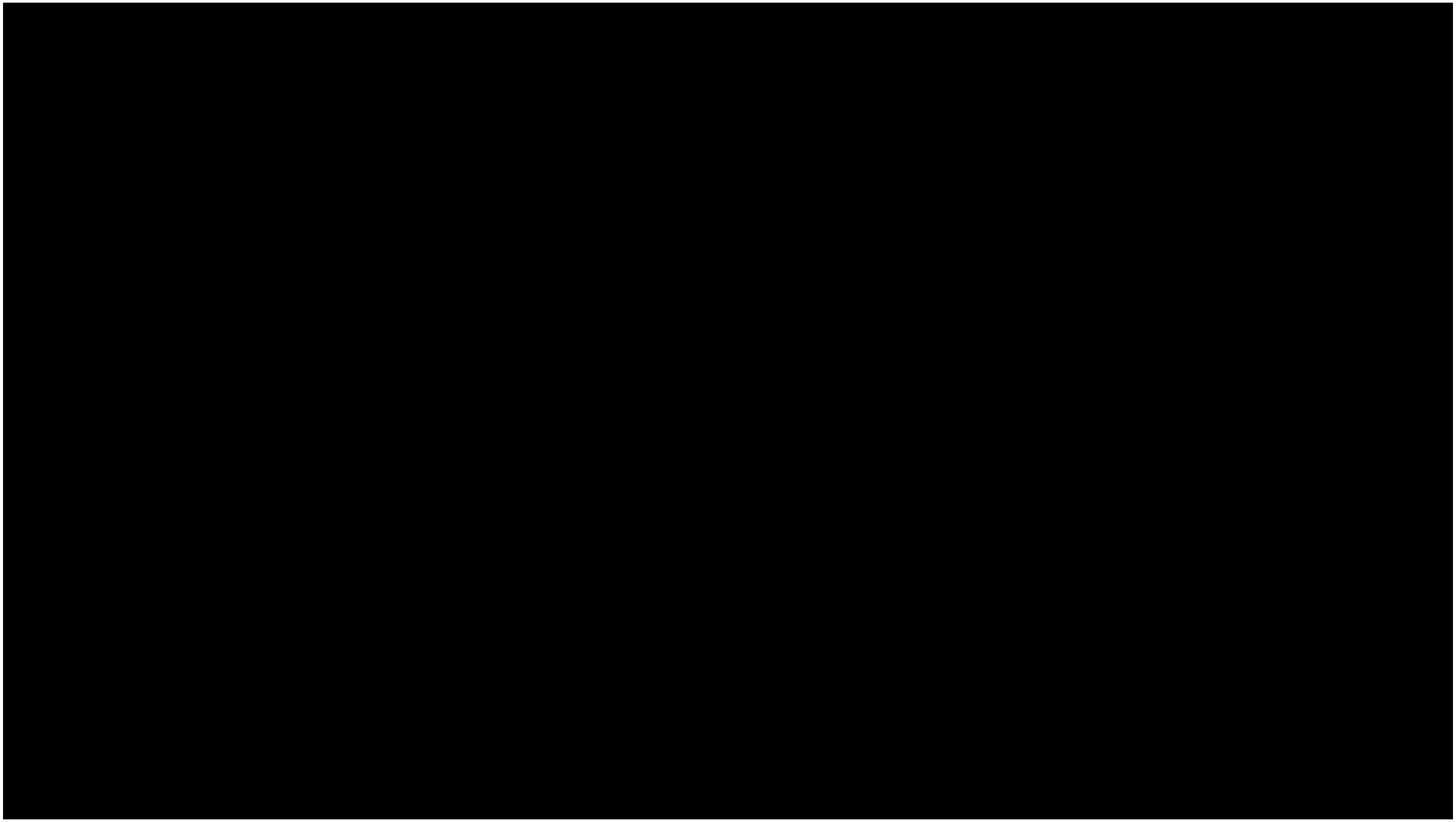
Effective Beam Emittance Growth Outside of Parametric Resonance

$$\frac{\varepsilon_{eff}}{\varepsilon} = 1 + \Phi\,\mathrm{ctg}\,\varphi_s \frac{\Omega^2}{4\Omega_{rs}^2 - \Omega^2}$$



Phase space of transverse oscillations in presence of RF field (from Kapchinky, 1985).

Details of RFQ Beam Dynamics



Dynamics of 35 mA proton beam in 201.25 MHz 4-rod RFQ (courtesy of Sergey Kurennoy).

Longitudinal - Transverse Parametric Resonance in RFQ

In RFQ, the Hamiltonian of particle motion is given by

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + qU_{ext} + qU_b \tag{1}$$

$$U_{ext} = \frac{U_L T}{\pi}[I_o(k_z r)\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s] + \frac{m\Omega_r^2 r^2}{2q} \tag{2}$$

where p_x and p_y are transverse momenta, $p_z = P_z - P_s$ is the deviation of longitudinal momentum from the momentum of the synchronous particle, $\zeta = z - z_s$ is the longitudinal deviation from the synchronous particle, U_L is the intervane voltage, $T = (\pi/4)A$ where A is the efficiency of acceleration, $k_z = 2\pi/\beta\lambda$ is the wave number, φ_s is the synchronous phase, and Ω_r is the transverse oscillation frequency without vane modulation:

$$\Omega_r = \frac{\omega}{\sqrt{2}\pi^2}\chi\frac{qU_L}{mc^2}(\frac{\lambda}{2a})^2 \tag{3}$$

where $\omega = 2\pi c/\lambda$ is the circular frequency, χ is the focusing efficiency, a is the radius of aperture, and U_b is the space charge potential.

For small oscillations,

$$H(\zeta, p) = (1/2)p^2 + (\Omega^2/2)I_0(kr)\zeta^2 + (\Omega^2/k \operatorname{tg}\varphi_s)[I_0(kr) - 1]\zeta. \tag{2.218}$$

We have introduced the following notation into the last expression:

$$\Omega^2 = \omega^2 \frac{eU_L T|\sin\varphi_s|}{\pi m_0 v_s^2}.$$

Longitudinal Particle Oscillations in RFQ

Injection of low-velocity particles into an RFQ results in dependence of the longitudinal oscillation frequency on transverse particle position. Neglecting space-charge forces, the equation of small-amplitude longitudinal oscillations for off-axis particles is given by :

$$\frac{d^2\zeta}{dt^2} + \Omega^2 I_o(k_z r)\zeta = \frac{\Omega^2}{k_z |tg\varphi_s|}[I_o(k_z r) - 1] \ . \qquad (12)$$

Averaged transverse oscillations can be approximated by $r = R\cos\Omega_{rs}t$. Periodic function $I_o(k_z R\cos\Omega_{rs}t)$ can be expanded in Fourier series:

$$I_o(k_z R\cos\Omega_{rs}t) = I_o^2(\frac{k_z R}{2}) + 2\sum_{m=1}^{\infty} I_m^2(\frac{k_z R}{2}) \cdot \cos 2m\Omega_{rs}t \qquad (13)$$

Because the amplitudes of the terms of the Bessel function drop off quickly, only the first two terms are important, resulting in the following equation of motion:

$$\frac{d^2\zeta}{dt^2} + \Omega^2\zeta[I_o^2(\frac{k_z R}{2}) + 2I_1^2(\frac{k_z R}{2}) \cdot \cos 2\Omega_{rs}t] = \frac{\Omega^2}{k_z tg\varphi_s}[I_o^2(\frac{k_z R}{2}) - 1 + 2I_1^2(\frac{k_z R}{2})\cos 2\Omega_{rs}t] \qquad (14)$$

Analysis of longitudinal parametric instabilities includes

(i) consideration of a Mathieu-type equation parametric resonance instability neglecting the right-side part of Eq. (14), and
(ii) external resonances, taking into account the right-hand external driving force of Eq. (14).

Longitudinal parametric resonances occur when the following condition is fulfilled:

$$\frac{\Omega_{rs}}{\Omega} = \frac{I_o(\frac{k_z R}{2})}{n} \qquad n = 1, 2, 3, \ldots \tag{15}$$

with the region of parametric instability defined as:

$$\frac{I_o^2(\frac{k_z R}{2})}{a_n} < (\frac{\Omega_{rs}}{\Omega})^2 < \frac{I_o^2(\frac{k_z R}{2})}{b_n} \tag{16}$$

where a_n, b_n are given by Eqs. (10), (11), and the parameter

$$q = (\frac{\Omega}{\Omega_{rs}})^2 I_1^2(\frac{k_z R}{2}). \tag{17}$$

The first significant parametric resonance area is when n = 1. This leads to the following resonance bandwidth defined by Eq. (16):

$$I_o^2(\frac{k_z R}{2}) - I_1^2(\frac{k_z R}{2}) < (\frac{\Omega_{rs}}{\Omega})^2 < I_o^2(\frac{k_z R}{2}) + I_1^2(\frac{k_z R}{2}). \tag{18}$$

An external resonance occurs when the transverse oscillation frequency is $\Omega_{rs} = \frac{\Omega}{2}\ I_o(k_z R/2)$.

Both external and parametric resonances can be avoided simultaneously when $\frac{\Omega_{rs}}{\Omega} > I_o(k_z a/2)$

Example of RFQ Dynamics

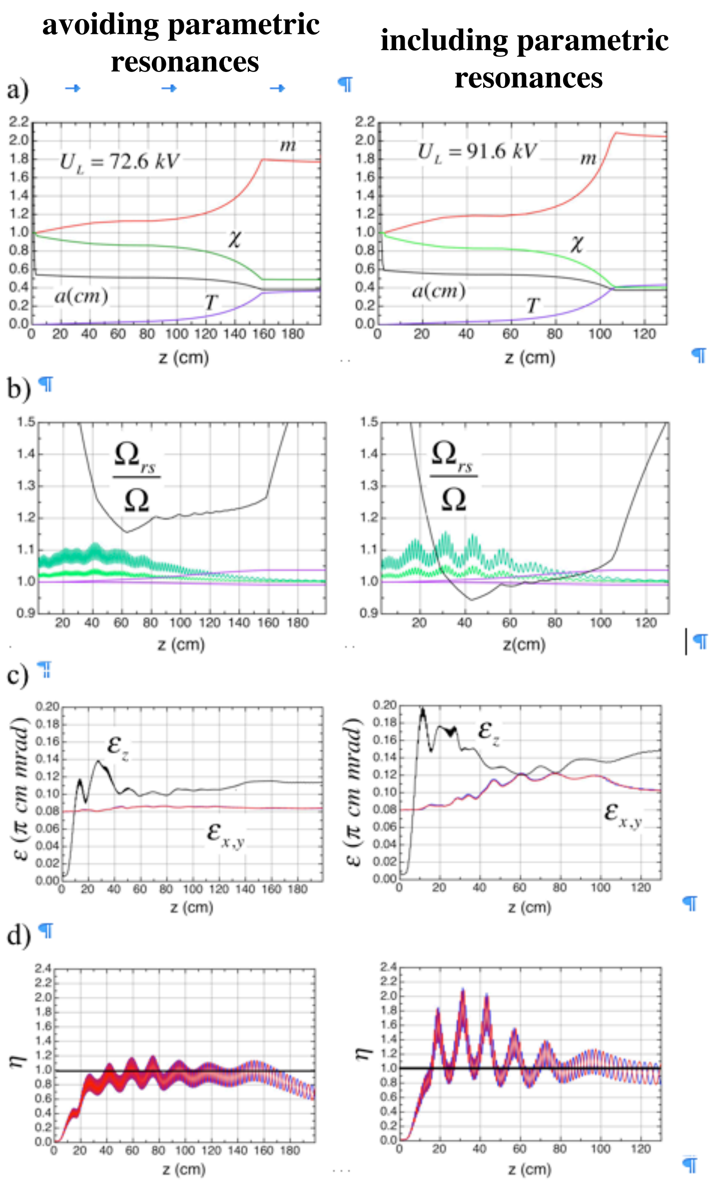
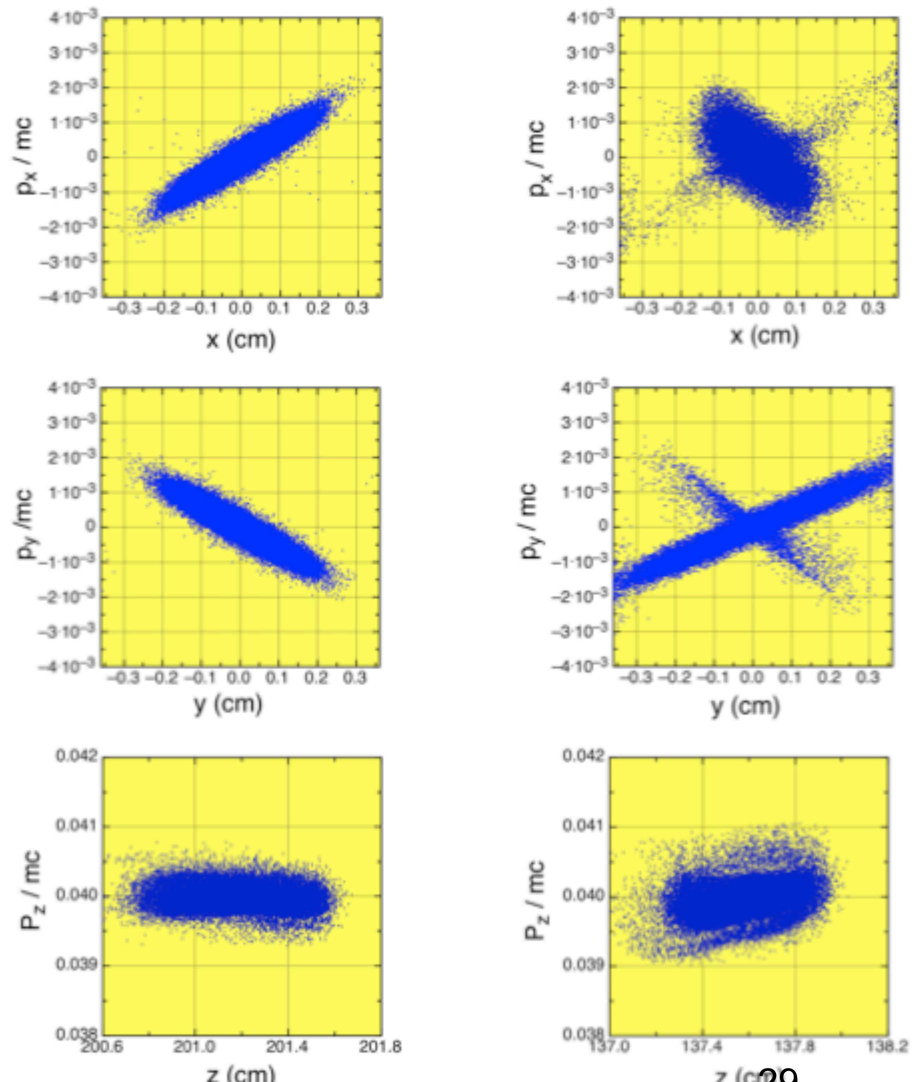


Fig. 4. Beam dynamics in RFQ with beam current $I = 35$ mA: (left) avoiding parametric resonances, (right) including parametric resonances: (a) RFQ parameters, (b) parametric resonance bandwidth: (green) Eq. (18), (red) Eq. (9), (c) beam emittances, (d) equipartitioning parameter, Eq. (5).

avoiding parametric resonances | **including parametric resonances**



RFQ Output beam phase space distributions

Required Transverse Focusing in Presence of RF field

Hamiltonian of particle motion in RF field with solenoid focusing

$$H = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m\gamma} + m\gamma\frac{r^2}{2}(\omega_L^2 - \frac{\Omega^2}{2}\frac{\sin\varphi}{\sin\varphi_s}) + q\frac{U_b}{\gamma^2}$$

Transverse oscillation frequency in presence of RF field

$$\Omega_r^2 = \omega_L^2 - \frac{\Omega^2}{2}\frac{\sin\varphi}{\sin\varphi_s}$$

Envelope equation

$$\frac{d^2R}{dz^2} - \frac{\text{э}^2}{R^3} + \frac{\Omega_r^2}{(\beta c)^2}R - \frac{2I}{I_c(\beta\gamma)^3 R} = 0$$

Beam equilibrium condition $\frac{d^2R_e}{dz^2} = 0$

$$\frac{\Omega_r^2}{(\beta c)^2}R_e + \frac{\text{э}^2}{R_e^3} - \frac{2I}{I_c(\beta\gamma)^3 R_e} = 0$$

$$\Omega_r^2 = (\frac{\beta c}{R_e})^2(\frac{\text{э}^2}{R_e^2} + \frac{2I}{I_c(\beta\gamma)^3})$$

Required magnetic field

$$B = \frac{2mc\beta\gamma}{qR_e}\sqrt{(\frac{\text{э}}{R_e})^2 + \frac{2I}{I_c(\beta\gamma)^3} + \pi(\frac{qE\lambda}{mc^2})\frac{\sin\varphi}{(\beta\gamma)^3}(\frac{R_e}{\lambda})^2}$$

Acceleration in Non-Ideal Accelerating Structure

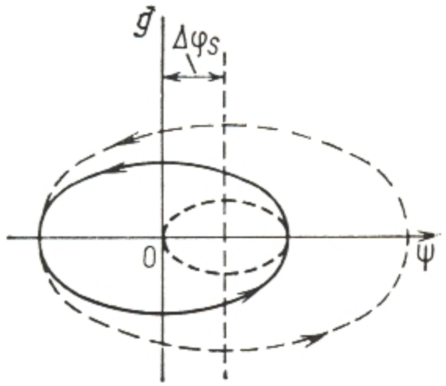


Fig. 1.10 Effect of an abrupt change of the equilibrium phase on the longitudinal oscillations of particles.

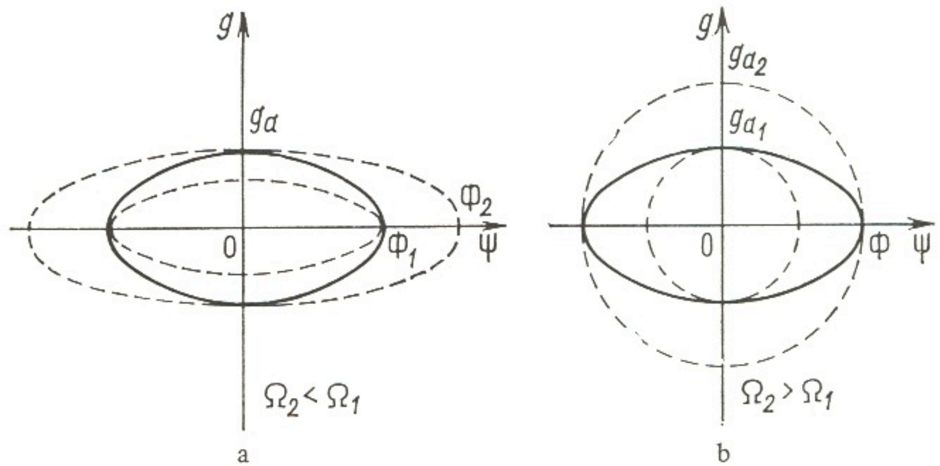


Fig. 1.12 Effect of an abrupt change in frequency on longitudinal oscillations of particles.

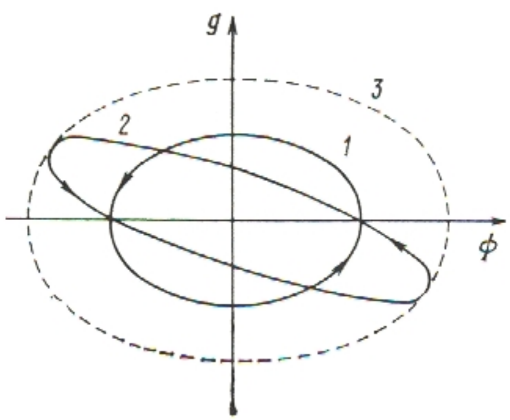


Fig. 1.11 Effect of an empty space on longitudinal oscillations of particles.

Acceleration in Non-Ideal Accelerating Structure (cont.)

Relative momentum deviation from synchronous particle

$$g = \frac{p - p_s}{p_s}$$

Dimensionless longitudinal oscillation frequency

$$\frac{\Omega}{\omega} = \sqrt{\left(\frac{qE\lambda}{mc^2}\right)\frac{|\sin\varphi_s|}{2\pi\beta\gamma^3}}$$

Dimensionless acceleration rate

$$W_\lambda = \frac{eE_o T\lambda\cos\varphi_s}{mc^2}$$

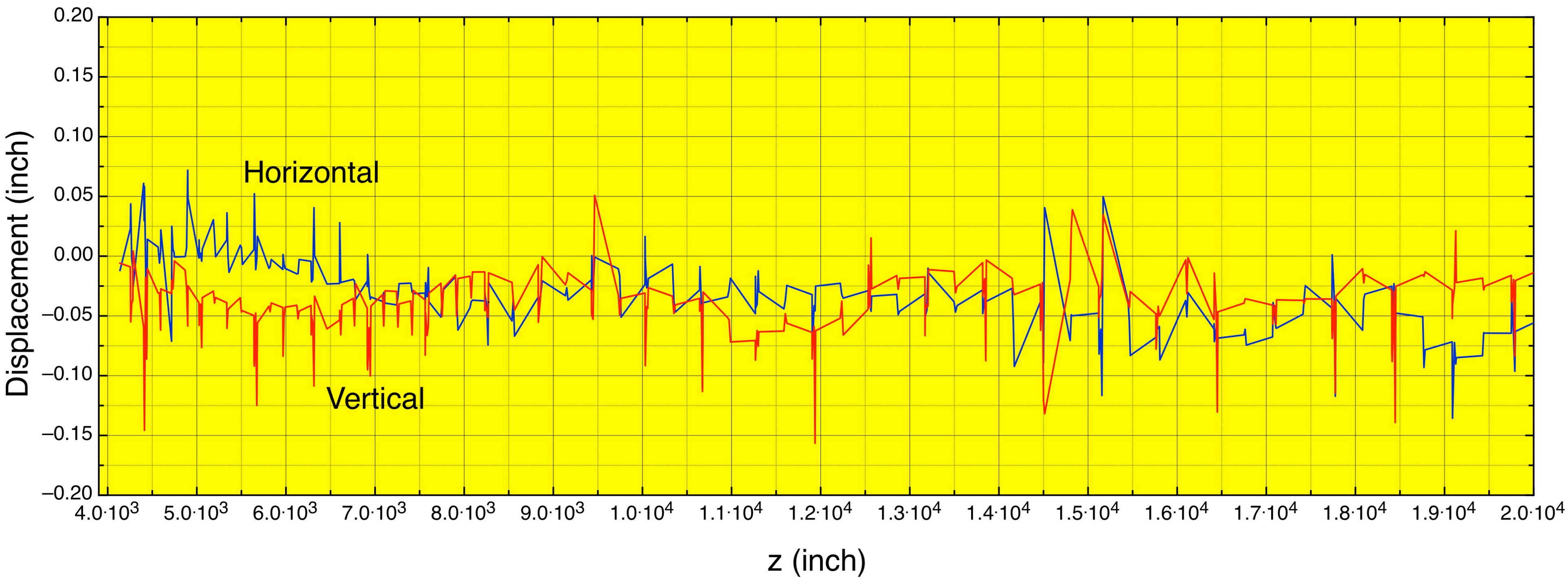
Increase in relative momentum spread

$$\langle\Delta g_a\rangle = \sqrt{\frac{N}{2}\left[\langle\delta g\rangle^2 + \left(\frac{\Omega}{\omega}\right)_N^2 \langle\delta\psi\rangle^2\right]},$$

$$\langle\delta\psi\rangle = 2\pi\left\langle\frac{\delta z}{\beta\lambda}\right\rangle;$$

$$\langle\delta g\rangle = \frac{kW_\lambda}{\beta_N}\sqrt{\left\langle\frac{\delta E_0}{E_0}\right\rangle^2 + 4\pi^2 \operatorname{tg}^2\varphi_s\left\langle\frac{\delta z}{\beta\lambda}\right\rangle^2}.$$

Transverse Displacement of Accelerating and Focusing Elements in 805 MHz LANSCE Linac



Transverse Oscillations in Non-Ideal Focusing Structure

Rms increase of amplitude of transverse oscillations

$$\langle \Delta A \rangle = \sqrt{\frac{N_\Phi}{2}\left[\Sigma\langle \Delta x^*\rangle^2 + \frac{1}{\nu_\Phi^2}\Sigma\langle \Delta \dot{x}^*\rangle^2\right]}.$$

1) slope of longitudinal axis of the lens

$$\langle \Delta x^*\rangle = a_1 K^2 \langle \Delta r_K\rangle; \quad \langle \Delta \dot{x}^*\rangle = b_1 K^2 \langle \Delta r_K\rangle;$$

2) parallel shift of axis of the lens

$$\langle \Delta x^*\rangle = a_2 K^2 \langle \Delta r_0\rangle; \quad \langle \Delta \dot{x}^*\rangle = b_2 K^2 \langle \Delta r_0\rangle;$$

3) rotation of transverse axes of the lens

$$\langle \Delta x\rangle^* = 4a_2 K^2 A \sqrt{\overline{(\Delta\psi)^4}}; \quad \langle \Delta \dot{x}^*\rangle = 4b_2 K^2 A \sqrt{\overline{(\Delta\psi)^4}};$$

For FODO Structure

$$a_1 = \frac{1}{3\sqrt{2}}\left[1 + \frac{K^2}{4}\left(1 + 2\frac{g}{D}\right)\right]^{1/2};$$

$$b_1 = \frac{K^4}{\sqrt{2}} 10^{-2}\left[1 + \left(1 + 6\frac{g}{D}\right)^2\right]^{1/2};$$

$$a_2 = \left[\left(1 + \frac{g}{D}\right)^2 - \frac{K^2}{6}\left(1 + \frac{5}{2}\frac{g}{D} + \frac{3}{2}\frac{g^2}{D^2}\right)\right]^{1/2};$$

$$b_2 = \sqrt{2}\left[1 - \frac{K^2}{4}\left(1 + 2\frac{g}{D}\right)\right]^{1/2}.$$

$$K = D\sqrt{\frac{qG}{mc\beta\gamma}}$$ Quadrupole strength

$\frac{g}{D}$ Ratio of drift space to lens length

$\upsilon_\phi \approx$ phase advanve

Δr_o shift of axis of the lens

Δr_k Shift of the end of magnetic axis

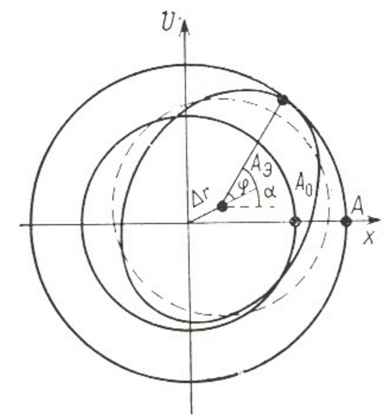
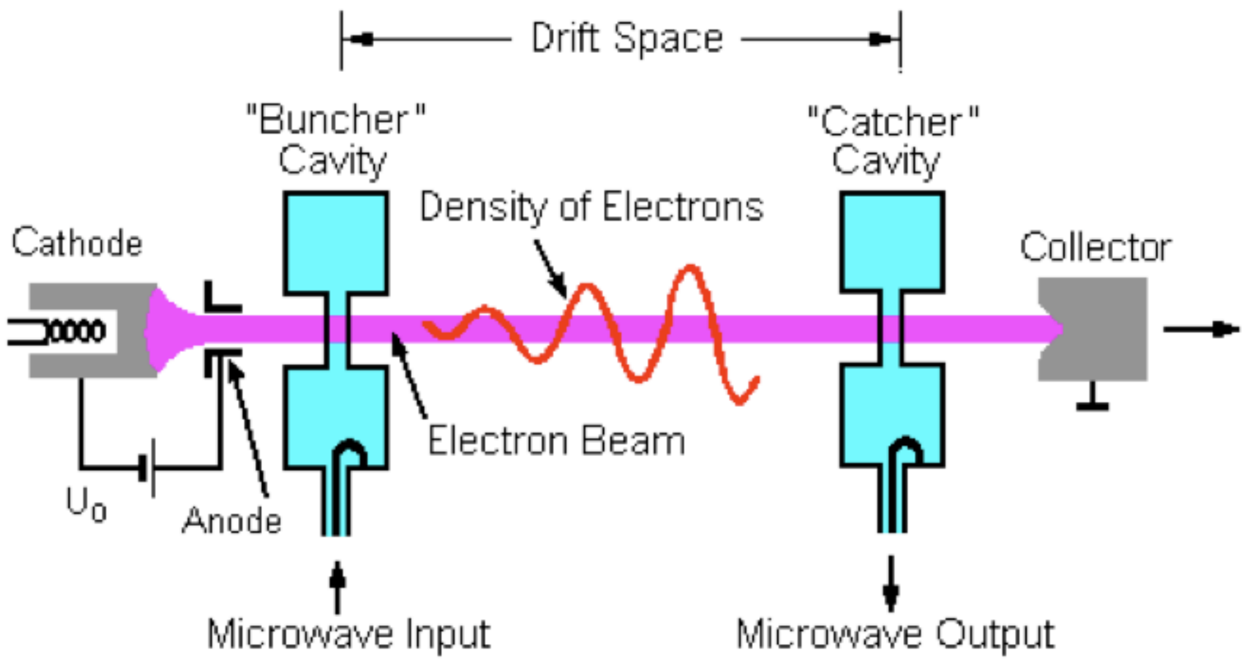
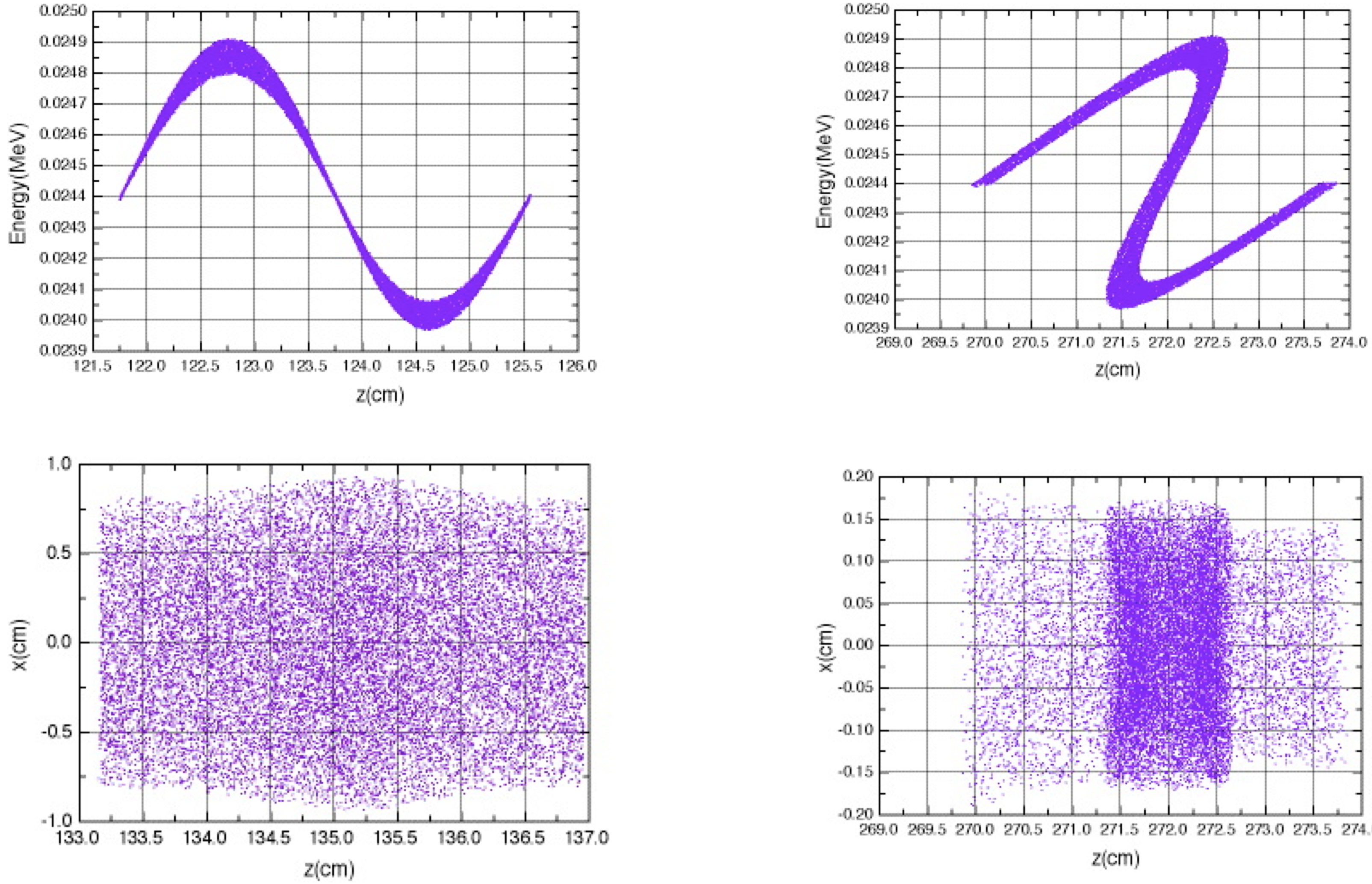


Fig. 2.15 Effective emittance caused by random beam perturbations.

Beam Bunching in RF field



Layout of klystron beam bunching scheme (from http://en.wikipedia.org/wiki/Klystron)



RF beam bunching scheme: (left) initial beam modulation in longitudinal momentum, (right) final beam modulation in density.

Initial particle velocity after extraction voltage U_o

$$v_o = \sqrt{\frac{2qU_o}{m}}$$

Equation of motion in RF gap of width d and applied voltage U_1

$$\frac{dv}{dt} = \frac{q}{m}\frac{U_1}{d}\sin\omega t$$

Longitudinal particle velocity in RF gap

$$v = v_o + \frac{q}{m}\frac{U_1}{d}\int_{t_{in}}^{t_{out}} \sin\omega t\, dt$$

Longitudinal particle velocity after RF gap

$$v = v_o + \frac{q}{m}\frac{U_1}{\omega d} 2\sin\left(\frac{\varphi_{in}+\varphi_{out}}{2}\right)\sin\left(\frac{\varphi_{out}-\varphi_{in}}{2}\right)$$

RF phase in the center of the gap

$$\frac{\varphi_{in}+\varphi_{out}}{2} = \omega t_1$$

Transit time angle through the gap

$$\theta_1 = \frac{\omega d}{v_o} \qquad \frac{\varphi_{out}-\varphi_{in}}{2} = \frac{\theta_1}{2}$$

Longitudinal particle velocity after RF gap

$$\boxed{v = v_o + v_1 \sin\omega t_1}$$

Amplitude of modulation of longitudinal velocity

$$v_1 = v_o \frac{U_1}{2U_0} M_1$$

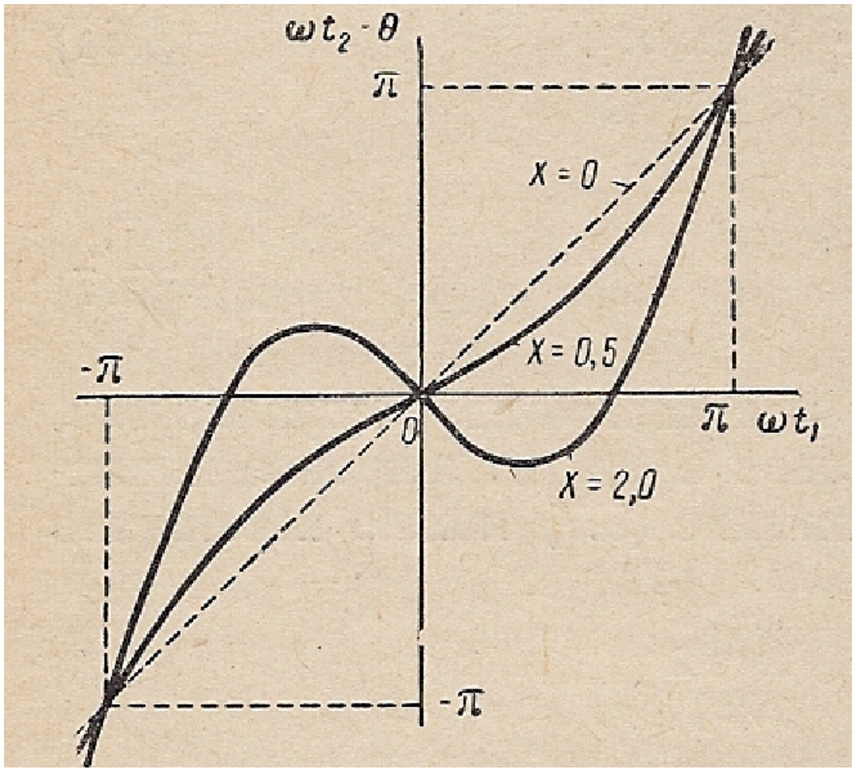
Transit time factor of RF gap

$$M_1 = \frac{\sin\frac{\theta_1}{2}}{\frac{\theta_1}{2}}$$

Time of arrival of particle to the second gap
$$t_2 = t_1 + \frac{z}{v_o + v_1 \sin \omega t_1} \approx t_1 + \frac{z}{v_o}(1 - \frac{v_1}{v_o} \sin \omega t_1)$$

Phase of arrival of particle into the second gap
$$\omega t_2 - \omega \frac{z}{v_o} = \omega t_1 - \omega \frac{z v_1}{v_o^2} \sin \omega t_1$$

$$\boxed{\omega t_2 - \theta = \omega t_1 - X \sin \omega t_1}$$



Transit angle between gaps $\theta = \omega \frac{z}{v_o}$

Bunching parameter
$$\boxed{X = \omega \frac{z v_1}{v_o^2} = \frac{U_1 M_1}{2U_o} \frac{\omega z}{v_o}}$$

Phase of arrival of particle into second gap as a function phase of the same particle in the first gap.

Conservation of charge

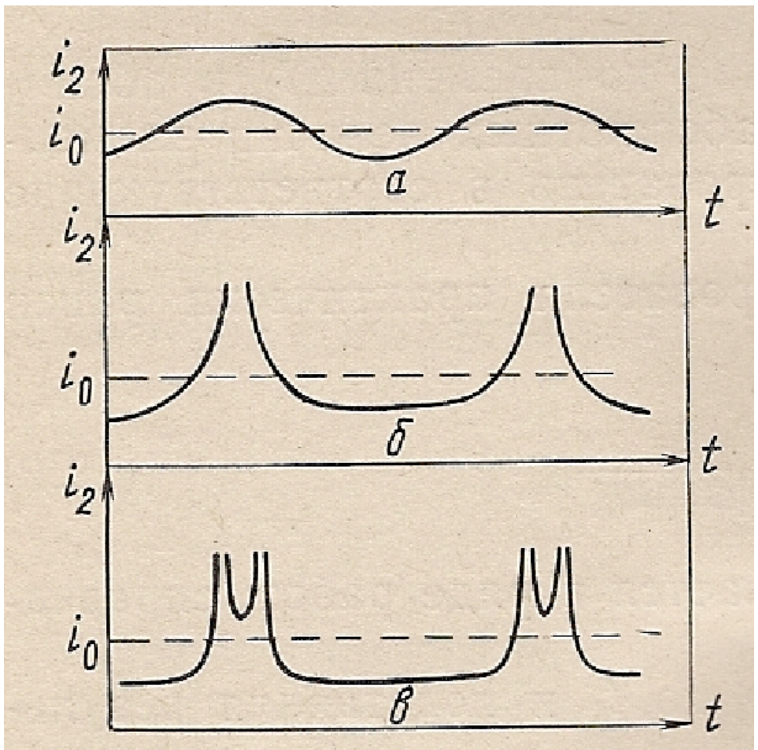
$$i_1 dt_1 = i_2 dt_2$$

Beam current in the second gap

$$i_2 = i_1 \frac{dt_1}{dt_2} = \frac{I}{\dfrac{dt_2}{dt_1}}$$

Beam current in the second gap as a function of RF phase
in the first gap and bunching parameter

$$i_2 = \frac{I}{1 - X\cos\omega t_1}$$



$X < 1$

$X = 1$

$X > 1$

Current in the second gap as a function of time.

Phase of arrival of particle into second gap

$$x = \omega t_2 - \theta = \omega t_1 - X \sin \omega t_1$$

Expansion of the current in the second gap in Fourier series

$$i_2(x) = A_o + \sum_{n=1}^{\infty} A_n \cos nx$$

Fourier coefficients

$$A_o = \frac{1}{\pi}\int_o^{\pi} i_2(x)\,dx \qquad A_n = \frac{2}{\pi}\int_o^{\pi} i_2(x)\cos nx\,dx$$

Differentiation of RF phase

$$dx = \omega dt_2$$

Constant in Fourier series

$$A_o = \frac{1}{\pi}\int_o^{\pi} I \frac{dt_1}{dt_2}\omega\, dt_2 = I$$

Other coefficients in Fourier series

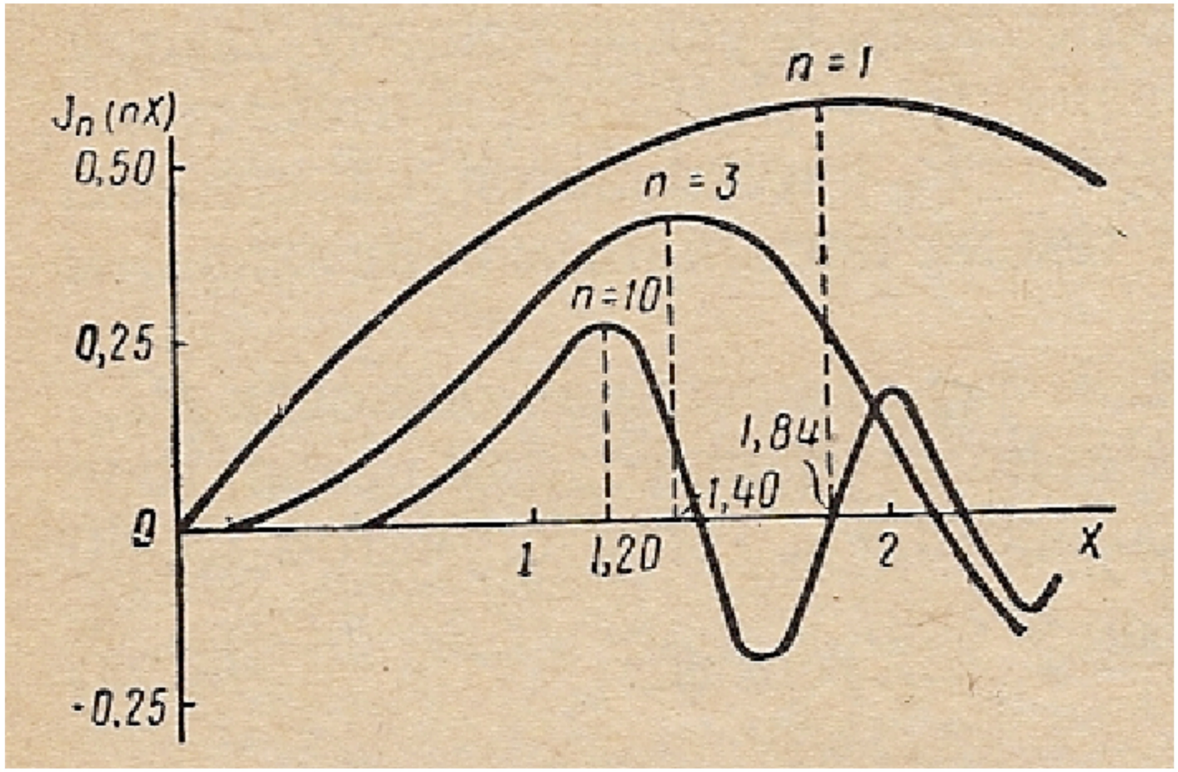
$$A_n = \frac{2I}{\pi}\int_o^{\pi} \cos(n\omega t_1 - nX \sin \omega t_1)\, d\omega t_1 = 2IJ_n(nX)$$

Bessel function (integral representation)

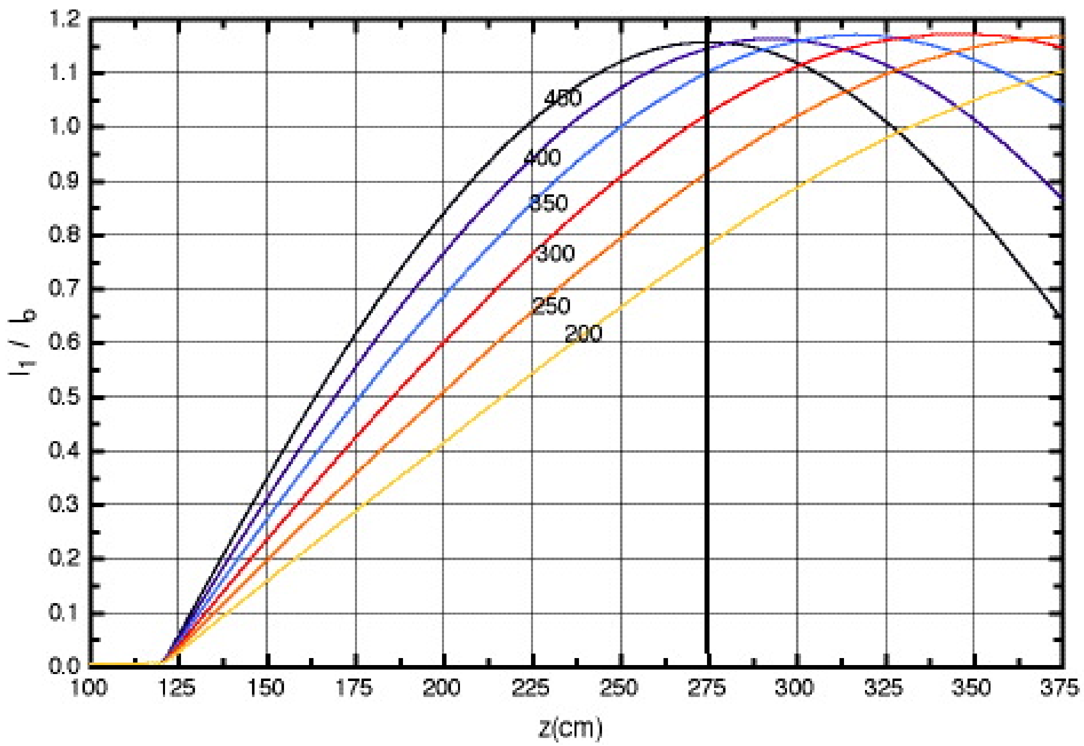
$$J_n(z) = \frac{1}{\pi}\int_o^{\pi} \cos(n\varphi - z \sin\varphi)\, d\varphi$$

Beam current in the second gap

$$i_2(x) = I + 2I\sum_{n=1}^{\infty} J_n(nX)\cos nx$$



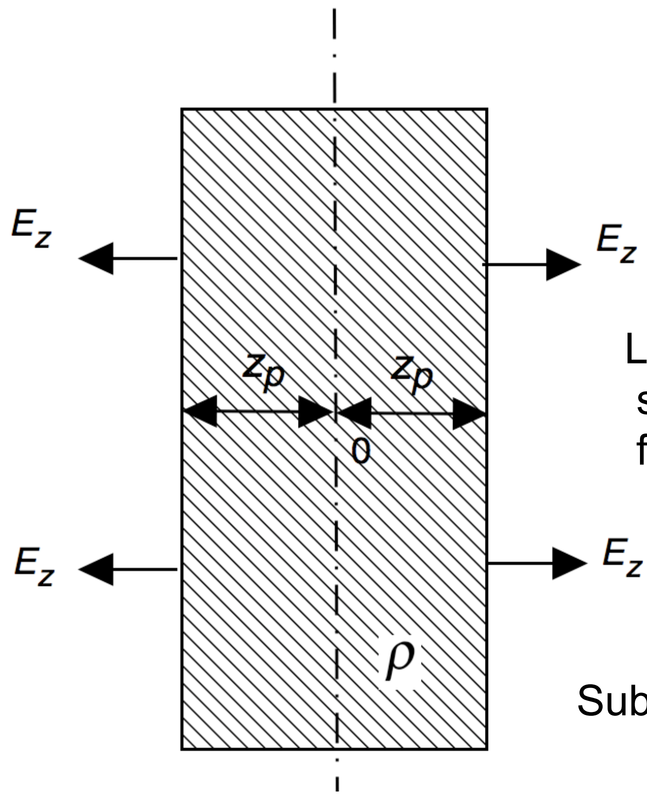
Bessel functions determine amplitude of the fist, third and tenth harmonics of induced current in two-resonator buncher.



The first harmonic of the induced beam current in the second gap as a function of z for different values of voltage at first gap. $\frac{I_1}{I} = 2J_1(X)$

The optimal value of bunching parameter is $X_{opt} = 1.84$.

Beam Bunching in Presence of Space Charge Forces



Gauss theorem $$2E_z = \frac{\rho}{\varepsilon_o} 2z_p$$

1D longitudinal space charge field $$E_z = \frac{\rho}{\varepsilon_o} z_p$$

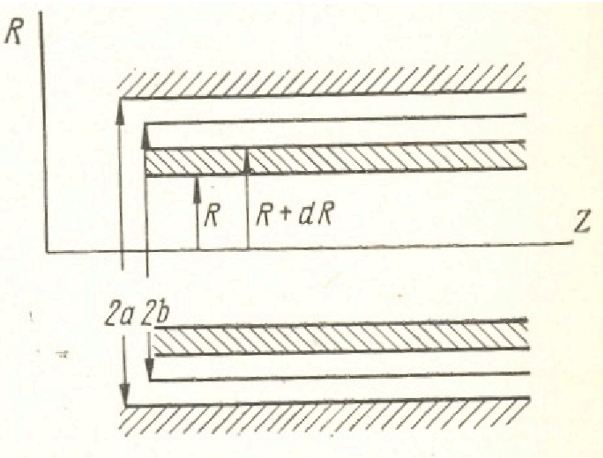
Longitudinal oscillation in presence of space charge field, E_z, and external field E_{ext} $$m\frac{d^2 z_p}{dt^2} = q(E_{ext} - E_z)$$

Substitution of space charge field gives: $$\frac{d^2 z_p}{dt^2} + \omega_p^2 z_p = \frac{q}{m} E_{ext}$$

Plasma frequency $$\omega_p = \sqrt{\frac{q\rho}{m\varepsilon_o}} = \frac{2c}{R}\sqrt{\frac{I}{I_c\beta}}$$

Space charge density of the beam $$\rho = \frac{I}{\pi R^2 \beta c}$$

Reduction of Beam Plasma Frequency in Presence of Conducting Tube



Reduced plasma frequency of the beam of radius *R* in the tube of radius *a* $\omega_q = \sqrt{F_p}\omega_p$

Plasma frequency reduction factor $F_p = 2.56 \dfrac{J_1^2(2.4\frac{R}{a})}{1+\dfrac{5.76}{(\frac{\omega a}{v_o})^2}}$

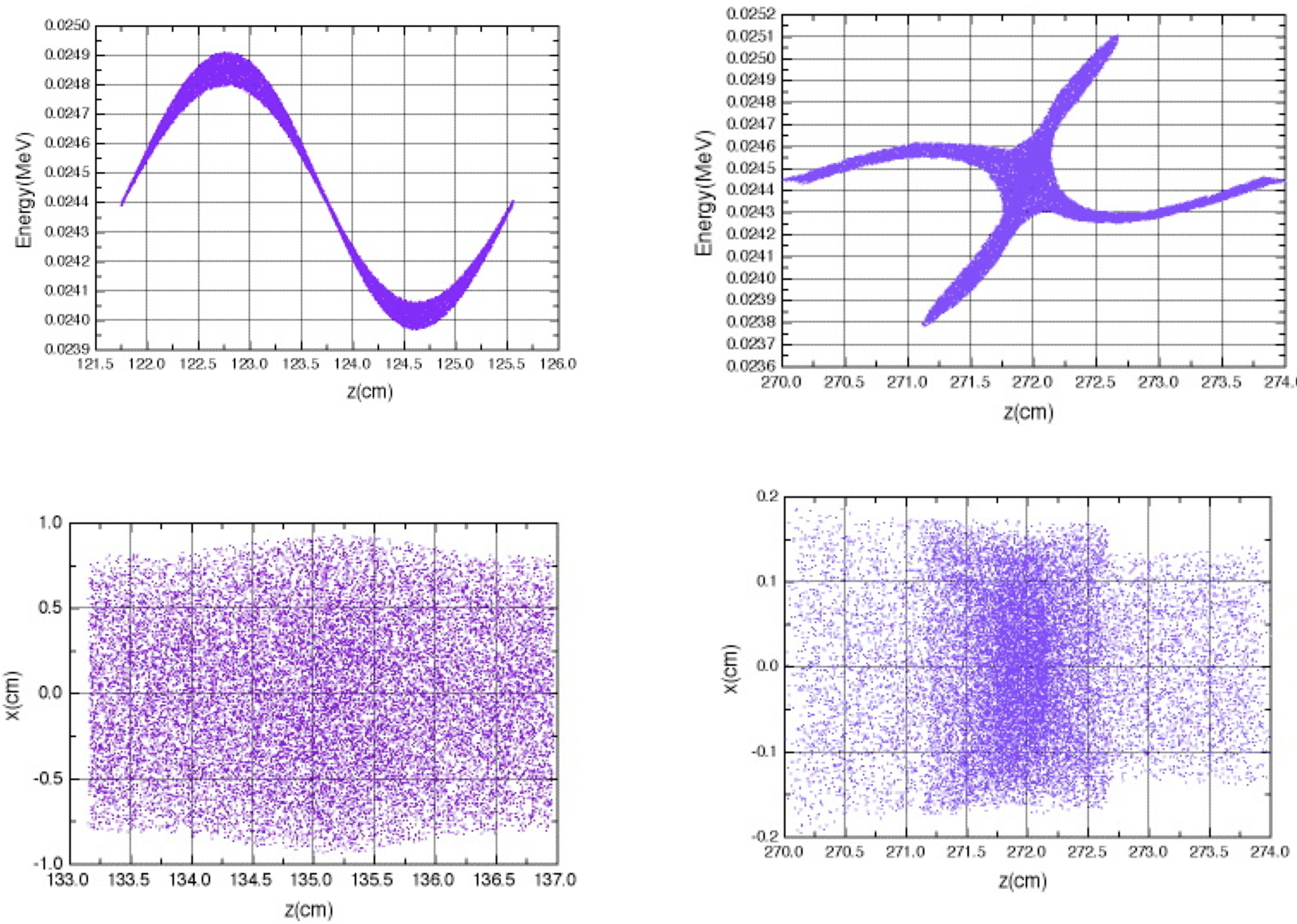
Longitudinal plasma oscillations in tube $\dfrac{d^2 z_p}{dt^2} + \omega_q^2 z_p = 0$

Longitudinal particle oscillations under space charge forces $z_p = B_o \sin\omega_q(t - t_1)$

Longitudinal velocity of particle oscillations under space charge forces: $\dfrac{dz_p}{dt} = B_o\omega_q \cos\omega_q(t - t_1)$

Constant B_o is defined from initial conditions for particle velocity after first RF gap: $\dfrac{dz_p}{dt}(t_1) = B_o\omega_q = v_1 \sin\omega t_1$

$$B_o = \frac{v_1}{\omega_q}\sin\omega t_1$$



Effect of space charge repulsion on beam bunching.

Finally, particle oscillations under space charge forces in the moving system

$$z_p = \frac{v_1}{\omega_q}\sin\omega_q(t-t_1)\sin\omega t_1$$

Particle drift

$$z = v_o(t_2-t_1)+z_p$$

$$z = v_o(t_2-t_1)+\frac{v_1}{\omega_q}\sin\omega_q(t_2-t_1)\sin\omega t_1$$

Multiply by ω

$$\frac{\omega z}{v_o} = \omega t_2-\omega t_1+\frac{\omega v_1}{\omega_q v_o}\sin\omega_q(t_2-t_1)\sin\omega t_1$$

RF phase in the second gap

$$\omega t_2-\theta = \omega t_1 - X'\sin\omega t_1$$

Modified bunching parameter in presence of space charge forces

$$X' = \frac{\omega v_1}{\omega_q v_o}\sin\omega_q(t_2-t_1)$$

$$\boxed{X' = X\frac{\sin(\omega_q \frac{z}{v_o})}{\omega_q \frac{z}{v_o}}}$$

Condition for maximum bunching:

$$\sin(\omega_q\frac{z}{v_o})=1 \qquad \omega_q\frac{z}{v_o}=\frac{\pi}{2}$$

$$X'_{opt} = \frac{U_1 M_1}{2U_o}\left(\frac{\omega}{\omega_q}\right) \qquad \frac{I_1}{I} = 2J_1(X'_{opt})$$

Bunched Beam in RF Field: Problems with Ellipsoidal Bunch Model

1. There is no 6D distribution function which results in 3D uniformly charged ellipsoid in linear field (see F.Sacherer Thesis, 1968).

2. RF field across separatrix is essentially non-linear.

3. There are special cases when ellipsoid is a self-consistent solution.

APPENDICES

A. The Nonexistence of Uniformly Charged Three-Dimensional Beams

We are given an ensemble of three-dimensional harmonic oscillators with the Hamiltonian

$$H(\vec{p}, \vec{q}) = p^2 + q^2 , \quad 0 \leqslant H \leqslant 1 . \qquad (A1)$$

Because of the inequality, the accessible region in phase space is a six-dimensional unit sphere; in configuration space it is a 3-sphere. Does there exist a spherically symmetric distribution $f(p^2 + q^2)$ that has a uniform projection onto the 3-sphere? The following necessary condition for the existence of such a distribution has been found by Maurice Neuman.

Theorem: The spherically symmetric distribution $f(p^2 + q^2)$ does not exist if its projection $\rho(q^2) = \int f(p^2 + q^2) d^3p$ violates any of the following inequalities:

$$\rho(\tau) \begin{cases} \leqslant \dfrac{4}{\pi^2}\left(\dfrac{3}{4\tau}\right)^{3/2} , & 0 \leqslant \tau \leqslant \dfrac{3}{4} , \\ \leqslant \dfrac{8}{\pi^2}\sqrt{1-\tau} , & \dfrac{3}{4} \leqslant \tau \leqslant 1 . \end{cases} \qquad (A2)$$

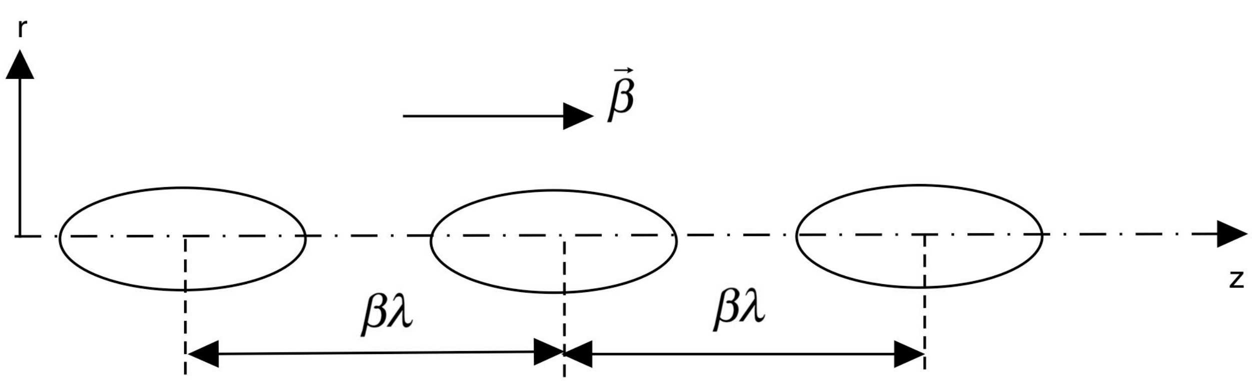
The maximum permissible value of $\rho(\tau)$, which corresponds to the equal sign, is shown in Fig. (A1). An immediate consequence of this theorem is the nonexistence of a spherically symmetric distribution $f(p^2 + q^2)$ with a uniform projection, $\rho(q^2) = \text{constant}$.

Space Charge Dominated Bunched Beam in RF Field*

Assumptions

1. Beam is accelerated in traveling wave with constant amplitude *E*
2. Beam is bunched at RF frequency $\omega = \frac{2\pi c}{\lambda}$. Particles between bunches are removed.
3. Focusing is provided by a continuous z-independent focusing structure
4. Beam is matched with the structure, i.e. there are no envelope oscillations (both transverse and longitudinal)

What is the self-consistent particle distribution within the bunch and what is the limited beam current?



Sequence of bunches in RF field.

* Y.B., NIM-A 483 (2002), 611-628.

Equation for Field of Moving Bunch

The space charge density distribution of a moving bunched beam has the form $\rho = \rho\,(x, y, z - v_s t)$. The moving bunch creates an electromagnetic field with a scalar potential $U_b = U_b\,(x, y, z - v_s t)$ and a vector potential $\vec{A}_b = \vec{A}_b\,(x, y, z - v_s t)$, which obey the wave equations:

$$\Delta U_b - \frac{1}{c^2}\frac{\partial^2 U_b}{\partial t^2} = -\frac{\rho}{\varepsilon_o}, \tag{5.50}$$

$$\Delta \vec{A}_b - \frac{1}{c^2}\frac{\partial^2 \vec{A}_b}{\partial t^2} = -\mu_o \vec{j}, \tag{5.51}$$

where $\vec{j} = \rho\,\vec{v}_s$ is the current density of the beam. The current density has only longitudinal component

$$j_x = j_y = 0, \qquad j_z = v_s\,\rho\,(x, y, z - v_s t)\,, \tag{5.52}$$

and, therefore, the vector potential has only a longitudinal component A_z.

In a moving coordinate system where particles are static, the vector potential of the beam is zero, $\vec{A} = 0$. According to the Lorentz transformation, the longitudinal component of the vector potential in the laboratory system is $A_z = \beta_s\,U_b / c$ while transverse components $A_x = 0$, $A_y = 0$. Therefore, to find solution of the problem it suffice to solve only equation for the scalar potential (5.50). Substitution of the value A_z into the wave equation (5.51) gives the equation for the scalar potential:

$$\frac{\partial^2 U_b}{\partial x^2} + \frac{\partial^2 U_b}{\partial y^2} + \frac{\partial^2 U_b}{\gamma^2 \partial \zeta^2} = -\frac{1}{\varepsilon_o}\,\rho\,(x, y, \zeta). \tag{5.53}$$

Self - Consistent Problem for Bunched Beam

Equation (5.53) has to be solved together with the Vlasov equation for the beam distribution function:

$$\frac{df}{dt} = \frac{1}{m\gamma}\left(\frac{\partial f}{\partial x}p_x + \frac{\partial f}{\partial y}p_y + \frac{\partial f}{\partial \zeta}p_z\right) - q\left(\frac{\partial f}{\partial p_x}\frac{\partial U}{\partial x} + \frac{\partial f}{\partial p_y}\frac{\partial U}{\partial y} + \frac{\partial f}{\partial p_z}\frac{\partial U}{\partial \zeta}\right) = 0 \qquad (5.54)$$

where $U = U_{ext} + \gamma^{-2} U_b$ is a total potential of the structure. Eqs (5.53), (5.54) define the self-consistent distribution of a stationary beam which acts on itself in such a way, that this distribution is conserved.

The general approach to find a stationary, self-consistent beam distribution function is to represent it as a function of Hamiltonian $f = f(H)$ and then to solve Poisson's equation. Because the Hamiltonian is a constant of motion for a stationary process, any function of Hamiltonian is also a constant of motion which automatically obeys Vlasov's equation. A convenient way is to use an exponential function $f = f_o \exp(-H / H_o)$:

$$f = f_o \exp\left(-\frac{p_x^2 + p_y^2}{2\, m\, \gamma H_o} - \frac{p_z^2}{2\, m\, \gamma^3 H_o} - q\frac{U_{ext} + U_b \gamma^{-2}}{H_o}\right). \qquad (5.55)$$

Beam Equipartitioning in RF field

Let us rewrite the distribution function, Eq. (5.55)

$$f = f_o \exp\left(-2\frac{p_x^2 + p_y^2}{p_t^2} - 2\frac{p_z^2}{p_l^2} - q\frac{U_{ext} + U_b\gamma^{-2}}{H_o}\right), \qquad (5.56)$$

where $p_t = 2\sqrt{<p_x^2>} = 2\sqrt{<p_y^2>}$ and $p_l = 2\sqrt{<p_z^2>}$ are double root -mean-square (rms) beam sizes in phase space. Transverse, ε_t, and longitudinal, ε_l, rms beam emittances are:

$$\varepsilon_t = 2\frac{p_t}{mc}\sqrt{<x^2>} = 2\frac{p_t}{mc}\sqrt{<y^2>}, \qquad (5.57)$$

$$\varepsilon_l = 2\frac{p_l}{mc}\sqrt{<\zeta^2>}. \qquad (5.58)$$

The value of H_o can be expressed as a function of the beam parameters:

$$16 \cdot H_o = \frac{mc^2}{\gamma}\frac{\varepsilon_t^2}{<x^2>} = \frac{mc^2}{\gamma}\frac{\varepsilon_t^2}{<y^2>} = \frac{mc^2}{\gamma^3}\frac{\varepsilon_l^2}{<\zeta^2>}. \qquad (5.59)$$

Equation (5.59) can be rewritten as

$$\frac{\varepsilon_t}{R} = \frac{\varepsilon_l}{\gamma\, l} \qquad (5.60)$$

where $R = 2\sqrt{<x^2>}$ is a beam radius and $l = 2\sqrt{<\zeta^2>}$ is a half-length of the bunch.

Self-Consistent Solution for Beam Distribution

The first approximation to self-consistent space charge dominated beam potential is:

$$V_b = - \frac{\gamma^2}{1+\delta} V_{ext}$$

where parameter $\delta \approx \frac{1}{b_\varphi k} << 1$

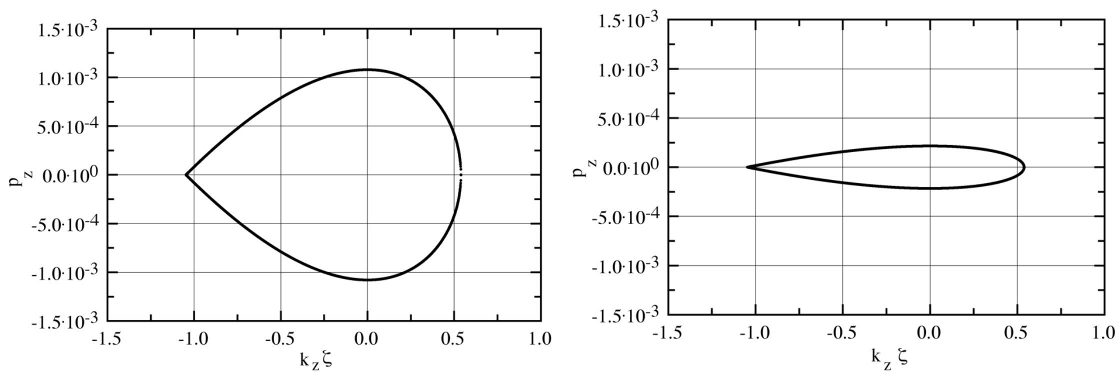
and b_φ is a dimensionless beam brightness of the bunched beam: $b_\varphi = \frac{2}{\beta\gamma} \frac{I}{BI_c} \frac{R^2}{\varepsilon_t^2}$

The Hamiltonian corresponding to the self-consistent bunch distribution is as follows:

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\, (\frac{\delta}{1+\delta})\, U_{ext}.$$

Equation (5.88) indicates that in the presence of an intense, bright bunched beam ($\delta << 1$) the stationary longitudinal phase space of the beam becomes narrow in momentum spread, while the phase width of the distribution remains the same in the first approximation.

Low brightness beam, $b << 1$ High brightness beam, $b >> 1$



Analogy with Plasma Physics: Debye Screening

screening. If a positive test charge of magnitude Ze is placed in a plasma, it attracts electons and repels ions in such a way that its Coulomb electrostatic potential $\phi_c \approx Ze/4\pi\varepsilon_0 r$ is attenuated at distances beyond a Debye length. To calculate this effect, we solve for the potential $\phi(r)$ generated by such a test charge. Assuming the plasma to be in thermal equilibrium, the distribution functions of electrons and ions are of the Maxwell–Boltzmann form

$$f(\boldsymbol{x}, \boldsymbol{v}) = n_0 \exp\left(-\frac{mv^2}{2k_B T} + \frac{e_j \phi}{k_B T}\right), \tag{1.8.1}$$

and the densities are $n_j(r) = n_0 \exp(e_j \phi(r)/k_B T)$. Here $\phi(r)$ is the potential generated by the test charge, which is as yet unknown. Since this potential must satisfy Poisson's equation

$$\nabla^2 \phi = \frac{1}{\varepsilon_0} \rho(r), \tag{1.8.2}$$

with the charge density $\rho(r) = \sum_j e_j n_j(r)$, it follows that, assuming spherical symmetry, ϕ satisfies the equation

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} r^2 \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{2n_0 e^2}{\varepsilon_0 k_B T} \phi; \tag{1.8.3}$$

here we have assumed that the potential is small enough that $e\phi/k_B T \ll 1$.

Taking the solution of Eq. (1.8.3) which vanishes as $r \to \infty$, we obtain

$$\phi = \frac{A}{r} \exp(-r/\lambda_D), \tag{1.8.4}$$

where $\lambda_D \equiv (\varepsilon_0 kT/2n_0 e^2)^{1/2}$ is known as the Debye length, and A is not yet determined. To evaluate the constant A, we must match the potential to the 'bare' Coulomb potential of the test charge, $\phi_c = Ze/4\pi\varepsilon_0 r$, at a distance r from the charge which is small compared to the average interparticle distance $n_0^{-1/3}$. The result is that $A = Ze/4\pi\varepsilon_0$, provided that $n_0^{-1/3} \ll \lambda_D$. Eq. (1.8.4) then shows that, at distances greater than a Debye length, the potential of a test charge in a plasma is exponentially attenuated below the value it would have in a vacuum. This cutoff of the potential has important implications for the collisional events in a plasma,

Kapchinsky Model for Self-Consistent Bunched Beam

<< 1. Restricting ourselves in the expansion of a modified Bessel function to the first two terms

$$I_0\left(\frac{\omega r}{\gamma v_s}\right) \approx 1 + \frac{\omega^2}{4\gamma^2 v_s^2} r^2,$$

we can write potential function (4.7) in the form

$$V(x, y, \zeta) = \frac{e v_s E}{\omega}\left[\sin\left(\varphi_s - \frac{\omega}{v_s}\zeta\right) + \frac{\omega\zeta}{v_s}\cos\varphi_s\right] + \frac{m_0\gamma}{2}\left[\Omega_r^2 + \frac{e\omega E}{2m_0\gamma^3 v_s}\sin\left(\varphi_s - \frac{\omega}{v_s}\zeta\right)\right] r^2.$$

By ignoring the dependence of the defocusing force produced by the accelerating wave on the variable component of the particle phase, we can represent the potential function as a sum of two terms $V(x, y, \zeta) = V_z(\zeta) + V_r(x, y)$. The

first term

$$V_z(\zeta) = \frac{e v_s E}{\omega}\left[\sin\left(\varphi_s - \frac{\omega}{v_s}\zeta\right) + \frac{\omega\zeta}{v_s}\cos\varphi_s\right], \quad (4.13)$$

which depends only on the longitudinal coordinate of the particle, coincides (to within a constant factor) with potential function (1.41). The second term

$$V_r(x, y) = (m_0\gamma/2)\,[\Omega_r^2 - e\omega E\,|\sin\varphi_s|/2m_0\gamma^3 v_s]\, r^2, \quad (4.14)$$

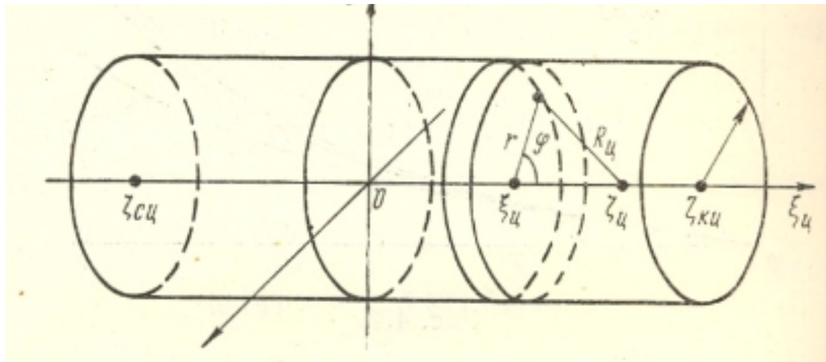
which depends only on the transverse coordinates, is the potential function for the equilibrium particle in a "smoothed out" external field. In Section 3.1 we showed by using a

With this simplifying assumption, the Coulomb potential of the bunch can be represented as a sum of two independent functions $U_C(x, y, \zeta) = U_z(\zeta) + U_r(x, y)$. Because of the axial symmetry of the fields, the potential U_r is a function of only the radius r. The two independent integrals of motion can be separated by using the simplifying assumptions discussed above;

$$H_z = \frac{p_z^2}{2m_0\gamma^3} + V_z(\zeta) + (e/\gamma^2)\, U_z(\zeta); \quad (4.15)$$

$$H_r = [(p_x^2 + p_y^2)/2m_0\gamma] + V_r(r) + (e/\gamma^2)\, U_r(r). \quad (4.16)$$

Representation of the Bunch as a Uniformly-Charged Cylinder with Variable Density Along *z*



Transverse distribution

326 I. M. KAPCHINSKIY

The microcanonical phase-density distribution $f_1(H_r) = \delta(H_r - H_1)$ can be used in four-dimensional transverse-oscillation phase space. In this case,

$$\rho(r, \zeta) = en_0 \int_{-\infty}^{\infty} f_2(H_z)\, dp_z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(H_r - H_1)\, dp_x\, dp_y.$$

Longitudinal distribution

side the separatrix. Specifically, we assume that the phase density on the ψ, p_ψ plane inside the separatrix is constant. Since $H_z < H_c$ for the phase trajectories inside the separatrix and $H_z > H_c$ for the phase trajectories outside it, we can write

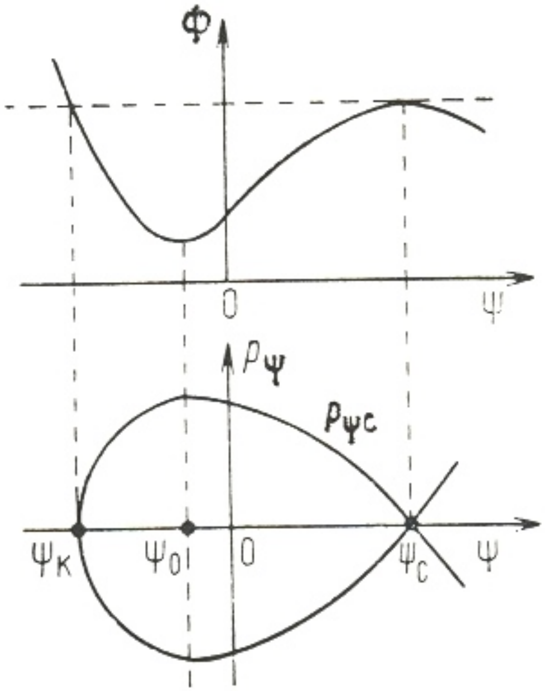
$$f_2(H_z) = \begin{cases} 1 \text{ for } H_z \leqslant H_c; \\ 0 \text{ for } H_z > H_c. \end{cases} \qquad (4.26)$$

Although the space-charge density in each beam cross section is constant, it nonetheless depends on the longitudinal coordinate. A bunch can be represented as a circular cylinder of finite length. Since the charge density inside the cylinder depends only on the longitudinal coordinate, the cylindrical bunch has flat end-faces. The cyl-

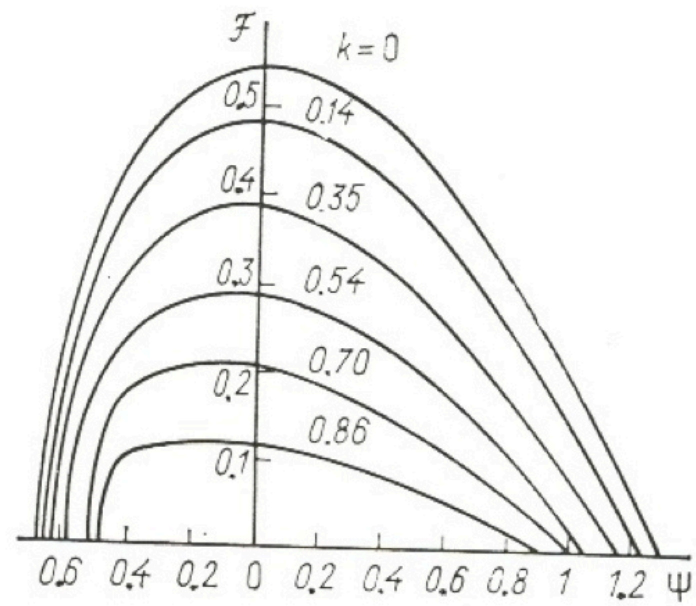
The law governing the charge-density distribution along the longitudinal axis of the bunch duplicates the behavior of the separatrix. The maximum charge density of a cylin-

Separatrix as a Function of Beam Current

Analysis based on Kapchinsky's model for beam distribution indicates that synchronous phase is shifted in space charge dominated beam and phase width of the bunch decreases with current but much slower than the vertical size of the separatrix.

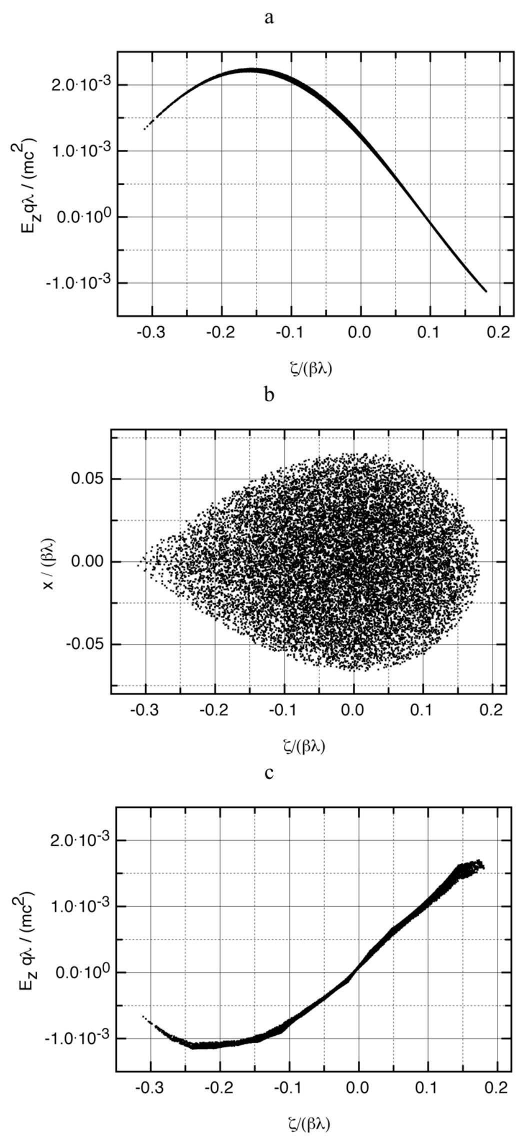


The potential function and separatrix of the beam with high space-charge density (from Kapchinsky, 1985).



The separatrix shape for different values of space charge parameter (from Kapchinsky, 1985).

Stationary Bunch Profile



Stationary self-consistent particle distribution in RF field, $\varphi_s = -60^o$, C=3.8: (a) RF field, (b) particle distribution, (c) space charge field of the beam.

Stationary bunch profile

$$I_o\left(\frac{k_z R}{\gamma}\right) sin(\varphi_s - k_z \zeta) + sin\varphi_s - (2\varphi_s - k_z\zeta) cos\varphi_s + C(k_z R)^2 = 0$$

Space charge density of stationary bunch is close to constant in space charge limit

$$\rho(r,\zeta) \approx 2\,\frac{\gamma^2}{1+\delta}\,G_t\,\varepsilon_o.$$

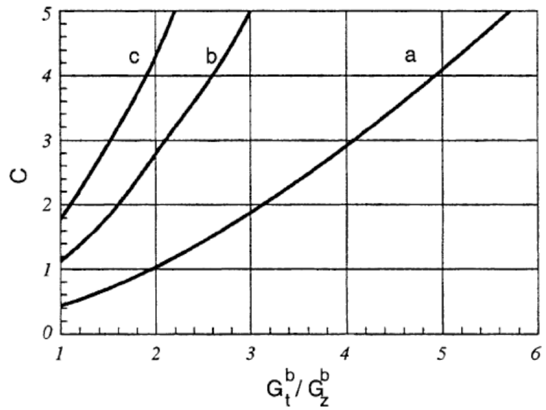
Bunch Profile as a Function of Accelerator Parameters

Parameter C can be expressed as a function of ratio of effective transverse gradient:

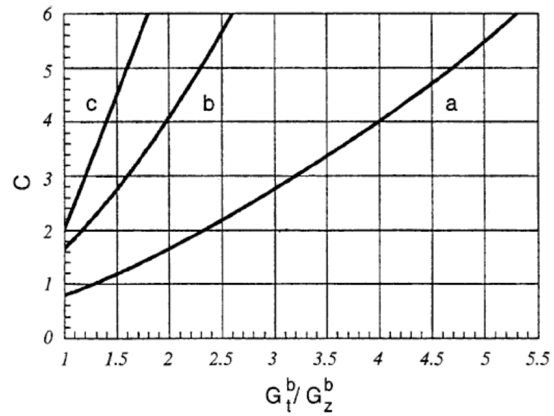
$$G_{t,\,eff} = G_t\left(1 - \frac{G_z}{2\,\gamma^2 G_t}\right)$$

and longitudinal gradient

$$G_z = 2\pi \frac{E\,|\sin \varphi_s|}{\beta\lambda}$$



Coefficient C in bunch shape for φ_s = -30^o as a function of ratio of transverse and longitudinal gradients of space charge field of the beam: a) γ = 1, b) γ= 3, c) γ= 6.



Coefficient C in bunch shape for φ_s = -60^o as a function of ratio of transverse and longitudinal gradients of space charge field of the beam: a) γ= 1, b) γ= 3, c) γ= 6.

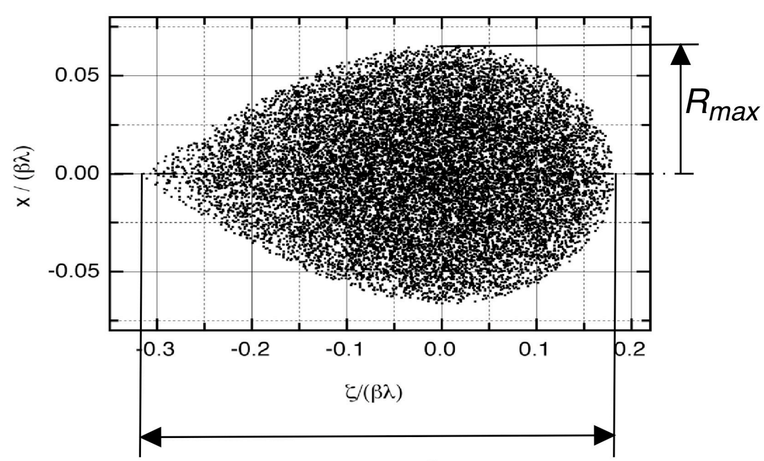
Transverse and Longitudinal Bunch Sizes

For a long bunch, $\beta\lambda >> R_{max}$, the Bessel function can be approximated as $I_o(\chi) \approx 1 + \chi^2/4$, and bunch boundary is given by:

$$R(\zeta) = \frac{\beta\lambda}{2\pi}\sqrt{\frac{(2\varphi_s - k_z\zeta)\cos\varphi_s - \sin\varphi_s - \sin(\varphi_s - k_z\zeta)}{C + \frac{1}{4\gamma^2}\sin(\varphi_s - k_z\zeta)}}. \qquad (5.96)$$

Transverse bunch size, R_{max}, is determined from the equation $\partial R(\zeta)/\partial\zeta = 0$:

$$R_{max} = \frac{\beta\lambda}{2\pi}\sqrt{\frac{2(\varphi_s\cos\varphi_s - \sin\varphi_s)}{C + \frac{1}{4\gamma^2}\sin\varphi_s}}. \qquad (5.97)$$

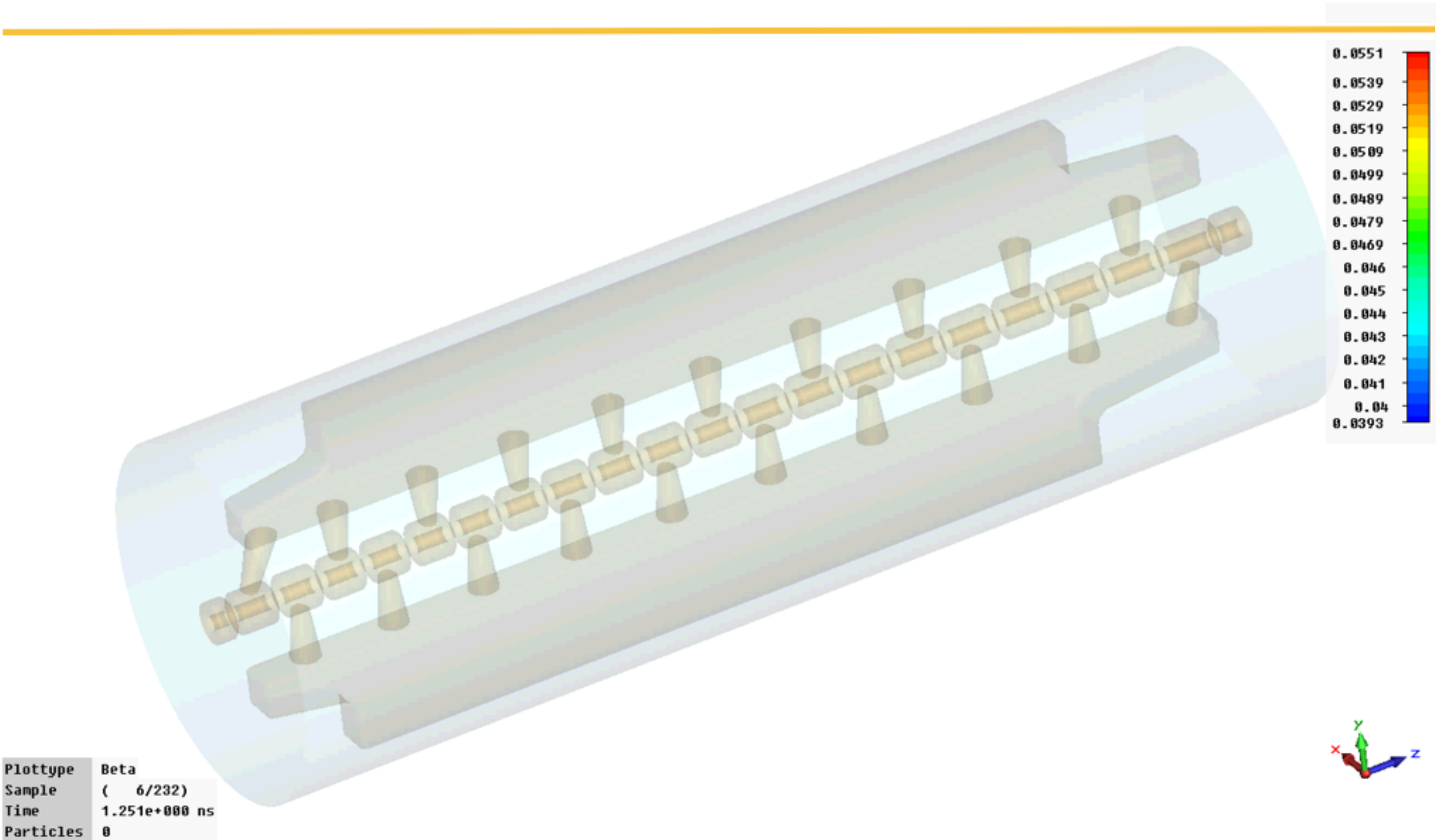


$$l_b \approx \beta\lambda\frac{3\varphi_s}{2\pi}$$

The ratio of transverse to longitudinal bunch sizes for a given value of synchronous phase, φ_s, is:

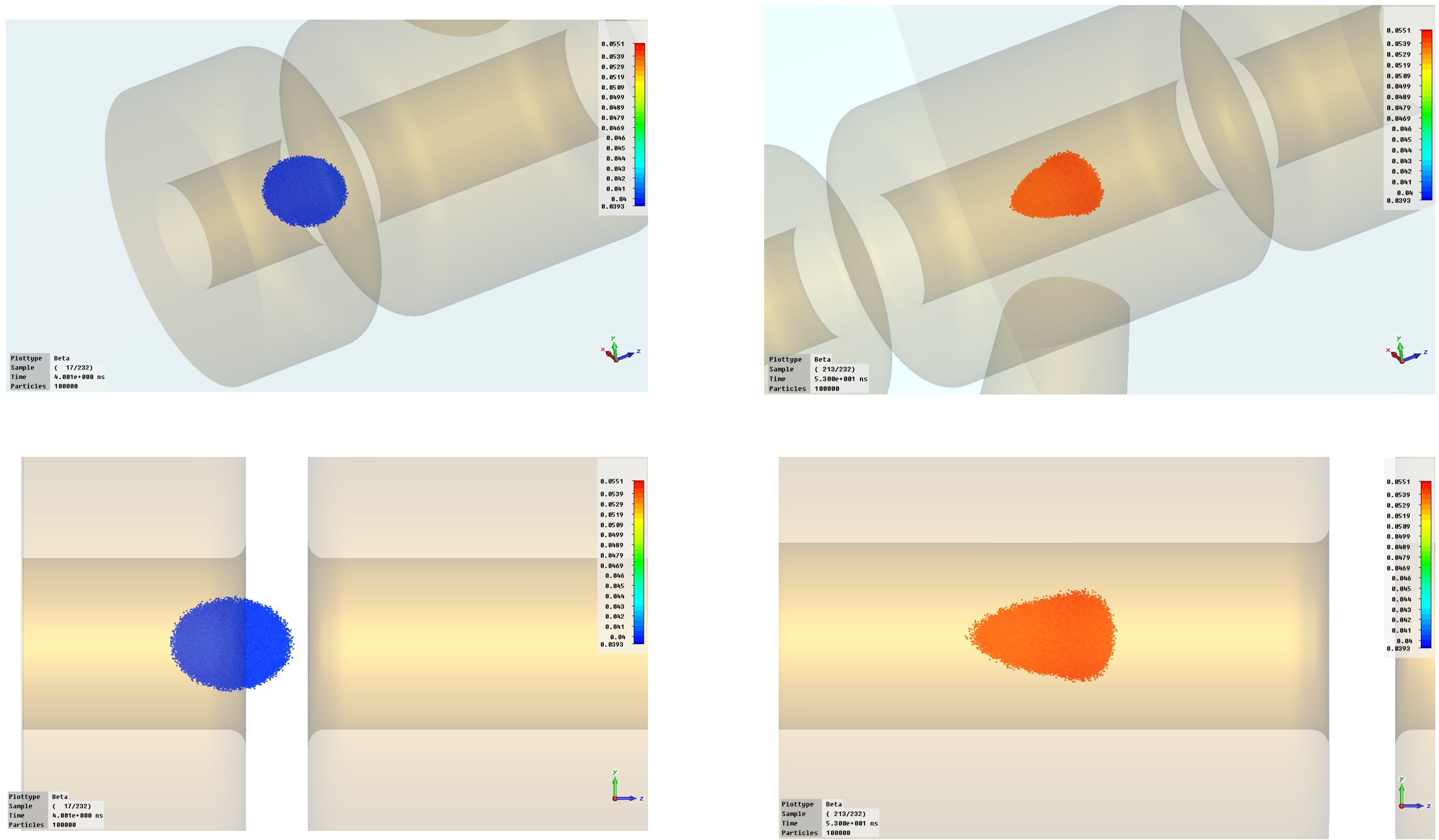
$$\frac{R_{max}}{l_b} = \frac{1}{3|\varphi_s|}\sqrt{\frac{2(\varphi_s\cos\varphi_s - \sin\varphi_s)}{C + \frac{1}{4\gamma^2}\sin\varphi_s}}. \qquad (5.98)$$

Bunch Evolution in RF field



Dynamics of elliptical beam injected into RF linac (courtesy of Sergey Kurennoy).

Initial and final bunch in RF field



(Left) initial and (right) final beam distribution in RF field. (Courtesy of Sergey Kurennoy.)

Maximum Beam Current

The volume of the bunch is defined by $V = \pi \int_{z_{min}}^{z_{max}} R^2(\zeta)\, d\zeta = \frac{\beta\lambda}{2} \int_{\varphi_s}^{-2\varphi_s} R^2(\psi)\, d\psi.$

For a long bunch, $\beta\lambda >> R_{max}$, the bunch volume:

$$V = \frac{(\beta\lambda)^3}{8\pi^2 C}\left(3\varphi_s \sin\varphi_s - \frac{9}{2}\varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s\right)$$

The total charge of the bunch is $Q = \rho \cdot V$ and the beam current, $I = \frac{Q}{2\pi}\omega$, is

$$I_{max} = I_c\left(\frac{\beta^3\gamma^2}{16\pi^3 C}\right)\left(\frac{G_t q\lambda^2}{mc^2}\right)\left[3\varphi_s \sin\varphi_s - \frac{9}{2}\varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s\right]$$

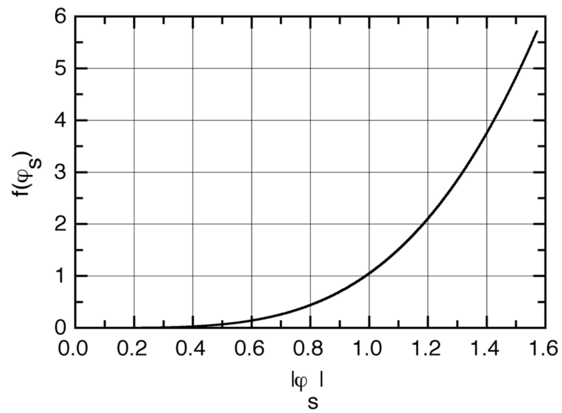
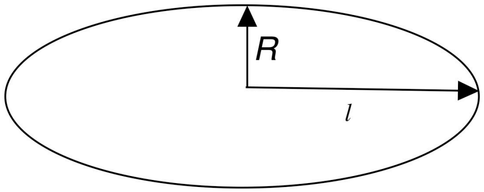


Fig. 5.7. Function $f(\varphi_s) = 3\varphi_s \sin\varphi_s - \frac{9}{2}\varphi_s^2 \cos\varphi_s + \cos\varphi_s - \cos 2\varphi_s$ in maximum beam current, Eq. (5.118).

Comparison with Ellipsoidal Model

Potential of a stationary bunch in the vicinity of the synchronous particle:

$$U_b = -\frac{\rho}{2\varepsilon_o G_t}\left(G_z \frac{\zeta^2}{2} + \frac{G_{t,\,eff}}{2} r^2\right) \qquad (5.121)$$



Potential of a uniformly populated ellipsoid: $U_b = -\frac{\rho}{2\varepsilon_o}\left[M\gamma^2\zeta^2 + \frac{1-M}{2} r^2\right]$ (5.124)

Where M is the function of semi-axes of an ellipsoid:

$$M(R, \gamma l) = \frac{R^2 \gamma l}{2}\int_o^{\infty} \frac{ds}{(R^2+s)\,(\gamma^2 l^2 + s)^{3/2}} \qquad (5.123)$$

Comparison gives

$$\boxed{M(R,\gamma l) = \frac{G_z}{2\gamma^2 G_t}} \qquad (5.125)$$

Volume of an ellipsoid is $V = (4/3)\pi\, R^2 l$,

Maximum bunched beam current, $I_{max} = \rho V\omega/(2\pi)$, which can be carried by an ellipsoid with space charge density $\rho = 2\gamma^2 G_t \varepsilon_o$

$$\boxed{I_{\max} = I_c \frac{2}{3}\gamma^2 \left(\frac{R^2 l}{\lambda^3}\right)\left(\frac{G_t q\lambda^2}{mc^2}\right)} \qquad (5.126)$$

Let us show that this expression give both transverse and longitudinal current limits.

Transverse Beam Current Limit

Zero-current phase advance, σ_o, of betatron oscillations per period $S = N\beta\lambda$ of a pure focusing structure (without RF field):

$$\sigma_o = \sqrt{\frac{q\,G_t}{m\,\gamma}}\;\frac{S}{\beta c} \qquad (5.127)$$

zero-current phase advance per period, $\sigma_{o,t}$, including both the focusing and RF defocusing term

$$\sigma_{o,t}^2 = \sigma_o^2\,(1 - M) \qquad (5.128)$$

The phase width of the bunch is approximately taken as $2\varphi_s$ and, therefore, half of the bunch length

$$l = \beta\lambda\varphi_s/(2\pi) \qquad (5.129)$$

Substitution into $I_{max} = I_c\,\frac{2}{3}\,\gamma^2(\frac{R^2 l}{\lambda^3})\,(\frac{G_t\,q\,\lambda^2}{m\,c^2})$ gives for the current limit

$$I_{max} = \frac{4}{3}\,\frac{mc^2}{Z_o\,q}\,\beta\gamma^3\,\frac{\varphi_s\,\sigma_{ot}^2}{(1 - M)\,N^2}\,(\frac{R}{\lambda})^2 \qquad (5.130)$$

where $Z_o = (c\varepsilon_o)^{-1} = 376.73\ \Omega$ is the impedance of free space. This is the well-known transverse current limit.

Longitudinal Beam Current Limit

Substitution of $M(R, \gamma l) = \frac{G_z}{2\gamma^2 G_t}$, $G_z = 2\pi \frac{E\,|\sin\varphi_s|}{\beta\lambda}$ into $I_{max} = I_c \frac{2}{3}\gamma^2 (\frac{R^2 l}{\lambda^3}) (\frac{G_t q \lambda^2}{m c^2})$ gives for current limit:

$$I_{max} = \frac{8\pi^2}{3Z_o} \frac{E \sin\varphi_s}{\beta M} \frac{R^2 l}{\lambda^2}, \quad (5.131)$$

which is the well-known expression for longitudinal current limit in a RF field.

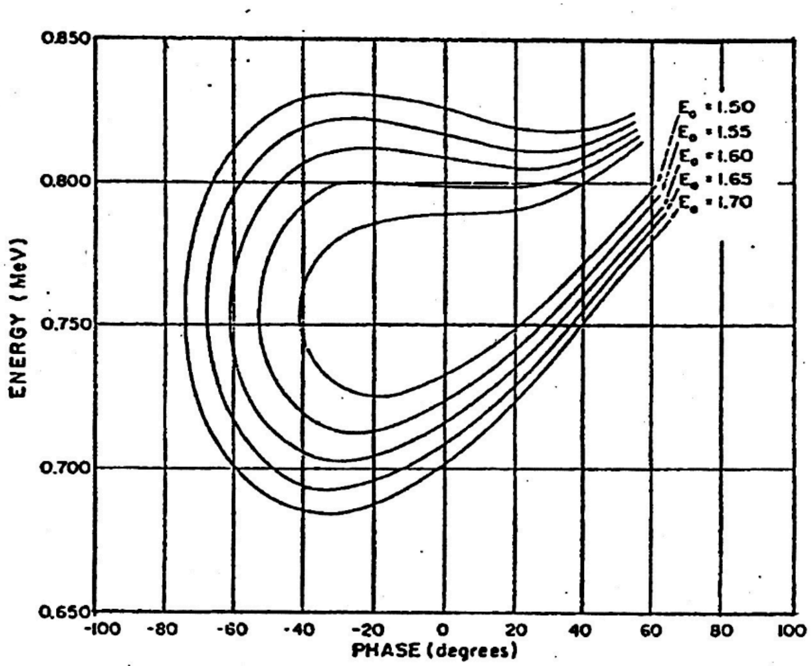
Usually the parameter M can be approximated as $M \approx R/(3\gamma l)$. With that approximation the longitudinal current limit is:

$$I_{max} = \frac{2\,\beta\gamma}{Z_o} E\,\varphi_s^2\,|\sin\varphi_s|\,R. \quad (5.132)$$

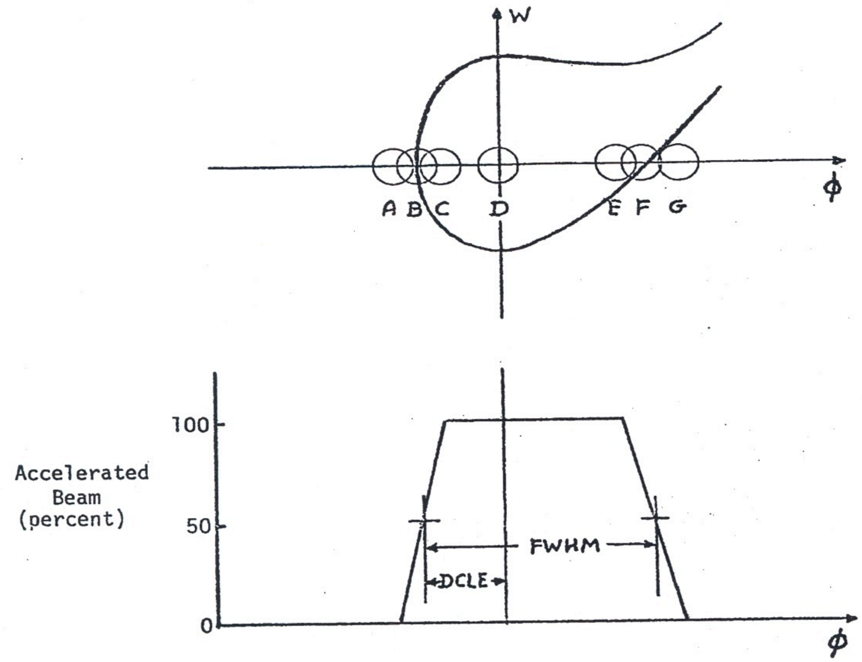
For small absolute values of synchronous phase one can assume $|\sin\varphi_s| \approx |\varphi_s|$, and the current limit, Eq. (5.132), is proportional to the cube of synchronous phase which is consistent with previous derivations.

Approximation of the bunched beam by an uniformly populated ellipsoid is valid for small bunches, $R << \beta_s\lambda$, $l << \beta_s\lambda$, while more general analysis results in a bunch shape, described above.

Phase Scans to Set the Phase and Amplitude of RF Linac

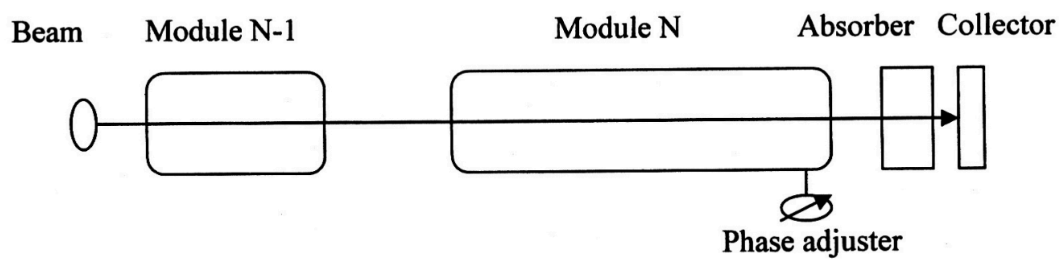


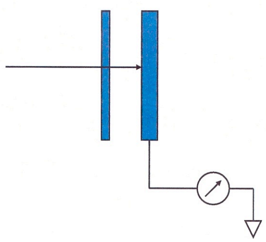
Longitudinal acceptance of RF linac for 5 different average axial field amplitudes.



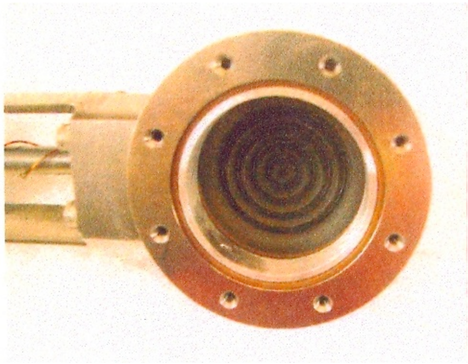
Accelerated beam as a function of beam phase

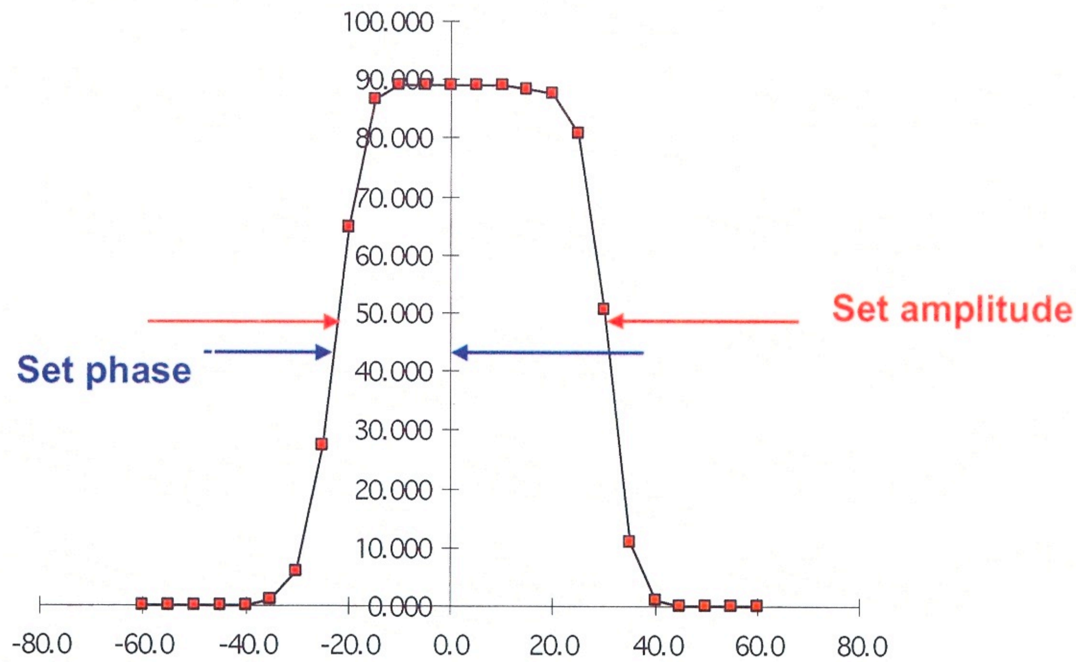
Phase Scans to Set the Phase and Amplitude of RF Linac (cont.)

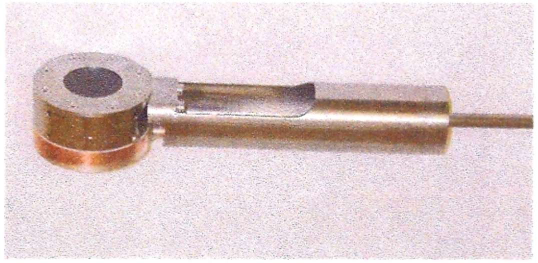




Schematic of the phase scan measurement setup

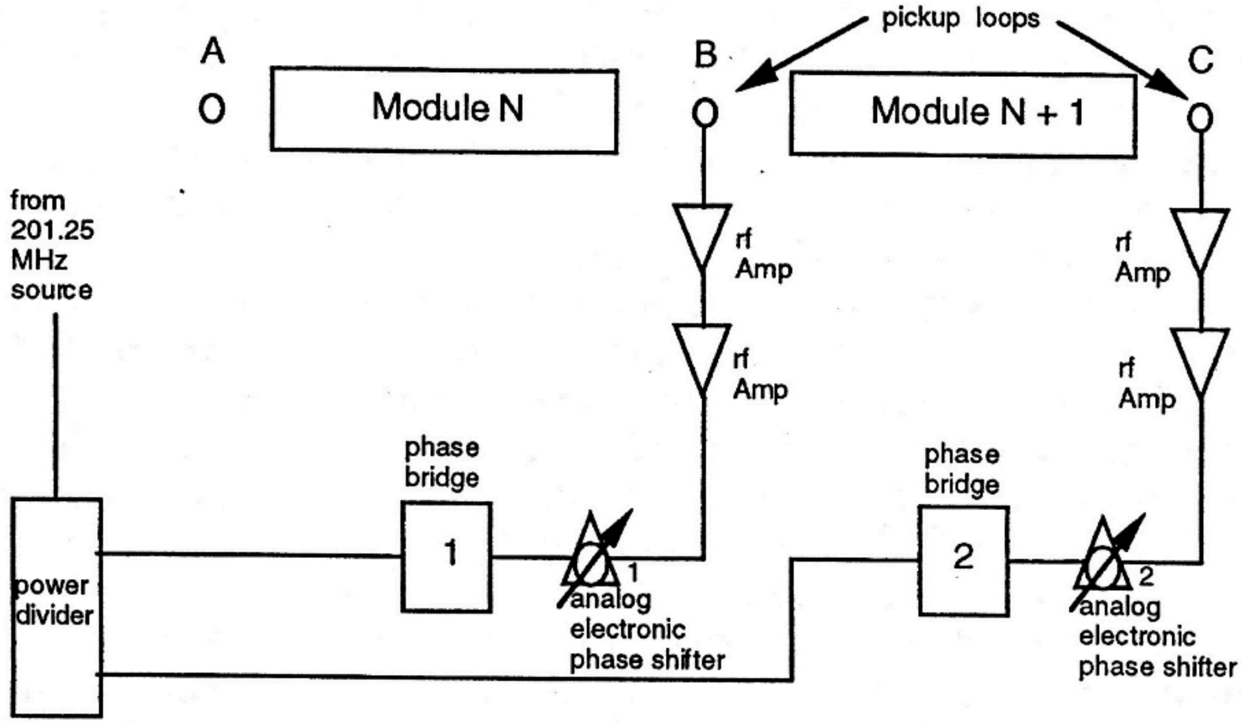


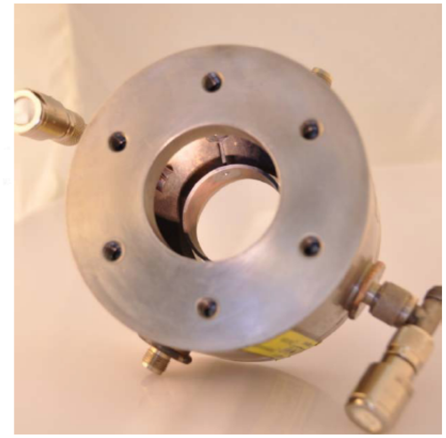




Results of phase scan

Delta-T Procedure to Set the Phase and Amplitude of RF Linac





Delta-t tranduser

Module N (being adjusted): ON and OFF
Module N+1: OFF

Let t_{AB} and t_{AC} be the time of flight of the beam "bunch" from locations A to B and A to C. The measurement of interest is the change in t_{AB} and t_{AC} when module N is brought in time. That is,

$$t_B = t_{AB\ OFF} - t_{AB\ ON}$$

$$t_C = t_{AC\ OFF} - t_{AC\ ON}$$

Differences with nominal values:

$$\Delta t_B = - D_{AB} \frac{\Delta v_A}{v_A^2} - (\Delta \phi_B - \Delta \phi_A)/\omega \ .$$

$$\Delta t_C = \Delta t_B - \frac{(D_2 - D_1)}{E_r c} \left[\frac{\Delta W_A}{\eta_A^3} - \frac{\Delta W_B}{\eta_B^3} \right] .$$

Longitudinal Beam Emittance Measurement (P.Strehl, 2010)

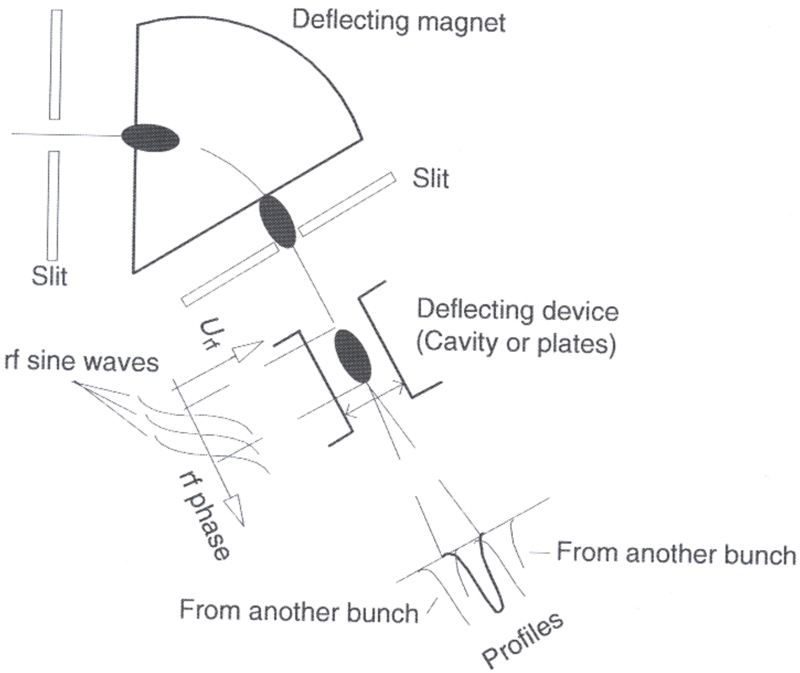
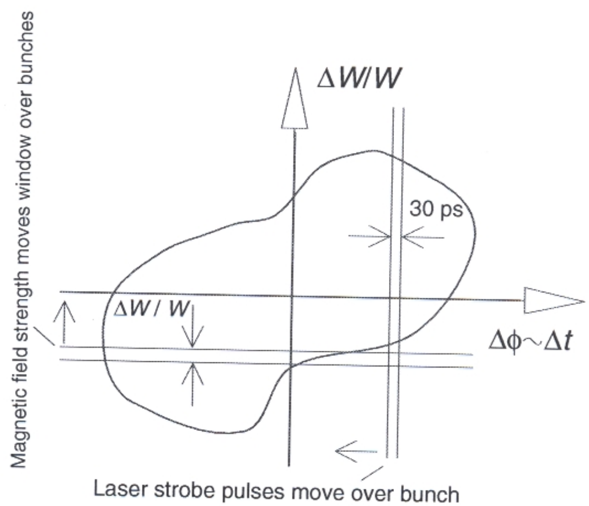
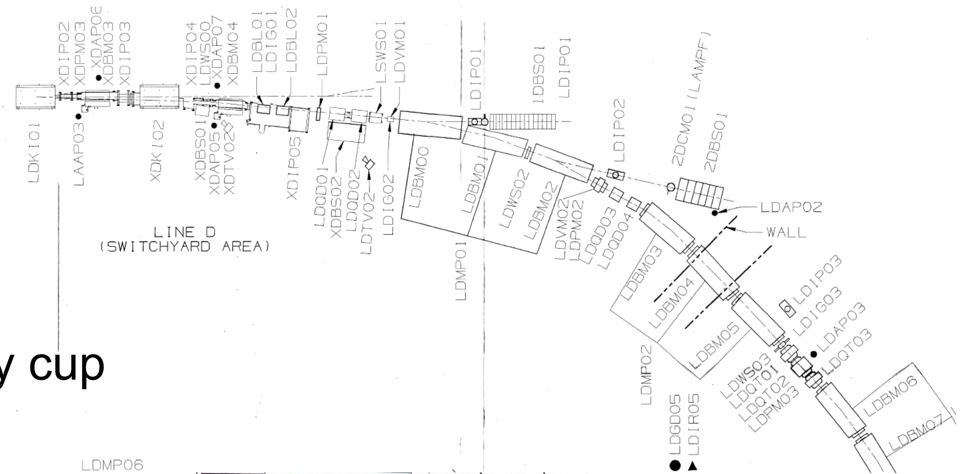


Fig. 7.1. Simplified scheme to measure longitudinal emittance

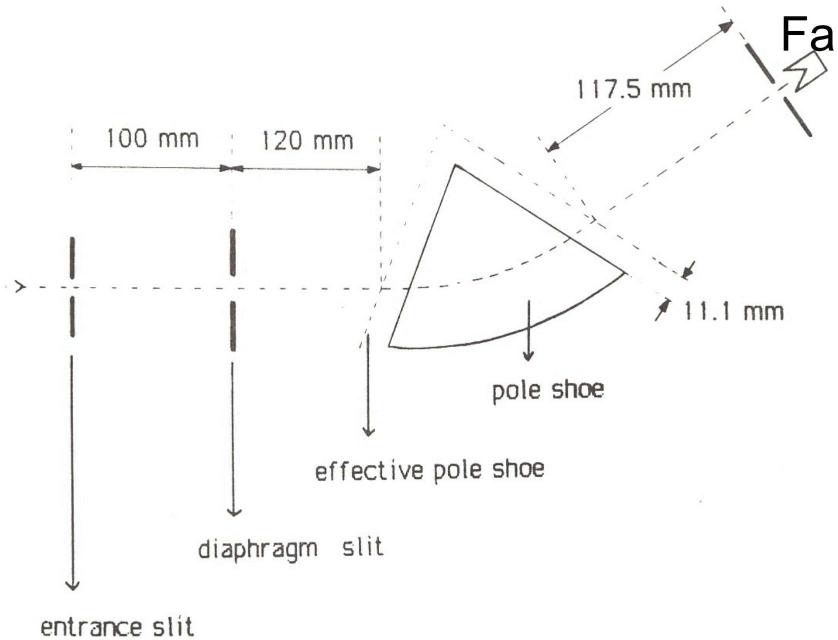


Measurement of Beam Energy Spread

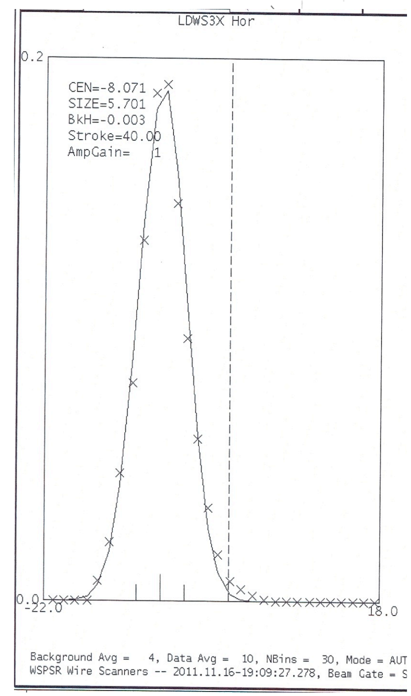
High-dispersive part of 800 MeV beamline



Faraday cup



Magnetic energy analyzer



Beam energy- spread-dependent wire scan

Bunch Shape Monitors (A.Feschenko, PAC 2001)

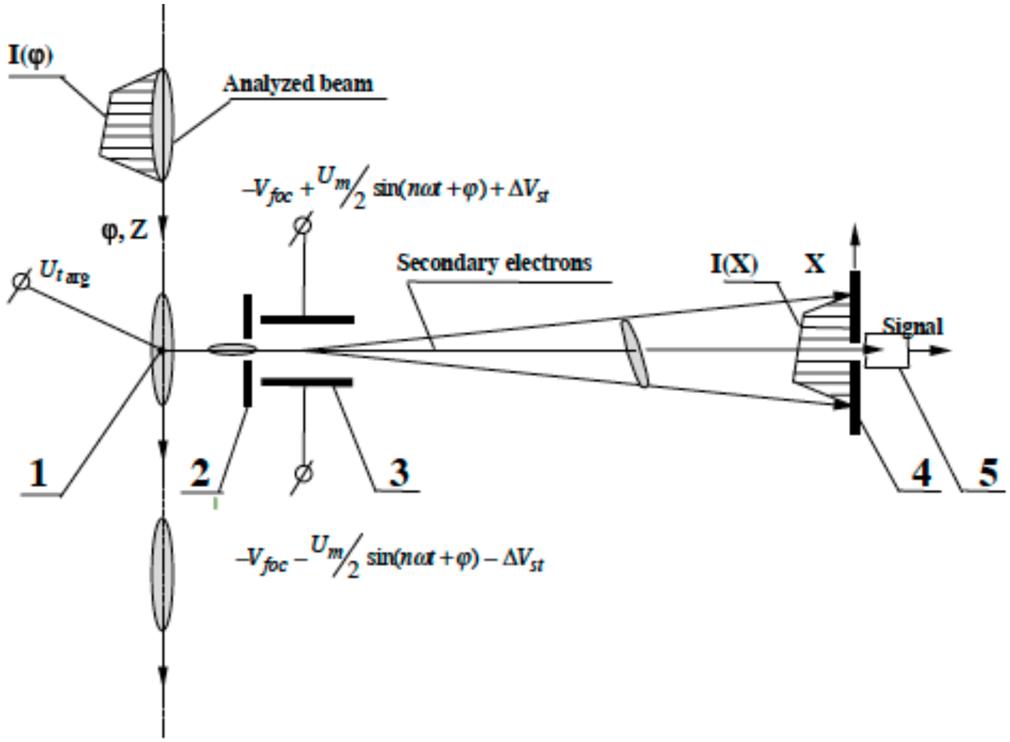


Figure 1: General configuration of Bunch Shape Monitor (1 –wire target, 2-input collimator, 3-deflector, 4-output collimator, 5-electron collector).

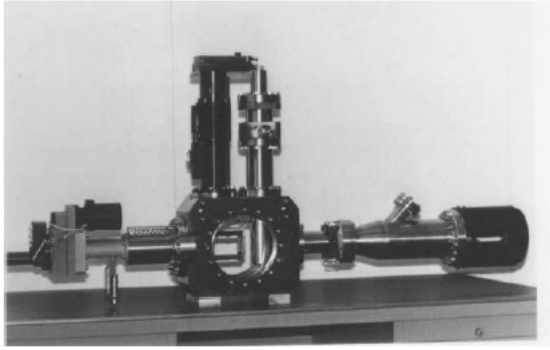


Figure 4: 3D-BSM for CERN Linac-2.

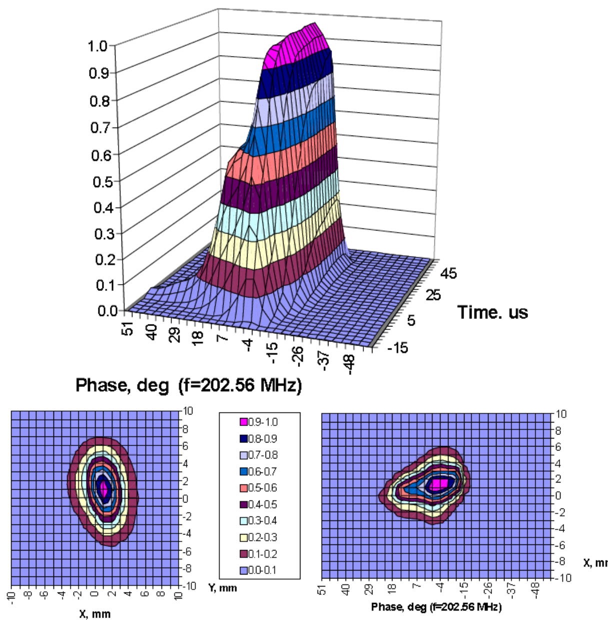


Figure 14: Behaviour of bunch shape in time, beam cross-section and longitudinally-transversal distribution measured at the exit of CERN Linac-2 with the 3D-BSM.

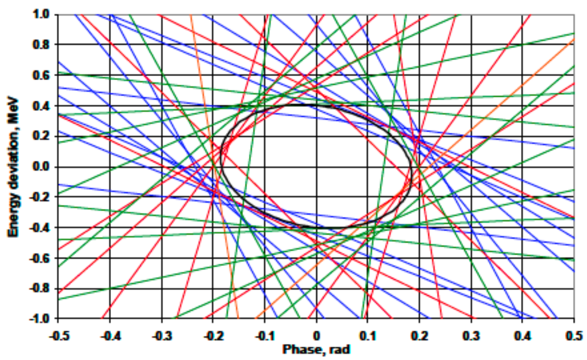


Figure 7: Bunch boundaries transformed to the entrance of CCL#1 and an equivalent phase ellipse.

Recommended Readings and References

1. M.Weiss, Fundamentals of Ion Linacs, CERN Accelerator School, , CERN 96-02, 4 March 1996, p. 39.
2. T.Wangler "RF Linear Accelerators", Wiley-VCH, 2008
3. I.M.Kapchinsky, Selected Topics in Ion Linac Theory, LA-UR-93-4192 (1993).
4. I.M.Kapchinsky, Theory of Resonance Linear Accelerators, Harwood, 1985.
5. M.Reiser, Theory and Design of Charged Particle Beams, Wiley, New York, 1994.
6. S.I.Molokovsky, A.D.Sushkov, Intense Electron and Ion Beams, Springer, Berlin, New York, 2005.
7. M.Conte, W.MacKay, An Introduction to the Physics of Particle Accelerators, World Scientific, 1991.
8. A. Septier, (Ed.) (1967). Focusing of Charged Particles, Volume 1, 2. Academic Press, New York, London, 1967.
9. J.D.Lawson, The Physics of Charged Particle Beams, Clarendon Press, Oxford, 1977.