Transverse & Longitudinal Dynamics: A Brief Survey

RF Linac for High Gain FEL Course
USPAS Summer 2014 Session

Wednesday, June 18

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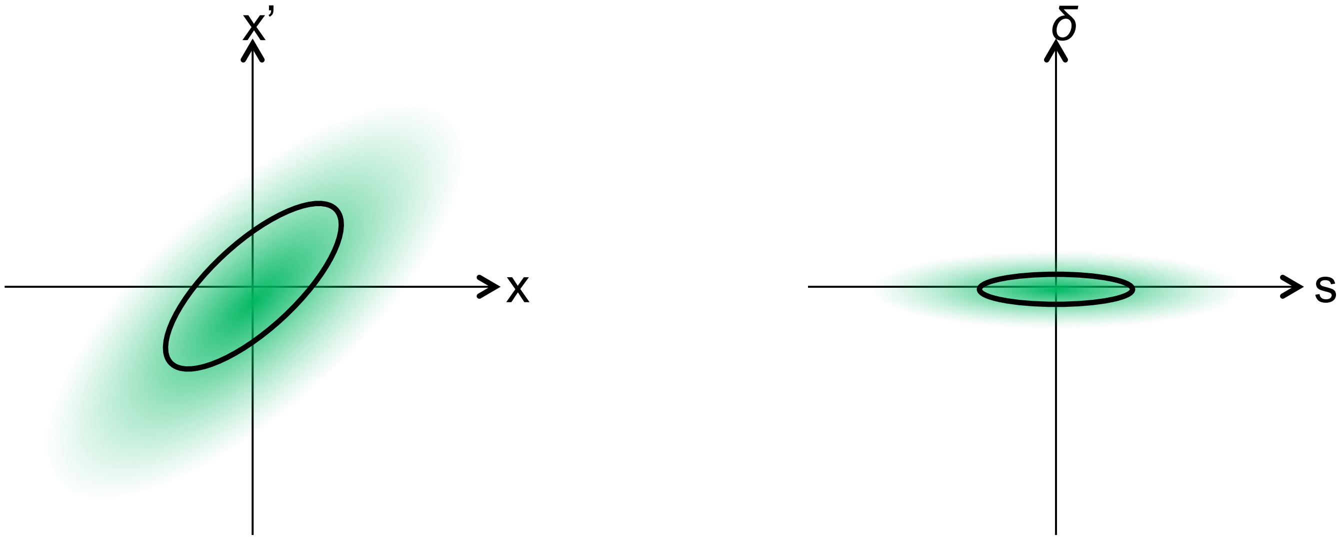
Phase Space

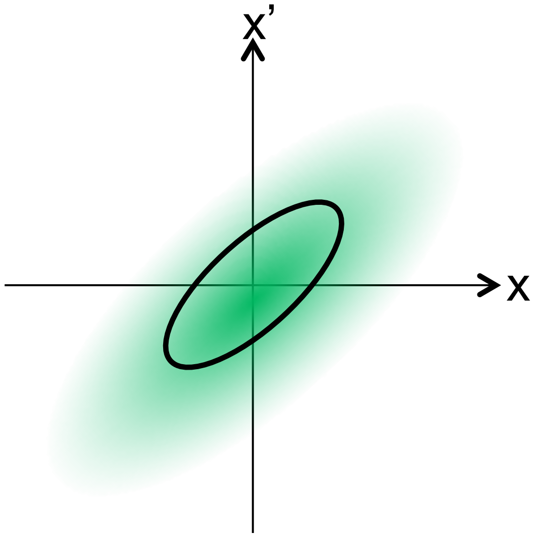
Using coordinates *(x, x′, y, y′, s, δ):*



Can visualize the beam using a 2D projection of the 6D phase space

Phase Space

- Commonly use transverse (*x*-*x′* or *y*-*y′*) and longitudinal (*s*-*δ*) projections
- Can describe these projections by using an rms ellipse.



$$\sigma_X = \langle X^2 \rangle^{\frac{1}{2}}$$

Separation of Transverse and Longitudinal Dynamics

- Often consider dynamics in the transverse and longitudinal phase space separately
- Many elements of a beamline have a dominant effect either in the direction of beam motion or perpendicular to it
- Dynamics can be separated provided no significant coupling between transverse and longitudinal degrees of freedom
- Not true of dipole magnets (e.g. spectrometer)!

Transverse and Longitudinal Dynamics

In matrix terms:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ s \\ \delta \end{pmatrix} = \begin{pmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{pmatrix} \begin{pmatrix} x_i \\ x'_i \\ y_i \\ y'_i \\ s_i \\ \delta_f \end{pmatrix}$$

i.e. transverse and longitudinal degrees of freedom are effectively decoupled.

Our Survey

Transverse Dynamics

- Envelope equation of motion
- Twiss parameters
- Betatron motion
- Emittance
- Space charge effects
- Nonlinear effects

Longitudinal Dynamics

- Energy chirp
- RF curvature
- Space charge effects
- Wake fields

Transverse Dynamics

Envelope equation of motion

- Beam envelope described by transverse rms parameters
- For focusing: $$x'' + K(s)x = 0$$
- For *K(s)* periodic: $$x(s) = \sqrt{\epsilon_x \beta}\cos(\phi(s) + \phi_x)$$
- β and φ are related: $$\phi(s) = \int \frac{ds}{\beta(s)}$$
- Two other functions of β also defined:

$$\alpha(s) = \frac{1}{2}\frac{d\beta(s)}{ds} \qquad \gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

See Wangler, p.213

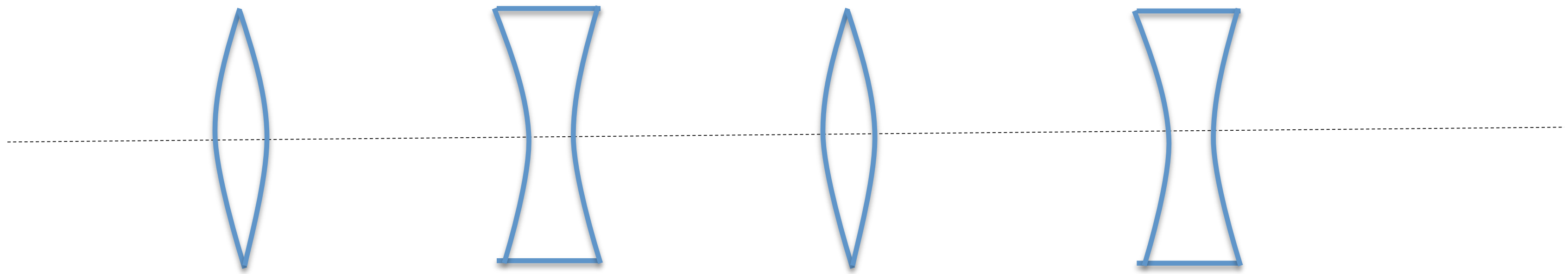
Twiss Parameters

- α, β and γ are called the *Twiss* or *Courant-Snyder* parameters
- α, β and γ are periodic functions with the same period as *K(s)* (for *K(s)* periodic)
- Then $$\gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 = \epsilon_x$$
- This is an ellipse with:
  - Center at the origin in *x-x'* phase space
  - Area: $$A_x = \pi\epsilon_x$$
- Only two of the three Twiss parameters are independent, as:

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}$$

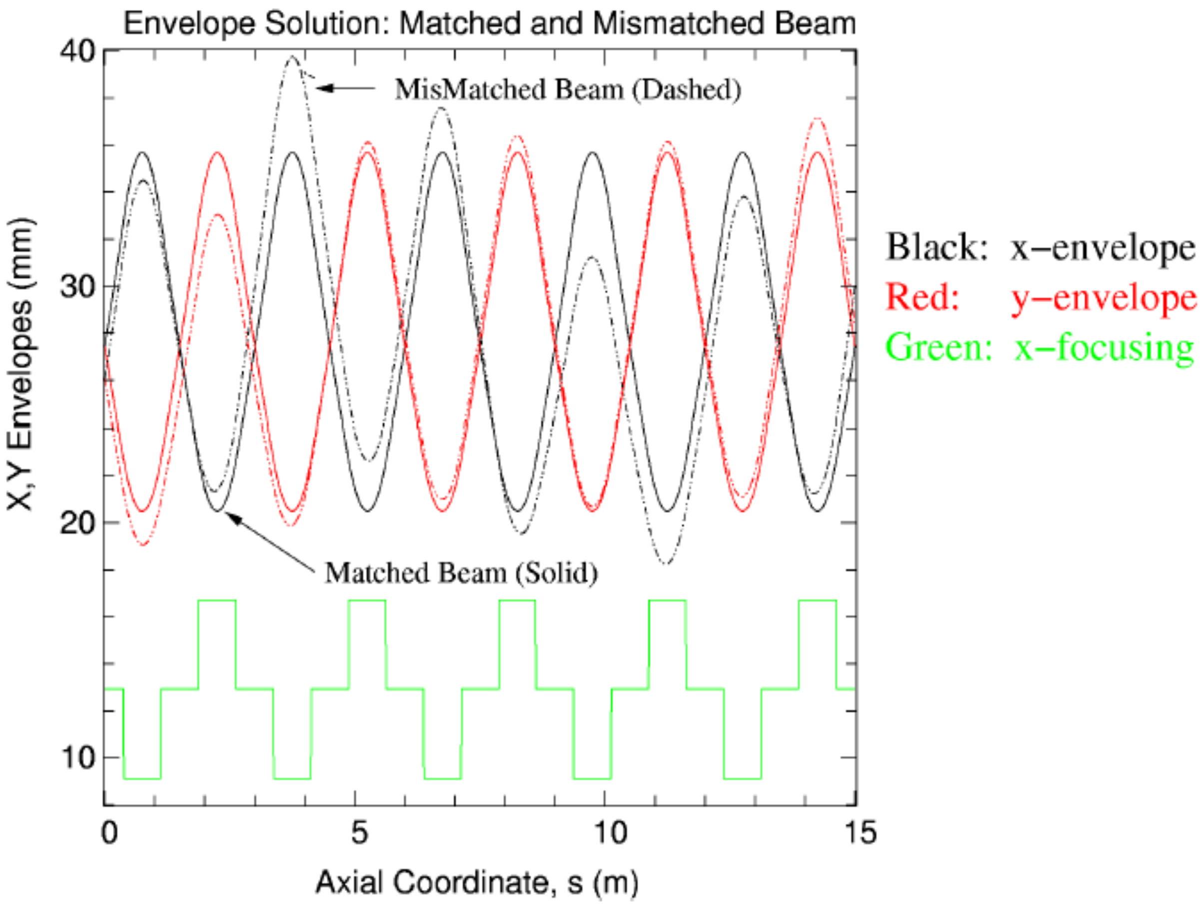
Betatron motion

- FODO lattice:



- Envelope size: $x_{max} = \sqrt{\epsilon_x \beta(s)}$
- For a matched beam, the envelope executes simple harmonic motion (or betatron motion).
- If the beam envelope is not matched to the FODO lattice on entry, there are oscillations around equilibrium (betatron oscillations).

Betatron motion and oscillations



Emittance

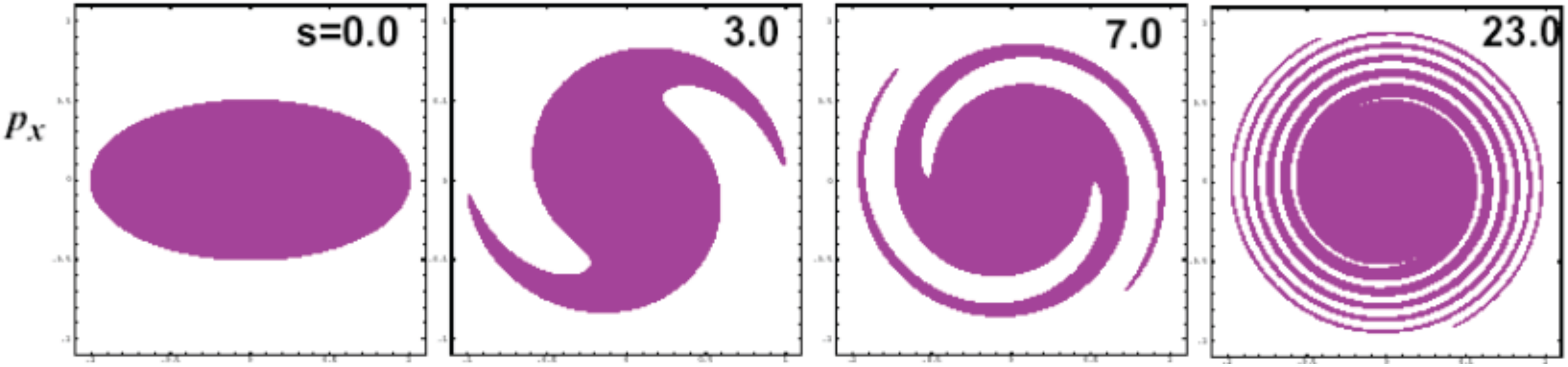
- The area of the rms phase space ellipse is proportional to the beam emittance, $A_x = \pi\epsilon_x$
- Emittance is a measure of beam quality.
- $\epsilon_x = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$
- When the beam is accelerated, $x' = dx/ds$ decreases.
- To compare beam quality along the entire beam path of an accelerated beam, we use the normalized emittance:

$$\epsilon_{x,n} = \gamma\beta\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

Emittance

Non-linear forces (e.g. $x'' = -k^2x + \varepsilon x^3$) $\Rightarrow$ position-dependent frequency

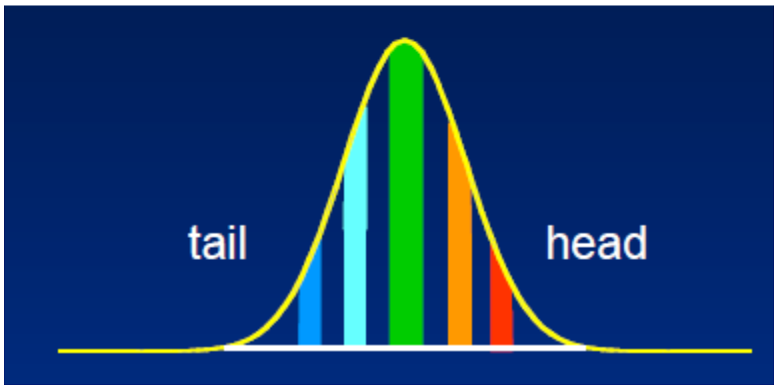
$\Rightarrow$ phase mixing, increasing effective area $\Rightarrow$ **Emittance increases if forces non-linear**



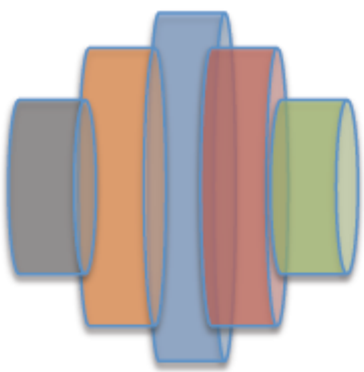
Emittance

Slice emittance

Divide the bunch into different slides

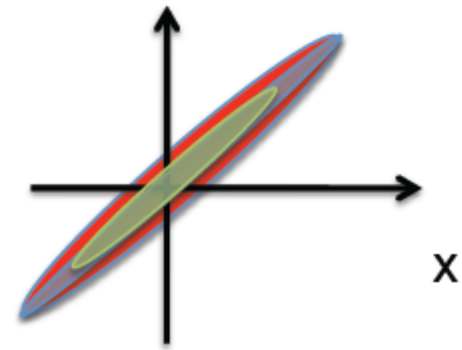


Represent each slice in x'x space

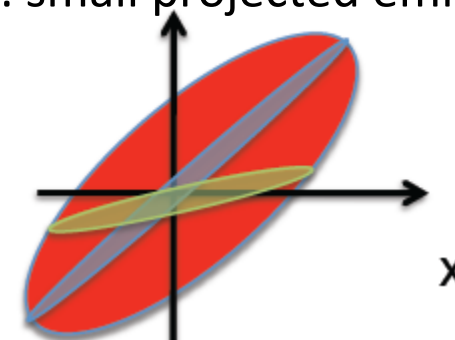


Projected emittance

Project the trace spaces onto x'x



Aligned: small projected emittance



Not aligned: large projected emittance

Space Charge

- Space charge is the force experienced by a particle in a bunch due to the electromagnetic forces in the rest of the bunch.
- Causes a beam to expand transversely.
- Nonlinear force.
- Typically causes emittance growth.
- Most significant at low energies.
- Additional force modifies equation of motion/envelope equation.

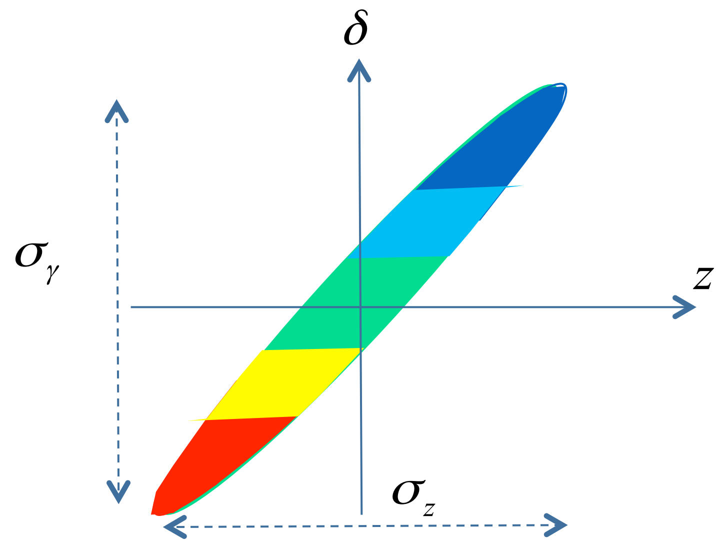
Space Charge

- Space charge introduces both additional electrostatic and magnetic (due to current) forces.
- Codes typically calculate space charge in the *rest frame* of the beam. This is chosen as the rest frame of either the beam longitudinal centroid or a reference particle.
- Motion of particles in the rest frame is then treated as non-relativistic – this can cause errors in computation.

Longitudinal Dynamics

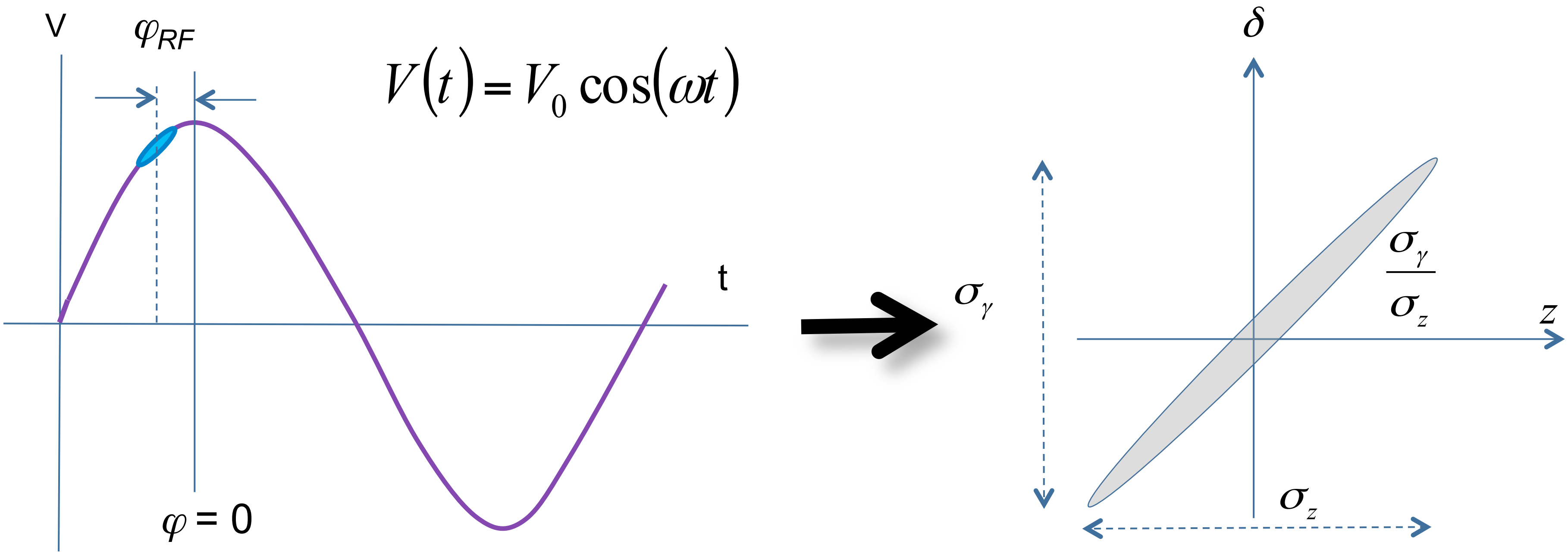
"Chirp" = energy-z correlation

Positively chirped bunches have low-energy (red) electrons at the head (left) with respect to high-energy (blue) ones at the tail (right).



Bunches are deliberately chirped before entering a bunch compressor.

Inducing an Energy Chirp



Chirper Cavity Transfer Matrix

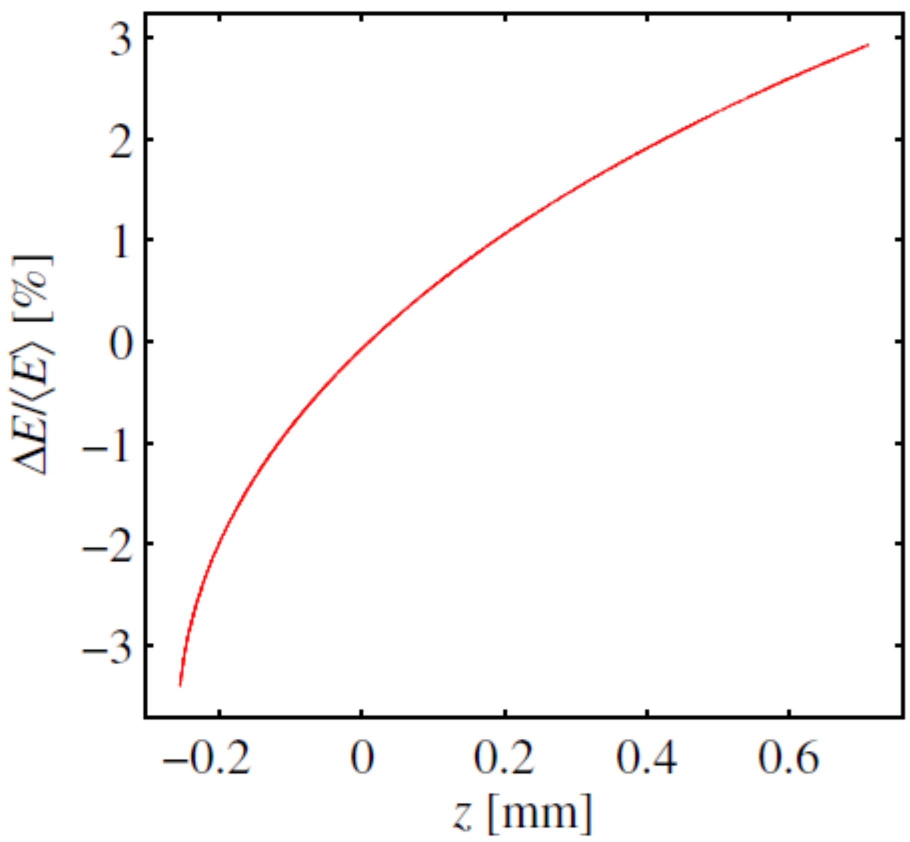
R_{65} matrix elements converts the particle's initial position within the bunch to its final energy deviation, thereby imposing an energy chirp.

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & R_{65} & R_{66} \end{bmatrix} \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}_0$$

Longitudinal 2x2 matrix

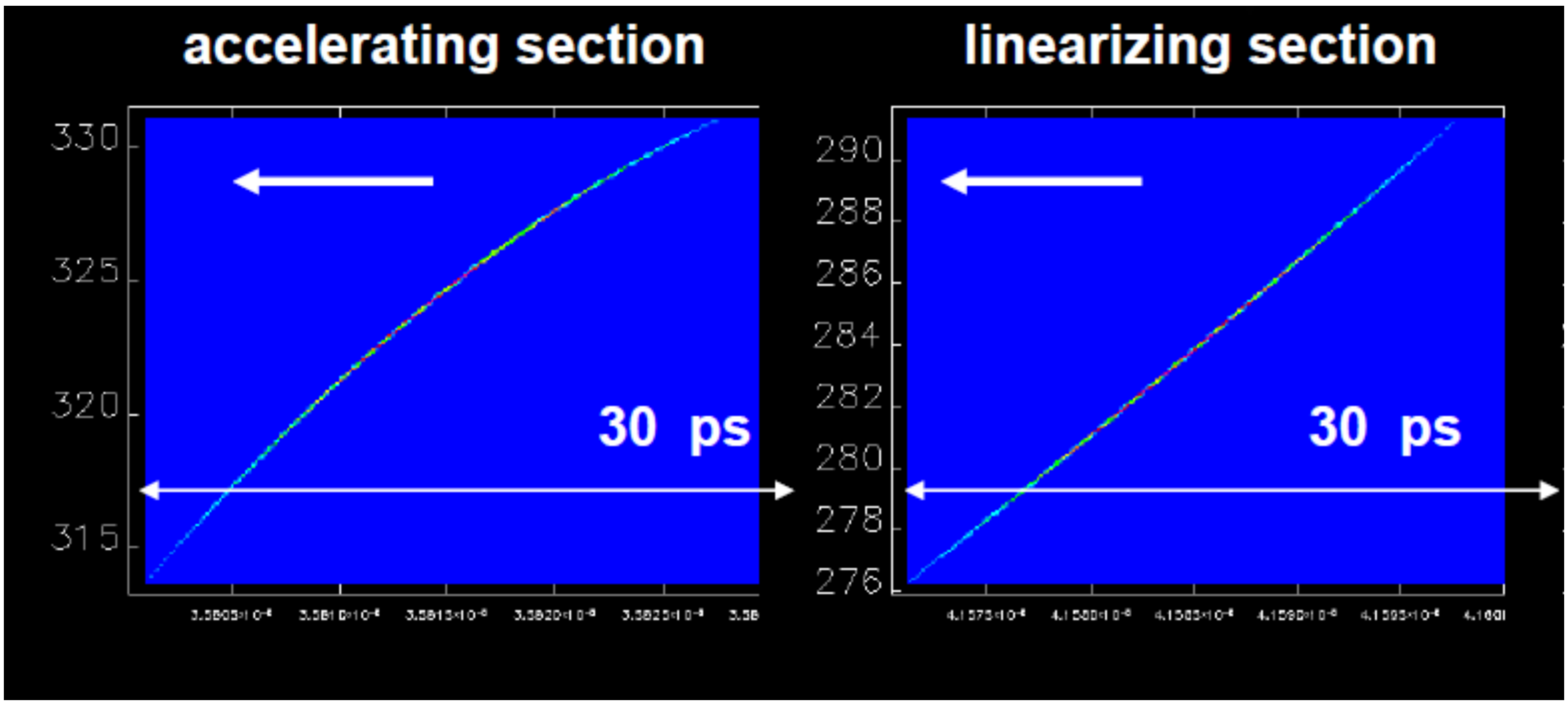
$$\begin{pmatrix} z_1 \\ \delta_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ R_{65} & R_{66} \end{pmatrix} \cdot \begin{pmatrix} z_0 \\ \delta_0 \end{pmatrix}$$

RF Curvature

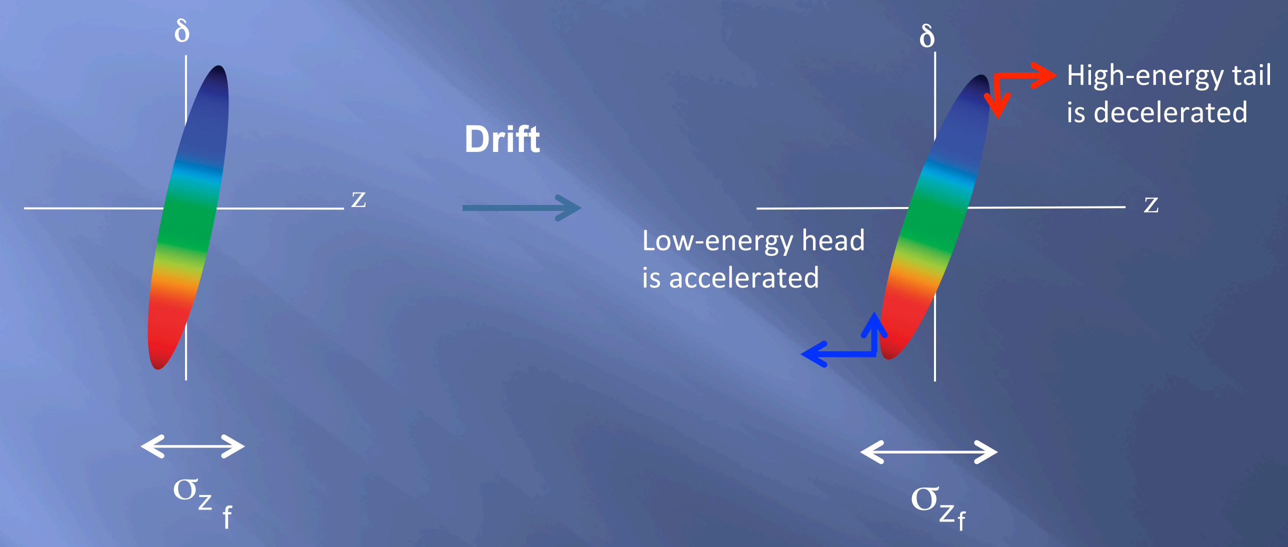


Harmonic Linearizer

Use the nonlinearity of a harmonic cavity to correct for RF curvature in the fundamental cavity and linearize the energy chirp.



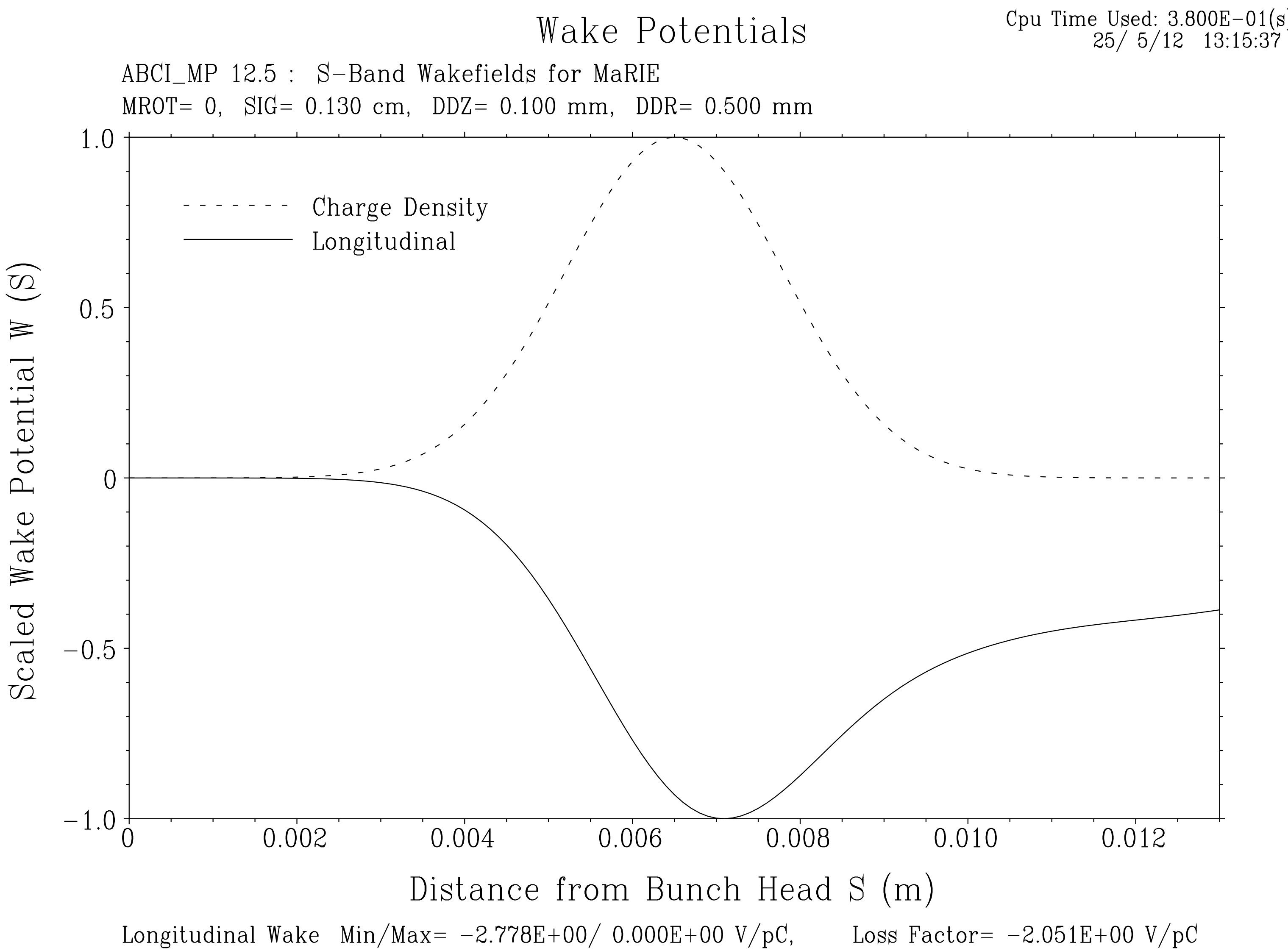
Space Charge Effects



Space charge stretches the bunch length and reduces the energy spread of a positively chirped electron bunch.

Conversely, space charge compresses the bunch length of a negatively chirped electron bunch and increases its energy spread

Longitudinal Wake Fields



Non-linear Effects

- Second order non-linearities = quadratic function of z.
- Third order non-linearities = cubic function of z (etc..).
- Longitudinal wake fields depress the energy of the bunch tail.
- Wake fields have second and third order non-linearities.
- RF curvature also causes second and third order non-linearities.
- These effects lead to non-linear chirp.