Transverse Dynamics & Optical Functions

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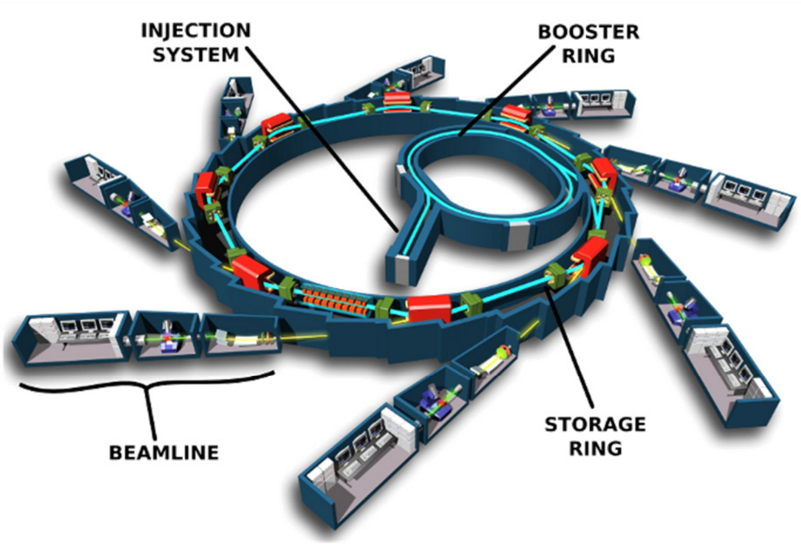
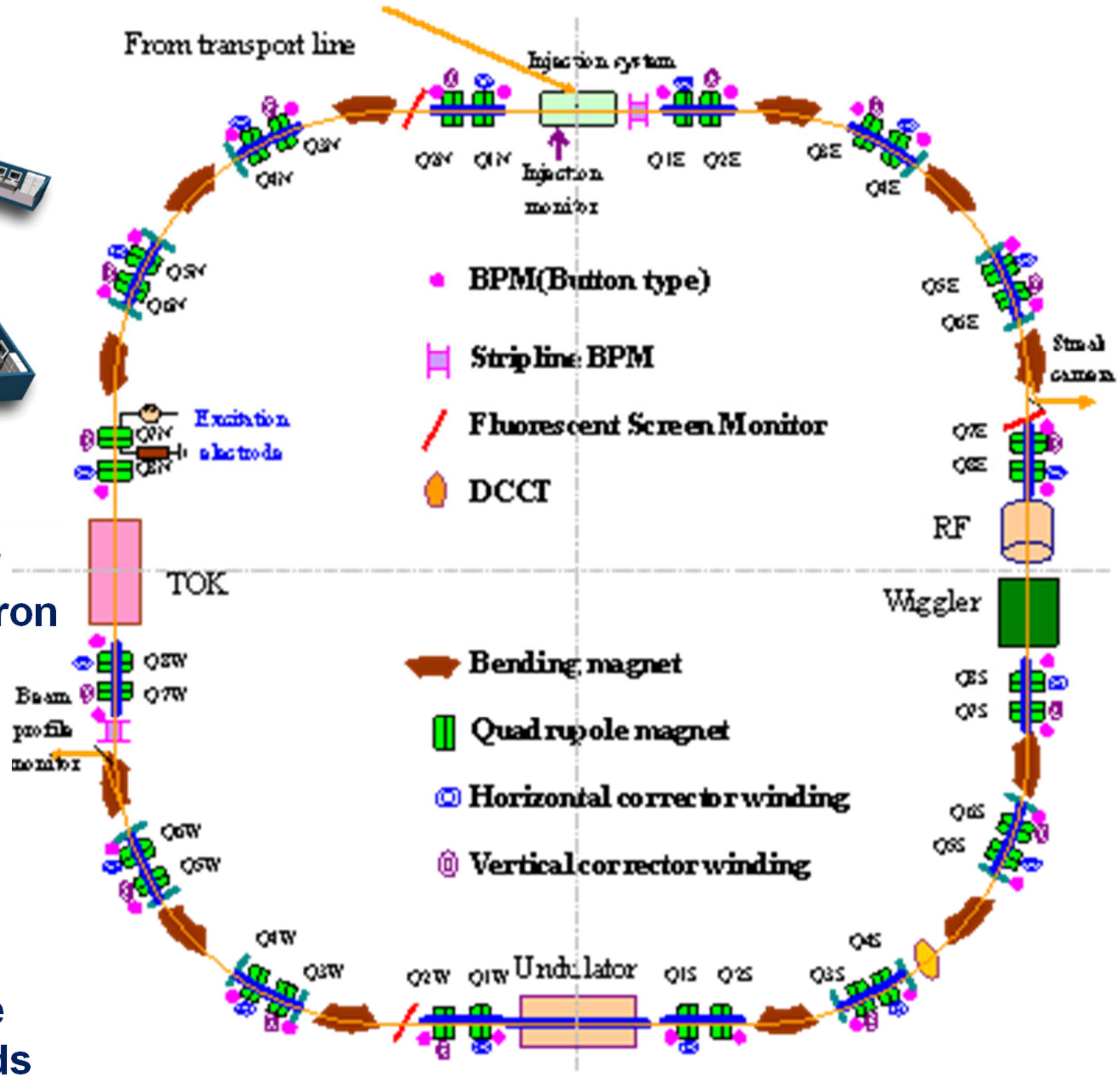
Outline

- Basic definitions
- Magnetic components in storage rings and their properties
- Linear optics
  - Hill's equations
  - Twiss parameters
  - Betatron tune
  - Matrix representation
- Off-momentum particles
  - Dispersion
  - Chromaticity
- Lattice evaluation and optimization techniques

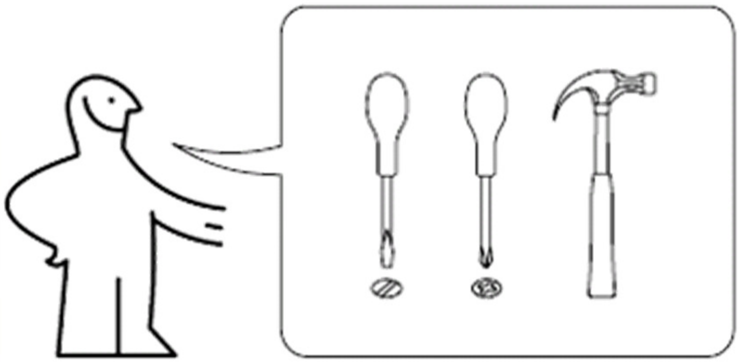
The Typical Storage Ring



- **RF**: accelerates the beam and/ or restores energy lost by synchrotron radiation (SR)
- **Magnets**: confine beam in the desired space
- **Insertion devices (ID)**: generates high quality SR
- **Injector**: generates the proper electron beam for the ring
- **Beamlines**: extracts and form the SR according to experiment needs

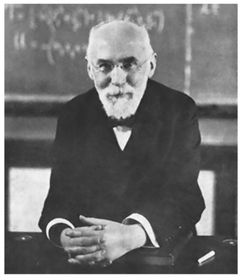


Transverse Dynamics. Required Tools



Lorentz force in relativistic regime:

$$\boxed{\vec{F} = -e\left(\cancel{\vec{E}} + \vec{v} \times \vec{B}\right)}$$



Electric fields to change the beam energy

Magnetic fields to force the particles on the desired trajectory

Maxwell equations to generate and manipulate the required fields:

$$\nabla \cdot \vec{E} = \rho / \varepsilon_0$$

$$\nabla \cdot \vec{B} = 0$$

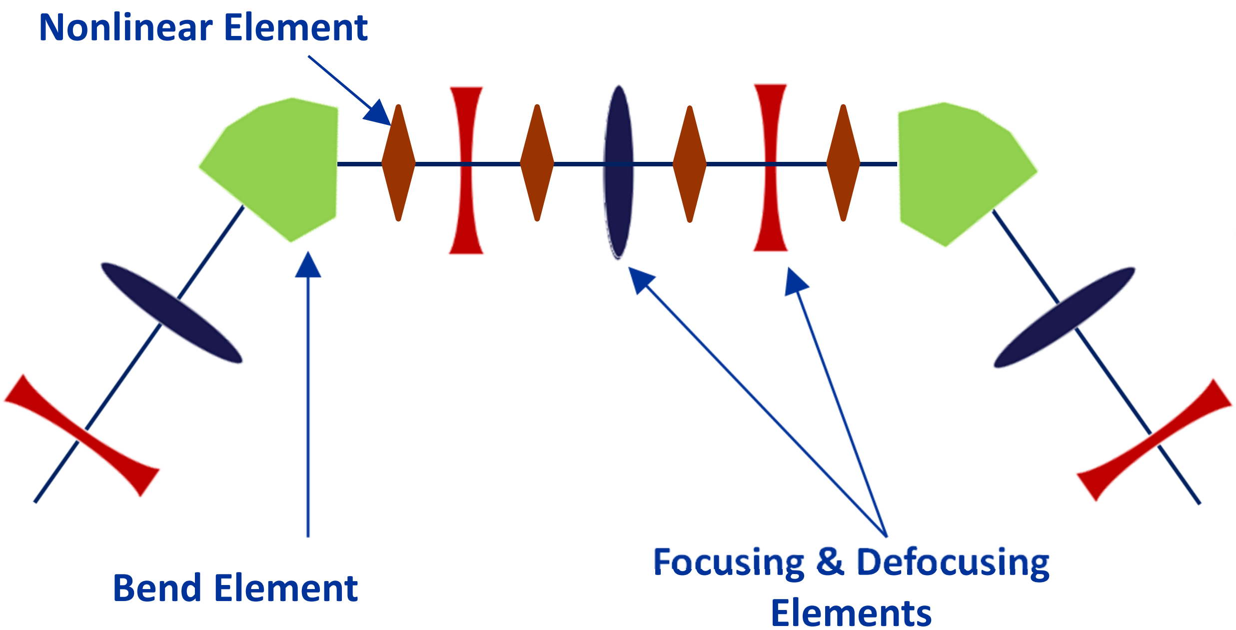
$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

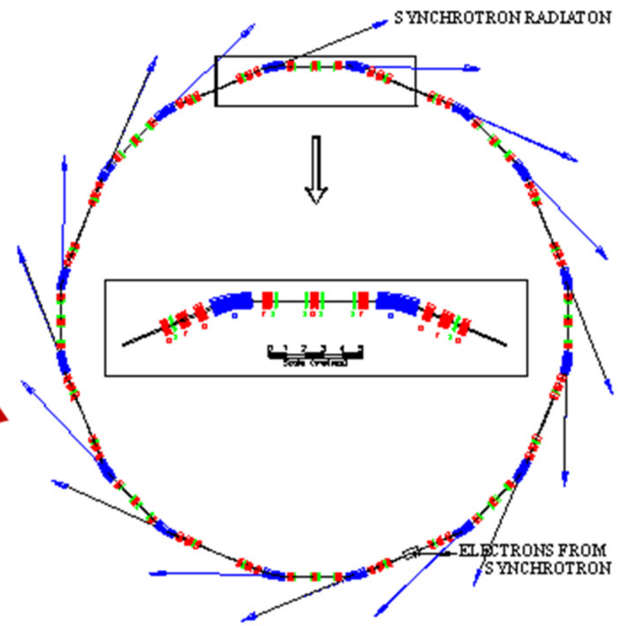
$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \, \partial \vec{E} / \partial t\right)$$

In the following transverse beam dynamics analysis we will consider the particle energy as a constant and assume that no electric field is present.

The Lattice

The Lattice in a storage ring is the distribution of discrete elements (typically magnetic) that guide and focus the beam.
Lattices are sometimes referred to as the beam optics.

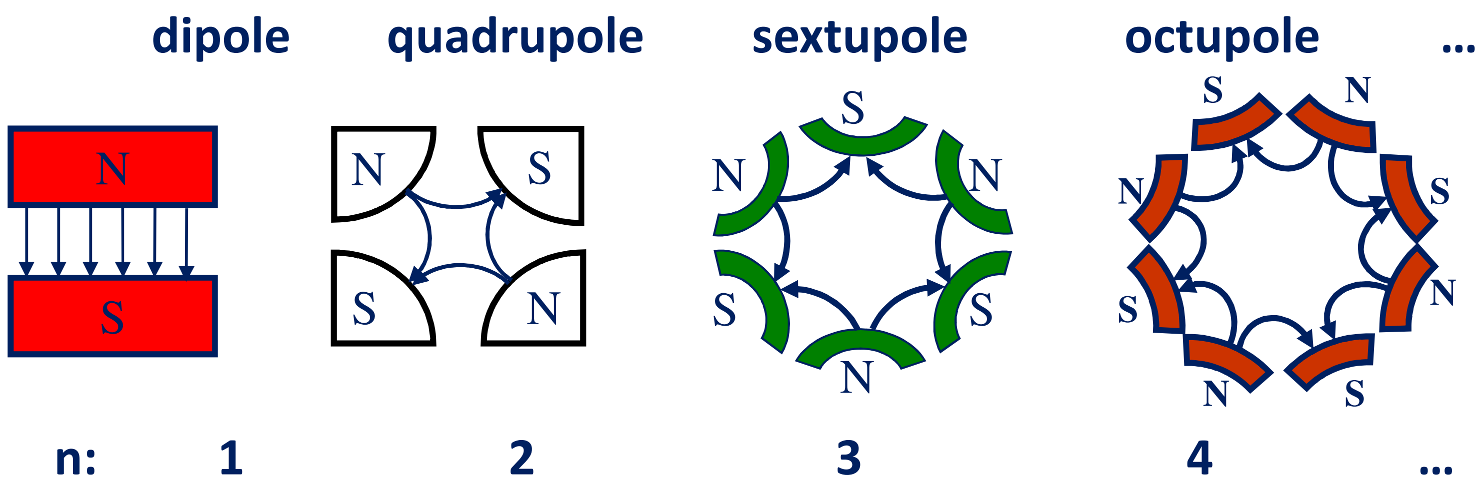




- **Ring are usually composed by identical lattice cells or periods.**
- **Sometimes bending and (de)focusing capabilities are combined in a single element (combined functions elements).**

Magnet Definitions

2n-pole:



Normal: gap appears at the horizontal plane

Skew: rotate around beam axis by $\pi/2n$ angle

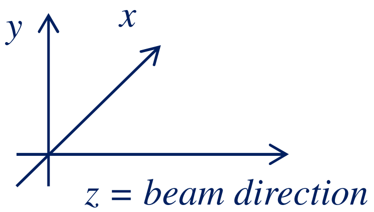
Symmetry: rotating around beam axis by π/n angle, the field is reversed (polarity flipped)

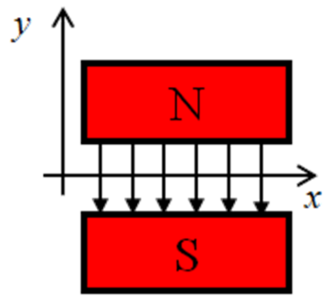
Properties of Typical Magnets

The most common magnet types used in storage rings are:

Dipoles → used for guiding, steering the beam

$$B_x = 0$$
$$B_y = B_0 = \text{constant}$$



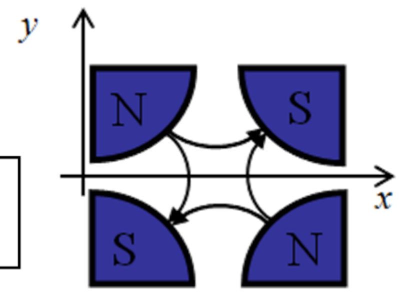


Quadrupoles → <u>linear</u> elements used for focusing

$$B_x = G\, y$$
$$B_y = G\, x$$

$$\text{G} = \text{constant}$$

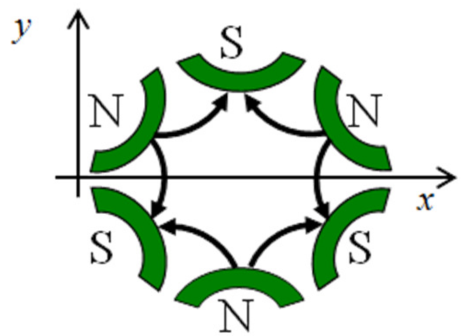
Note: quadrupoles are focusing in one plane and defocusing in the other.



Sextupoles → <u>nonlinear</u> elements used for chromatic correction and control of nonlinear dynamics (*dynamic aperture*)

$$B_x = 2S\, x\, y$$
$$B_y = S\left(x^2 - y^2\right)$$

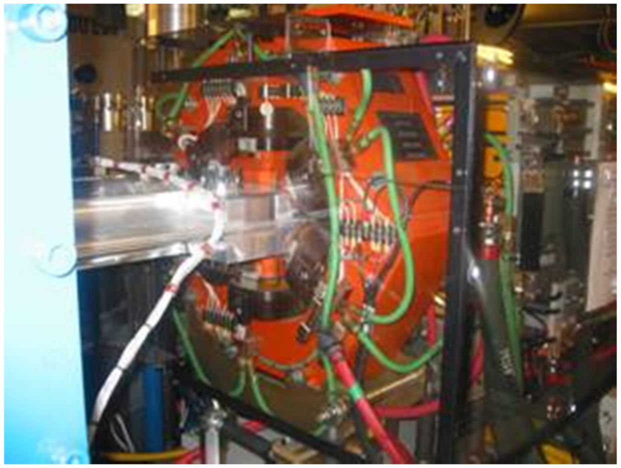
$$\text{S} = \text{constant}$$



Real Magnet Examples

Dipoles

Quadrupoles

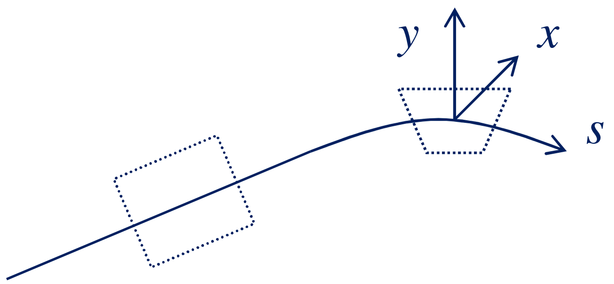


Sextupoles

Linear Optics (Lattice) Calculation

The first step in calculating a lattice is to consider only the **linear components** of it (quadrupoles and dipoles). Non linear effects and chromatic aberration corrections will be evaluated subsequently.

The trajectory of the **reference particle** (the particle with nominal energy and initial position and divergence set to zero) along the optics is calculated.
All the other beam particles are represented in a **frame moving along the reference trajectory**, and where the reference particle is always in the center.



Coordinate systems used to describe the motion are usually locally Cartesian or cylindrical (typically the one that allows the easiest field representation)

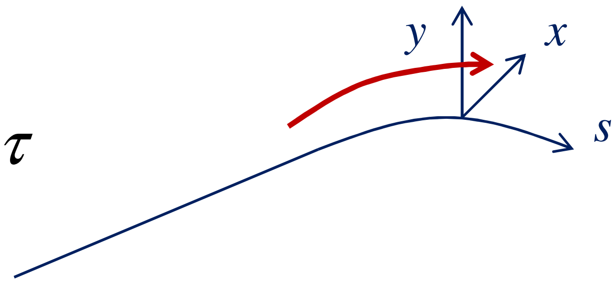
In integrating the equation of motion it is often convenient to change the independent integration variable from t to s

$$s = \beta c t$$

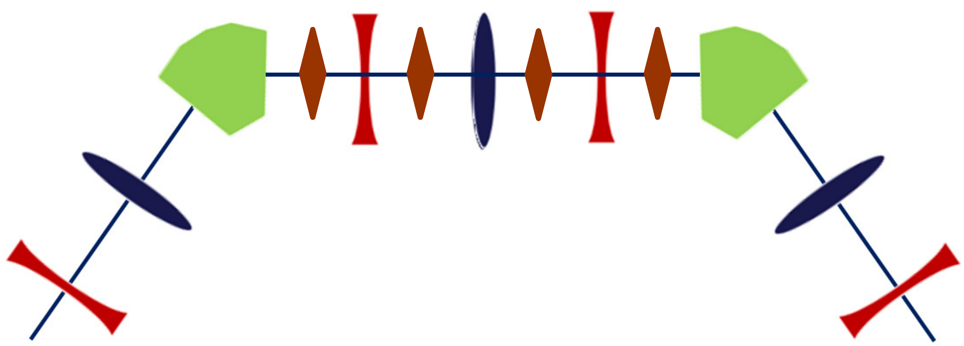
Integrating Along the Lattice

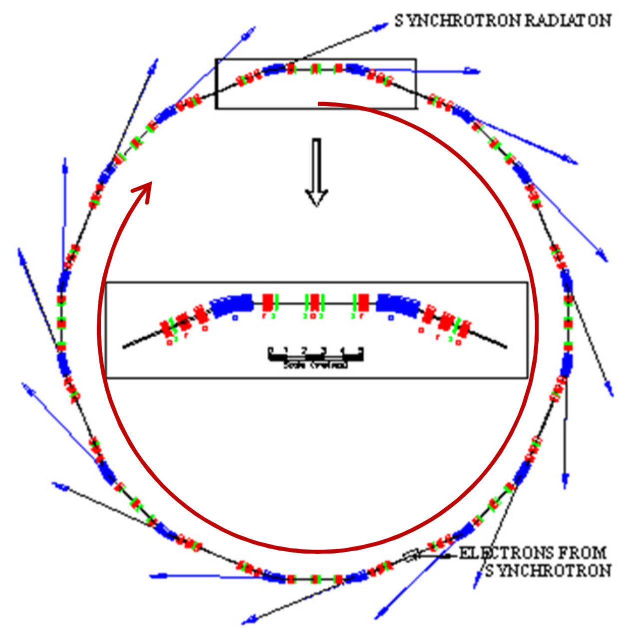
In the new reference frame each particle will be represented by the variables:

$$x,\ x' = \frac{dx}{ds} \cong \frac{p_x}{p_0},\ y,\ y' = \frac{dy}{ds} \cong \frac{p_y}{p_0},\ \delta = \frac{p - p_0}{p_0},\ \tau$$



And we need to integrate the equation of motion along the lattice elements





Quadrupoles

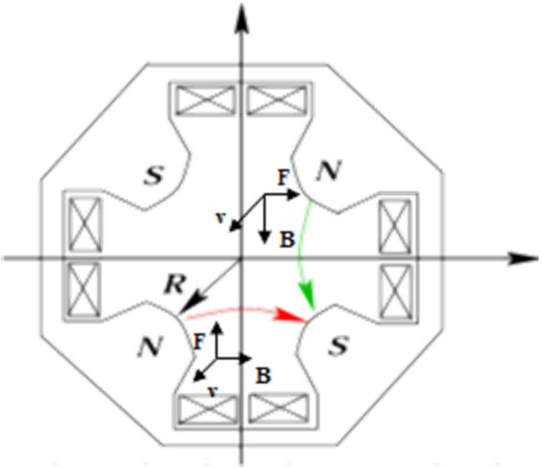
$$B_x = G\,y \qquad B_y = G\,x \qquad \mathrm{G} = \mathrm{constant}$$

$$F_x = -e\beta c B_y = -e\beta c G\,x \quad \text{focusing}$$

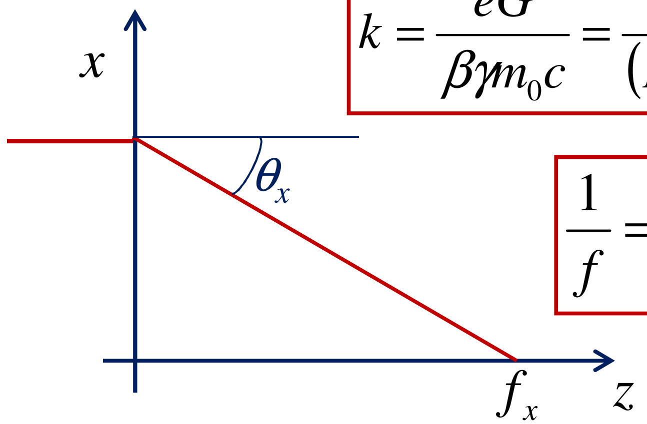
$$F_y = e\beta c B_x = e\beta c G\,y \quad \text{defocusing}$$

$$p_x \approx F_x \Delta t \approx F_x \frac{l}{\beta c} = -eGl\,x$$

$$\tan\theta_x = x' = \frac{p_x}{p} \approx -\frac{eG}{\beta\gamma m_0 c} l\,x = -k\,l\,x$$



$$k = \frac{eG}{\beta\gamma m_0 c} = \frac{G}{(B\rho)} = quadrupole\ strength\ [m^{-2}]$$



$$\frac{1}{f} = \frac{\tan\theta_x}{x} = \frac{x'}{x} = k\,l = \frac{eG}{\beta\,\gamma\,m_0 c} l$$

Thin lens model →

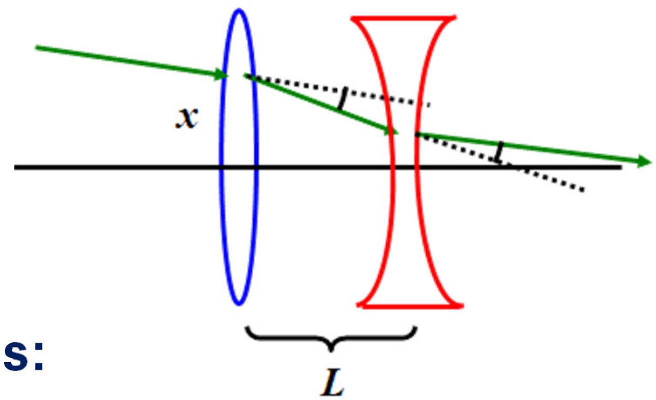
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

Remark: the focal length depends on the particle energy (chromatic aberrations).

Quad Doublet & Alternating Gradient Focusing

Consider a quadrupole doublet, i.e. two quadrupoles with focal lengths -f_1 and f_2 separated by a distance L.



In thin lens approximation the transport matrix is:

$$M_{doublet} = \begin{pmatrix} 1 & 0 \\ 1/f_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-L/f_1 & L \\ 1/f_2 - L/f_1 f_2 - 1/f_1 & 1+L/f_2 \end{pmatrix}$$

with the total focal length:

$$\frac{1}{\tilde{f}} = \frac{1}{f_2} - \frac{1}{f_1} - \frac{L}{f_1 f_2}$$

$$if\ \ f_1 = f_2 = f \quad \Rightarrow \quad \frac{1}{\tilde{f}} = -\frac{L}{f^2} < 0 \quad \text{always focusing}$$

Alternating gradient focusing seems overall focusing (in both planes!)

Warning: this is only valid in thin lens approximation!!!

Dipoles

In the presence of **constant magnetic field B**, an electron with momentum p will describe a circular trajectory with the **Larmor radius ρ** :

$$\rho = \frac{p}{e\,B} = \frac{\beta\,\gamma\,m_0 c}{e\,B} \quad with \quad \begin{array}{c} \beta = v/c \\ \gamma = E_T / m_0 c^2 \end{array}$$

In a storage ring the circular trajectory is broken to allow the insertion of focusing elements and of the other components (sextupoles, RF cavities, insertion devices for the production of light, …).

For example, if N equal dipoles are used, their individual bending angle and length will be given by:

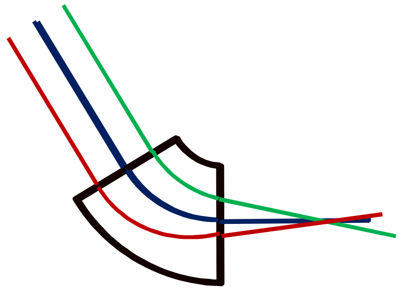
$$\theta = \frac{2\pi}{N} \qquad\qquad l = \rho\theta \quad (along\ \ the\ \ trajectory)$$

If normal conducting magnets are used the field is practically limited to ~1.5 T. Within this limit, higher energies require larger radius (larger rings).

Dipoles Can Also Focus!

Let's consider a typical type of dipole magnet, the **sector magnet**.
The reference particle enter and exit the magnet perpendicularly to its faces.

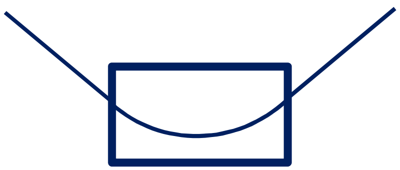
And let's also have two other particles with the same energy of the reference particle but moving along trajectories parallel to the reference one .



Inside the magnet, the three particles will move along circular trajectories with the same radius, but because of the geometry of the magnet they leave the exit face with different angles → **focusing effect in the plane of the magnet**!

It can be shown, that the effect is linear and that the **focusing strength of a sector magnet** is

$$\boxed{k = -\frac{1}{\rho^2} = -\left(\frac{eB}{\beta\gamma mc}\right)^2}$$



Rectangular (parallel face) magnet (easier to fabricate

Question: Is there any similar focusing effect in the rectangular magnet?

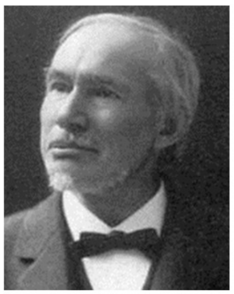
Remark: In the previous analysis we did not consider fringe fields. In real magnets they add (de) focusing effects in both planes.

Hill's Equations

We now have the focusing terms for the ring linear components (dipole and quadrupoles) to use in the Lorentz force for integrating Newton second law of motion:

$$\frac{d\vec{p}}{dt} = \vec{F} = -e\left(\cancel{\vec{E}} + \vec{v} \times \vec{B}\right)$$

$$w' = \frac{dw}{ds} \quad w'' = \frac{d^2 w}{ds^2} \quad w = x, y$$



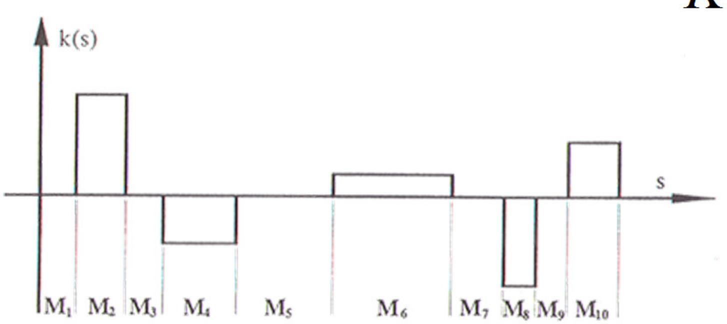
$$\boxed{\begin{aligned} x'' + K_x(s)\,x = 0 \\ y'' + K_y(s)\,y = 0 \end{aligned}}$$

$$K_x(s) = -\left[k(s) - 1/\rho^2(s)\right]$$
$$K_y(s) = k(s)$$

Hill's equations of linear transverse particle motion

In a storage ring:

$$\mathcal{M} = \mathcal{M}_{10} \ldots \mathcal{M}_5 \mathcal{M}_4 \mathcal{M}_3 \mathcal{M}_2 \mathcal{M}_1$$



$$K_x(s) = K_x(s + C)$$
$$K_y(s) = K_y(s + C)$$

$$C \equiv ring \;\; circumference$$

and where we assume that the focusing varies "piecewise" around the ring

Harmonic Oscillator

Hill's equations are very similar to those of the well known harmonic oscillator. The only difference is that in the harmonic oscillator $K = k_0$ is constant.

Hill's:

$$x'' + K_x(s)x = 0$$
$$y'' + K_y(s)y = 0$$

Harmonic oscillator:

$$u'' + k_0 \ u = 0$$

with general solution:

$$\begin{aligned} u(s) &= C(s)\ u(0) + S(s)\ u'(0) \\ u'(s) &= C'(s)u(0) + S'(s)u'(0) \end{aligned}$$

$$\begin{aligned} C(s) &= \cos(\sqrt{k_0} s) \ , \quad S(s) = \frac{1}{\sqrt{k_0}} \sin(\sqrt{k_0} s) \\ C(s) &= \cosh(\sqrt{|k_0|} s) \ , \ S(s) = \frac{1}{\sqrt{|k_0|}} \sinh(\sqrt{|k_0|} s) \end{aligned}$$

Or in matrix form:

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$$

Solution of Hill's Equation

We will search the general solution of Hills equation in this shape (Floquet Theorem):

$$u'' + K_u(s)u = 0 \qquad u = x, y$$

$$\boxed{u(s) = A_u w_u(s)\cos\left[\varphi_u(s) - \varphi_u(0)\right]}$$

where A_u and $\phi_u(0)$ are constants. This is the equation of an harmonic oscillator with frequency and amplitude changing along s.

Using the Floquet solution in Hill's equation we find that the following two differential equations need to be satisfied:

$$w_u'' - w_u \varphi_u'^2 + K_u w_u = 0$$

$$2w_u'\varphi_u' + w_u\varphi_u'' = 0$$

The second equation can be immediately integrated:

$$w_u \times (2w_u'\varphi_u' + w_u\varphi_u'') = 2w_u w_u'\varphi_u' + w_u^2\varphi_u'' = \left(w_u^2\varphi_u'\right)' = 0 \qquad \Rightarrow \varphi_u(s) = \int_0^s \frac{ds}{w_u^2} + \varphi_u(0)$$

Which used in the first equation:

$$w_u^3(w_u'' + K_u w_u) = 1$$

Twiss Parameters

Courant and Snyder introduced the following parameters, often referred as optical functions or Twiss parameters:

$$\beta_u(s) = w_u^{\;2}(s)$$

$$\alpha_u(s) = -\frac{1}{2}\frac{d\beta_u(s)}{ds}$$

$$\gamma_u(s) = \frac{1+\alpha_u^{\;2}(s)}{\beta_u(s)}$$

Courant

Snyder

Using these new definitions in the Hill's equation solutions, and defining ε_u as a constant, we obtain:

$$u(s) = \sqrt{\varepsilon_u \beta_u(s)}\cos[\varphi_u(s) - \varphi_u(0)]$$

$$u'(s) = -\sqrt{\frac{\varepsilon_u}{\beta_u(s)}}\{\alpha_u(s)\cos[\varphi_u(s)-\varphi_u(0)] + \sin[\varphi_u(s)-\varphi_u(0)]\}$$

$$u = x, y$$

$$\varphi_u(s) = \int_0^s \frac{ds}{\beta_u} + \varphi_u(0)$$

betatron phase

$$\frac{1}{2}\beta_u\beta_u'' - \frac{1}{4}\beta_u'^{\,2} + \beta_u^{\;2}K_u = 1$$

"*Envelope equation*"

To solve the linear transverse dynamics (a.k.a. betatron motion) in a lattice with a given $K_u(s)$ is necessary and sufficient to evaluate the betatron function $\beta_u(s)$.

Phase Space Trajectory

We can now combine the expressions for u and u' to remove the sine and cosine terms and obtain:

$$\gamma_u u^2 + 2\alpha_u u u' + \beta_u u'^2 = \varepsilon$$
$$u = x, y$$

Being ε a constant, the last expression is telling us that the LHS of the equation is a motion invariant (a.k.a. **Courant-Snyder invariant**)

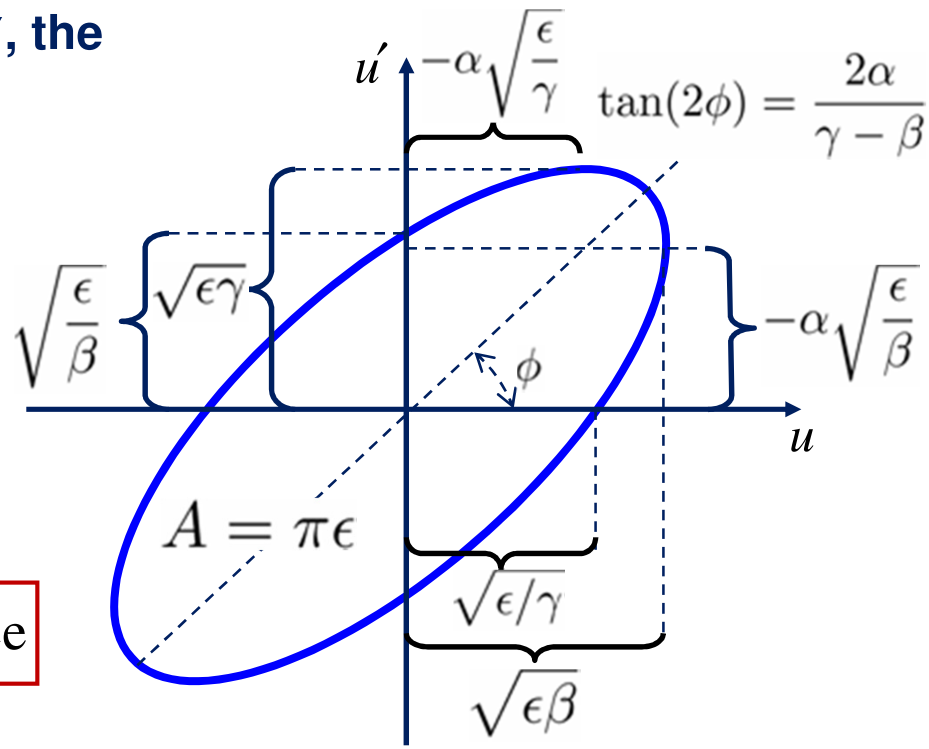
But in the phase space u, u', that expression is also the equation of an ellipse with area $\pi\varepsilon$.

- In other words, in the phase space u, u', the particle moves along an ellipse
  → **betatron oscillations**

- And the Twiss parameters assume significant meanings:

$$E_u(s) = \sqrt{\varepsilon \beta_u} \equiv \text{trajectory envelope}$$

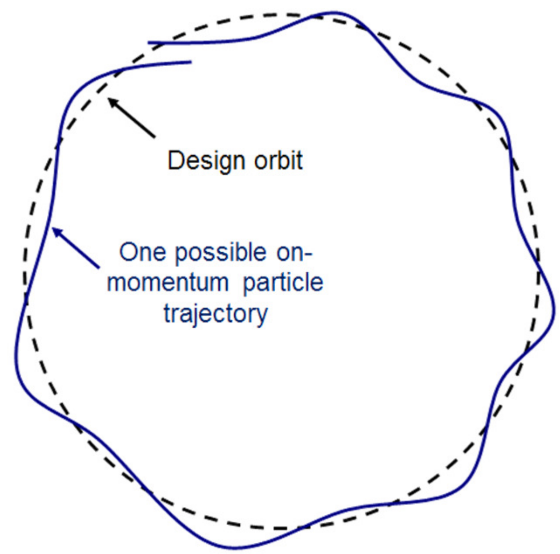
$$A_u(s) = \sqrt{\varepsilon \gamma_u} \equiv \text{trajectory max divergence}$$



Betatron Tune

The **betatron tune ν** (horizontal and vertical) is the number of betatron oscillations that a particle makes about the design trajectory over one ring turn.
(In nowadays storage rings $\nu > 1$ for both transverse planes)

$$\boxed{\nu_u = \frac{1}{2\pi}\oint_C \frac{ds}{\beta_u(s)}}$$



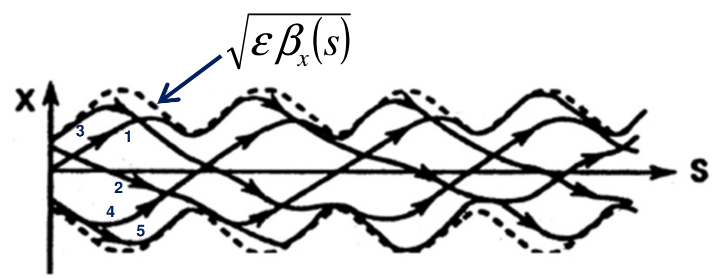
If f_0 is the revolution frequency of a particle around the ring then the **betatron frequencies** are defined as:

$$\boxed{f_{\beta_u} = \nu_u f_0 \qquad u = x, y}$$

It can be demonstrated that, in order for the trajectory to be confined, the **betatron tune cannot be an integer or a half-integer number.**
(Typically, any rational number is in general avoided)

Because of that, at a fixed s, a particle in subsequent turns does not pass at the same transverse position, but samples positions following the function:

$$u(s = s^*) = \sqrt{\varepsilon \beta_u(s^*)}\cos\left(2\pi\, f_{\beta_u} t\right)$$



Hill's Solution in Matrix Shape

Starting from Hill's solution:

$$u(s)=\sqrt{\varepsilon\beta(s)}\cos[\varphi(s)-\varphi_i],\quad u'(s)=-\sqrt{\frac{\varepsilon}{\beta(s)}}\{\alpha(s)\cos[\varphi(s)-\varphi_i]+\sin[\varphi(s)-\varphi_i]\}\quad u=x,y$$

we want now calculate how u and u' transform from $s=s_0$ to $s=s_1$

$$\Rightarrow\quad \cos[\varphi(s)-\varphi_i]=\frac{1}{\sqrt{\varepsilon\beta(s)}}u(s),\quad \sin[\varphi(s)-\varphi_i]=-\sqrt{\frac{\beta(s)}{\varepsilon}}u'(s)-\frac{\alpha(s)}{\sqrt{\varepsilon\beta(s)}}u(s)$$

Expanding the trigonometric functions and assuming $\varphi_{i\,s0}=0$ and $\varphi_{i\,s1}=\varphi(s_0)$, we can find :

$$u_1=\sqrt{\frac{\beta}{\beta_0}}[\cos(\Delta\varphi)+\alpha_0\sin(\Delta\varphi)]u_0+\sqrt{\beta\beta_0}\sin(\Delta\varphi)u_0'\qquad with\ \ \Delta\varphi=\varphi(s_1)-\varphi(s_0)$$

$$u_1'=\frac{1}{\sqrt{\beta\beta_0}}[(\alpha_0-\alpha)\cos(\Delta\varphi)-(1+\alpha_0\alpha)\sin(\Delta\varphi)]u_0+\sqrt{\frac{\beta_0}{\beta}}[\cos(\Delta\varphi)-\alpha\sin(\Delta\varphi)]u_0'$$

This is equivalent to the matrix form (general Hill's solution in matrix form):

$$\boxed{\begin{pmatrix}u_1\\u_1'\end{pmatrix}=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos(\Delta\varphi)+\alpha_0\sin(\Delta\varphi)] & \sqrt{\beta\beta_0}\sin(\Delta\varphi)\\ \frac{1}{\sqrt{\beta\beta_0}}[(\alpha_0-\alpha)\cos(\Delta\varphi)-(1+\alpha_0\alpha)\sin(\Delta\varphi)] & \sqrt{\frac{\beta_0}{\beta}}[\cos(\Delta\varphi)-\alpha\sin(\Delta\varphi)]\end{pmatrix}\begin{pmatrix}u_0\\u_0'\end{pmatrix}}$$

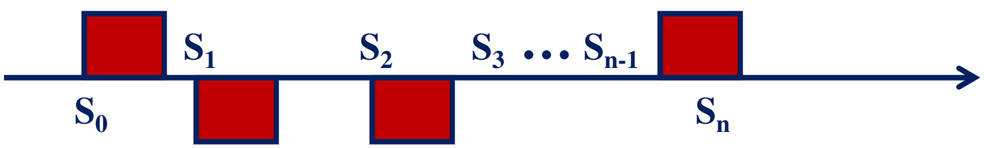
One Turn Transfer Matrix

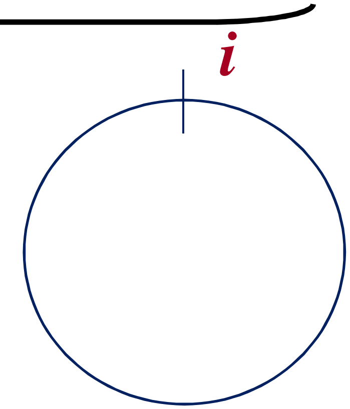
Single elements matrix for quadrupoles, dipoles, drifts can be now calculated and combined together to obtain a full ring transfer matrix.

$$\mathcal{M}(s_n|s_0) = \mathcal{M}(s_n|s_{n-1})\ldots\mathcal{M}(s_3|s_2)\cdot\mathcal{M}(s_2|s_1)\cdot\underbrace{\mathcal{M}(s_1|s0)}_{\text{from } s_0 \text{ to } s_1}$$

from s_0 to s_2

from s_0 to s_n





After a full turn because of the periodicity, we have $\alpha = \alpha_0$ and $\beta = \beta_0$ and the total phase advance is 2π the tune. So the one turn matrix M_C becomes:

$$\boxed{\begin{pmatrix} u_{i+1} \\ u'_{1+1} \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu) + \alpha\sin(2\pi\nu) & \beta\sin(2\pi\nu) \\ -\gamma\sin(2\pi\nu) & \cos(2\pi\nu) - \alpha\sin(2\pi\nu) \end{pmatrix}\begin{pmatrix} u_i \\ u'_i \end{pmatrix}}$$

and

$$\boxed{Trace\ \ M_C = 2\left|\cos(2\pi\nu)\right| \le 2 \quad solution \quad existence \quad condition}$$

Transfer matrix of a drift

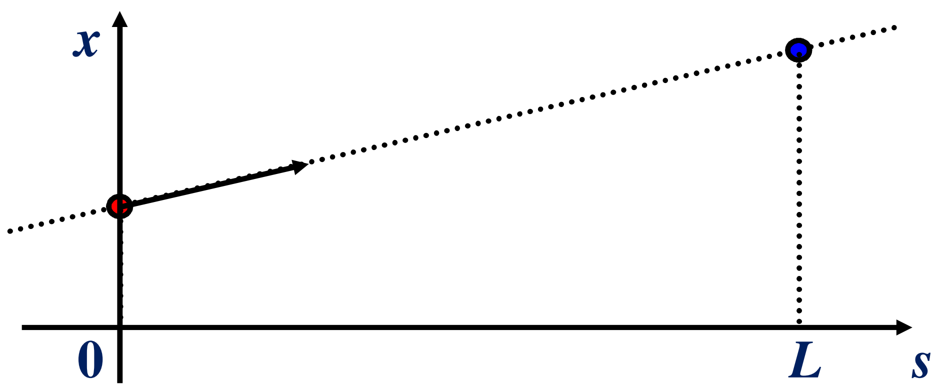
- **Consider a drift (no magnetic elements) of length $L = s - s_0$**

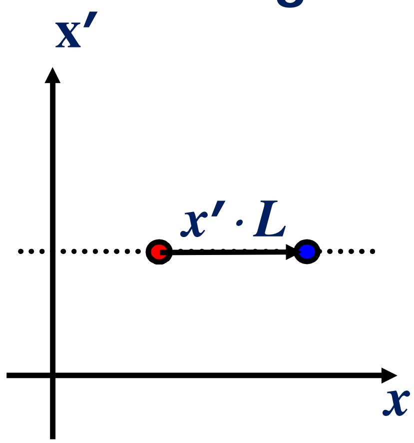
$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & s - s_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\boxed{\mathcal{M}_{\mathrm{drift}}(s|s_0) = \begin{pmatrix} 1 & s - s_0 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{aligned} u(s) &= u_0 + (s - s_0)u'_0 = u_0 + Lu'_0 \\ u'(s) &= u'_0 \end{aligned}$$

Position changes if there is a slope. Slope remains unchanged





Focusing – Defocusing Thin Lenses

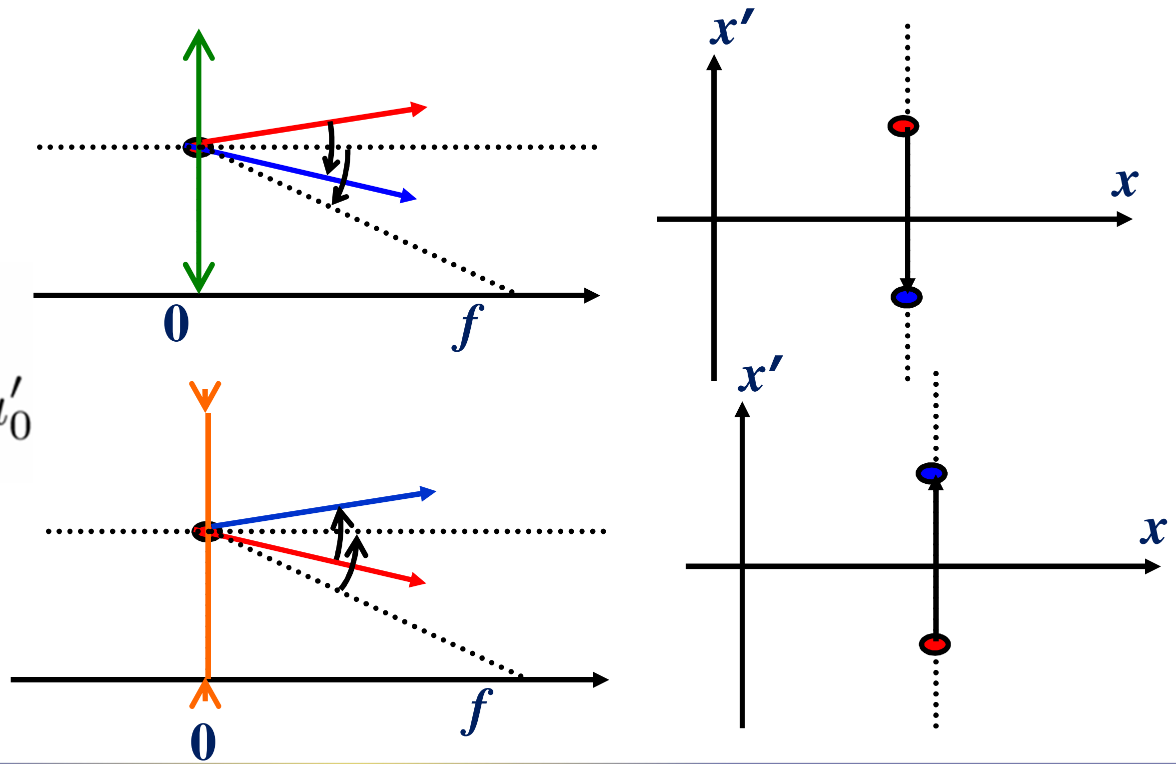
- Consider a lens with focal length $\pm f$

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$

$$\boxed{\mathcal{M}_{\mathrm{lens}}(s|s_0) = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}}$$

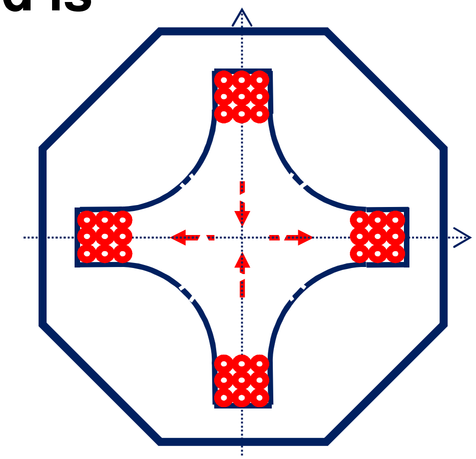
- **Slope diminishes (focusing) or increases (defocusing). Position remains unchanged**

$$\begin{aligned} u(s) &= u_0 \\ u'(s) &= u_0 \mp \frac{1}{f} u'_0 \end{aligned}$$



Real Quadrupole

- Consider a quadrupole magnet of length L. The field is



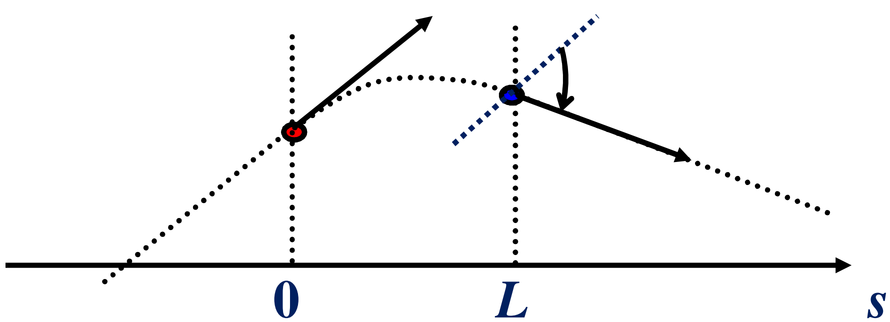
$$B_y = -G(s)x \;, \quad B_x = -G(s)y$$

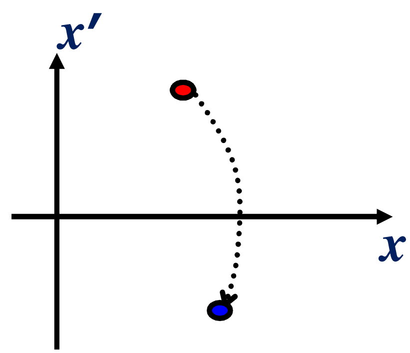
- with normalized quadrupole gradient (in m^{-2})

$$k = \frac{G}{B_0\rho}$$

- The transport through a quadrupole is

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{k}(s-s_0)) & \frac{1}{\sqrt{k}}\sin(\sqrt{k}(s-s_0)) \\ \sqrt{k}\sin(\sqrt{k}(s-s_0)) & \cos(\sqrt{k}(s-s_0)) \end{pmatrix} \begin{pmatrix} u_0 \\ u'_0 \end{pmatrix}$$





Off-Momentum Particle

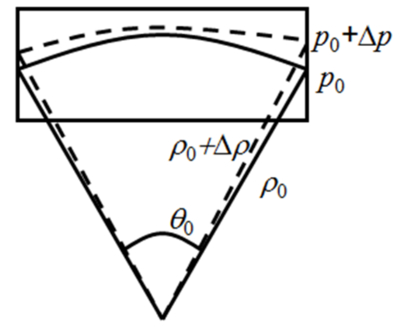
Magnets are chromatic elements:

Dipoles:

$$\rho = \frac{p}{eB} \Rightarrow \frac{\Delta\rho}{\rho_0} = \frac{\Delta p}{p_0}$$

$$\rho = \frac{L}{\theta} \Rightarrow \frac{\Delta\rho}{\rho_0} = -\frac{\Delta\theta}{\theta_0}$$

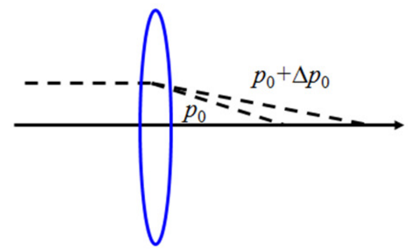
$$\boxed{\Delta\theta = -\theta_0 \frac{\Delta p}{p_0}}$$



Quadrupoles:

$$K = \frac{G}{B\rho} = \frac{Ge}{p}$$

$$\boxed{\Delta K = -K_0 \frac{\Delta p}{p_0}}$$



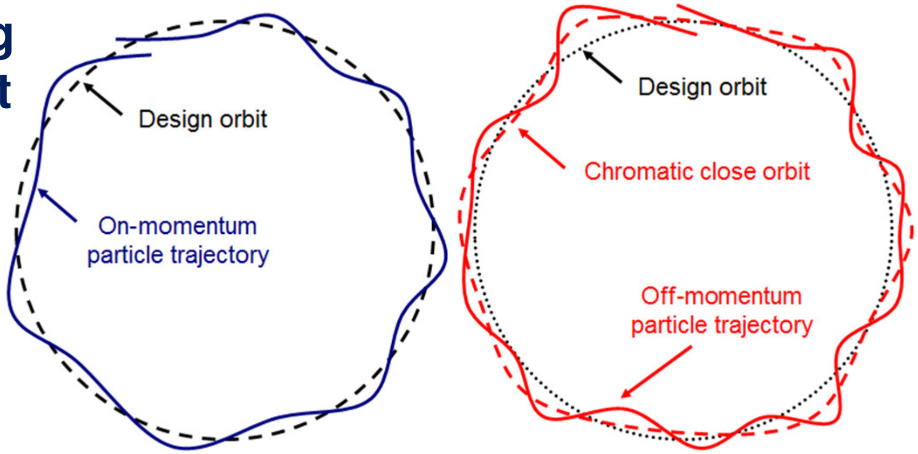
This implies that off-momentum particles will have different reference orbit and different optical functions

Because in a typical accelerator: $\dfrac{\Delta p}{p_0} << 1$

Such chromatic effects can be treated as perturbations of the on-energy case.

Chromatic Closed Orbit

Off-momentum particles are not oscillating around design orbit, but around a different closed orbit (chromatic closed orbit).



The displacement between the design and chromatic orbits is regulated by the dispersion function $D(s)$

$$\boxed{\begin{array}{l} u_{\Delta p/p_0} = D_u(s)\dfrac{\Delta p}{p_0} \\ u = x, y \end{array}}$$

The dispersion is the trajectory of a 100% off-momentum particle.

In the perturbation approach, the Hill's equation for the off-momentum particle assumes the form:

$$u'' + K_u(s)\,u = \frac{1}{\rho(s)}\frac{\Delta p}{p_0}$$

$$\rho \equiv local\ curvature\ radius$$

$$+ \quad u = u_{Homogenous} + D_u(s)\frac{\Delta p}{p_0} \quad \Rightarrow$$

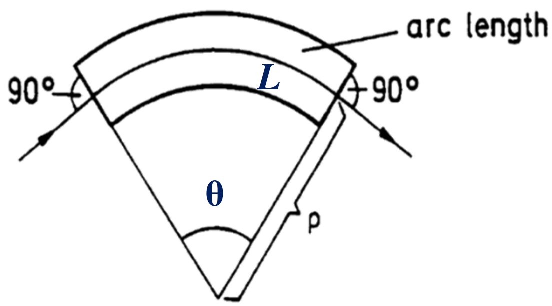
Dispersion equation.

$$\boxed{D_u''(s) + K_u(s)D_u(s) = \frac{1}{\rho(s)}}$$

At first order, in a planar ring the dispersion is nonzero only in the ring plane.

Example: Sector Magnet Case

Let's consider the motion through a sector dipole with constant bending radius ρ (sharp edge model)



The dispersion equation becomes: $D''(s) + \frac{1}{\rho^2} D(s) = \frac{1}{\rho}$

The solution, given by the homogeneous equation solution plus a particular solution of the nonhomogeneous one, can be easily calculated as:

$$\begin{aligned} D(s) &= D_0 \cos(\frac{s}{\rho}) + D_0' \rho \sin(\frac{s}{\rho}) + \rho(1 - \cos(\frac{s}{\rho})) \\ D'(s) &= -\frac{D_0}{\rho} \sin(\frac{s}{\rho}) + D_0' \cos(\frac{s}{\rho}) + \sin(\frac{s}{\rho}) \end{aligned}$$

The matrix formalism can be extended to include the dispersion terms:

$$\begin{pmatrix} x(s) \\ x'(s) \\ \Delta p/p \end{pmatrix} = \mathcal{M}_{3\times 3} \begin{pmatrix} x(s_0) \\ x'(s_0) \\ \Delta p/p \end{pmatrix} \qquad \mathcal{M}_{3\times 3} = \begin{pmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix}$$

So for the sector dipole case:

$$\boxed{\mathcal{M}_{\text{sector}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}}$$

Chromatic Aberration Correction

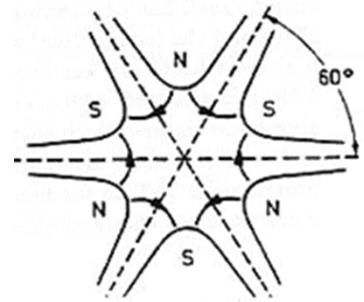
Chromatic focusing of quadrupoles generates distortion of optical functions and tune variation for off-momentum particles.

The amplitude of the effect is measured in terms of chromaticity:

$$\xi_u = \frac{\Delta\nu_u / \nu_u}{\Delta p / p_0} \qquad u = x, y$$

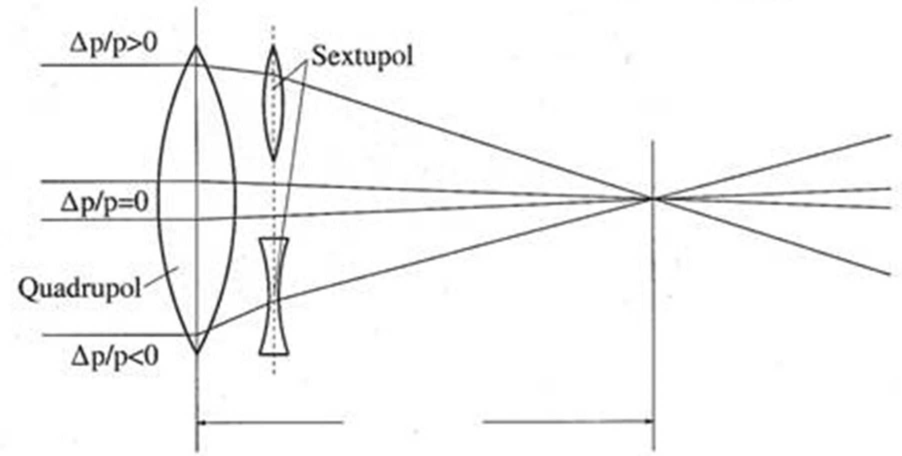
By including sextupoles in nonzero dispersion areas, it is possible to compensate (to first order) for chromaticity.

$$B_x = 2S\,x\,y$$
$$B_y = S\left(x^2 - y^2\right)$$



The sextupole gives a "position-dependent quadrupole":

$$D \neq 0 \quad region$$



Sextupoles are nonlinear elements that can affect lifetime and injection efficiency performance in a ring. Their use is necessary but their effects must be carefully evaluated.

Optics or Lattice Design Set Beam Properties

Choice of the lattice design depends upon the goal for the storage-ring based light source:

Performance goals:

- High photon brightness (implies high electron beam brightness)
  - Small beam spot
  - Small divergence
  - High coherence
- Short bunches
- High luminosity

Other goals:

- Obey certain physical constraints (building or tunnel).
- …

Direct and Inverse Problems

Direct problem:

"Given an existing lattice, determine the properties of the beam".

This first problem is in principle straight-forward to solve.

Inverse problem:

"For a desired set of beam properties, determine the design of the lattice."

This problem, which is the one of main interest, is not straight-forward, it is actually a bit of an art.

Numerical and Analytical

We pointed out earlier that to completely solve a lattice characterized by $K(s)$ is sufficient to evaluate the optical function $\beta(s)$

$$\frac{1}{2}\beta_u\beta_u'' - \frac{1}{4}\beta_u'^2 + \beta_u^{\,2}K_u = 1$$

Unfortunately, the equation can be solved analytically only in few cases.
In the large majority of the cases the solution must be found numerically

Typically, in computer codes the particle trajectory along the lattice (whole ring or periodical cell of it) is numerically calculated. In this way the generic matrix of the lattice is found

$$\begin{pmatrix} u_{i+1} \\ u'_{1+1} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} u_i \\ u'_i \end{pmatrix}$$

If $m_{11} + m_{22} \le 2$ *stability criterion* then by comparison with

$$\begin{pmatrix} u_{i+1} \\ u'_{1+1} \end{pmatrix} = \begin{pmatrix} \cos(2\pi\nu) + \alpha\sin(2\pi\nu) & \beta\sin(2\pi\nu) \\ -\gamma\sin(2\pi\nu) & \cos(2\pi\nu) - \alpha\sin(2\pi\nu) \end{pmatrix}\begin{pmatrix} u_i \\ u'_i \end{pmatrix}$$

we can calculate:

$$\cos(2\pi\nu) = \frac{m_{11}+m_{22}}{2}, \quad \beta = \frac{m_{12}}{\sin(2\pi\nu)}, \quad \gamma = -\frac{m_{21}}{\sin(2\pi\nu)}, \quad \alpha = \frac{m_{11}-m_{22}}{2\sin(2\pi\nu)}$$

The last step is to calculate the closed orbit solving

$$\begin{pmatrix} u_i \\ u'_i \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\begin{pmatrix} u_i \\ u'_i \end{pmatrix}$$

and find the dispersion orbit $D(s)$ around it.

Multi-Objective Optimization

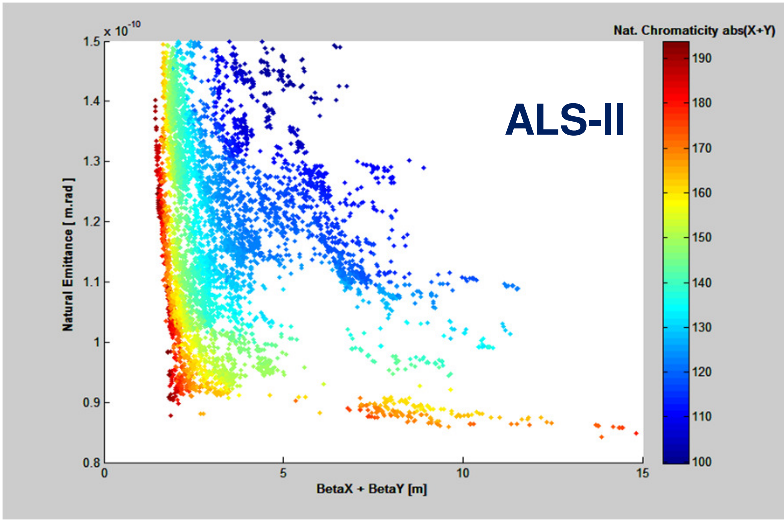
In the design of a real ring, a number of parameters (quadrupole strengths, position, drift length, …) can be varied to optimize the desired objectives.

Multiple objectives (brightness, beam sizes, …) are often simultaneously optimized.

In other words, we are dealing with a multi-objective optimization process. Systematic scanning when the number of parameters increase, can become difficult (or impossible) in terms of computer power and more efficient methods are required.

In relatively recent times, multi-objective genetic algorithms (MOGA) became very popular among lattice designers. See for example:

Lingyun Yang, et al.. *Global optimization of an accelerator lattice using multiobjective genetic algorithms*, Nuclear Instruments & Methods in Physics Research A **609**, 50 (2009).



Acknowledgements

Wish to thank Y. Papaphilippou, N.Catalan-Lasheras and David Robin for sharing some material that was used in this lecture

L3: Homework

1) Based on the information in slide 14, determine if a rectangular magnet is focusing in the horizontal plane for on energy particles

2) In the lecture, the accent was placed on the transverse dynamics in storage rings.
Can the same formalism be applied to transfer-lines (non-periodic beamlines)?
In particular, are the Hill's equations still valid?
Can the Twiss parameter description be used?
Can the matrix formalism be used?

3) At the Advanced Light Source (ALS) in Berkeley, an electron beam of 1.9 GeV is stored in a storage ring with 197 m of "circumference".
The ring horizontal tune is 19.2 while the vertical is 10.25.
Calculate the hor. and vert. betatron frequencies in Hz for the ALS.

Backup Slides

General Matrix Derivation

$$u(s)=\sqrt{\varepsilon\beta(s)}\cos[\psi(s)-\psi_0],\quad u'(s)=-\sqrt{\frac{\varepsilon}{\beta(s)}}\{\alpha(s)\cos[\psi(s)-\psi_0]+\sin[\psi(s)-\psi_0]\}$$

$$\cos[\psi(s)-\psi_0]=\frac{1}{\sqrt{\varepsilon\beta(s)}}u(s),\quad \sin[\psi(s)-\psi_0]=-\sqrt{\frac{\beta(s)}{\varepsilon}}u'(s)-\frac{\alpha(s)}{\sqrt{\varepsilon\beta(s)}}u(s)$$

$$\cos(\psi)\cos(\psi_0)+\sin(\psi)\sin(\psi_0)=\frac{u}{\sqrt{\varepsilon\beta}},\quad \sin(\psi)\cos(\psi_0)-\cos(\psi)\sin(\psi_0)=-\sqrt{\frac{\beta}{\varepsilon}}u'-\frac{\alpha}{\sqrt{\varepsilon\beta}}u$$

$$\cos(\psi_0)=\frac{u_0}{\sqrt{\varepsilon\beta_0}},\quad \sin(\psi_0)=\sqrt{\frac{\beta_0}{\varepsilon}}u_0'+\frac{\alpha_0}{\sqrt{\varepsilon\beta_0}}u_0\quad when\ \ \psi(s)=0$$

$$u=\sqrt{\varepsilon\beta}\cos(\Delta\psi)\frac{u_0}{\sqrt{\varepsilon\beta_0}}+\sqrt{\varepsilon\beta}\sin(\Delta\psi)\left[\sqrt{\frac{\beta_0}{\varepsilon}}u_0'+\frac{\alpha_0}{\sqrt{\varepsilon\beta_0}}u_0\right]$$

$$u=\sqrt{\frac{\beta}{\beta_0}}[\cos(\Delta\psi)+\alpha_0\sin(\Delta\psi)]u_0+\sqrt{\beta\beta_0}\sin(\Delta\psi)u_0'$$

$$u'=\frac{1}{\sqrt{\beta\beta_0}}[(\alpha_0-\alpha)\cos(\Delta\psi)-(1+\alpha_0\alpha)\sin(\Delta\psi)]u_0+\sqrt{\frac{\beta_0}{\beta}}[\cos(\Delta\psi)-\alpha\sin(\Delta\psi)]u_0'$$

4x4 Matrices

- **Combine the matrices for each plane**

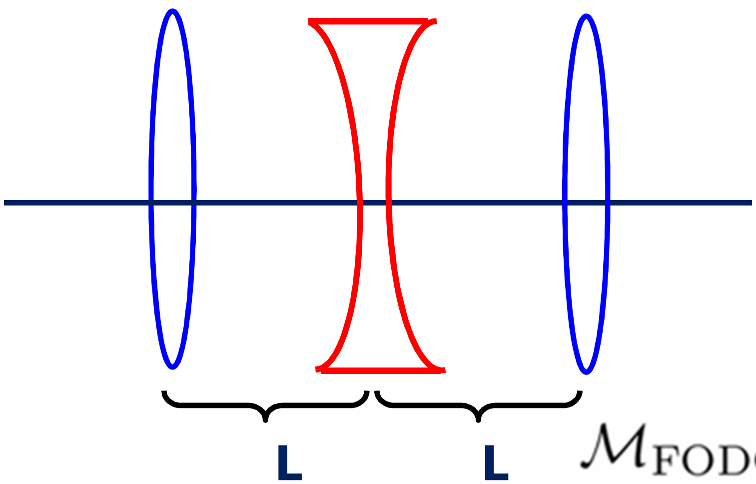
$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) \\ C'_x(s) & S'_x(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_y(s) & S_y(s) \\ C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}$$

to get a total 4x4 matrix

$$\begin{pmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \\ y_0 \\ y'_0 \end{pmatrix}$$

FODO Cell



- Consider a defocusing quadrupole "sandwiched" by two focusing quadrupoles with focal lengths **f**.
- The symmetric transfer matrix from center to center of focusing quads

$$\mathcal{M}_{\mathrm{FODO}} = \mathcal{M}_{\mathrm{HQF}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{QD}} \cdot \mathcal{M}_{\mathrm{drift}} \cdot \mathcal{M}_{\mathrm{HQF}}$$

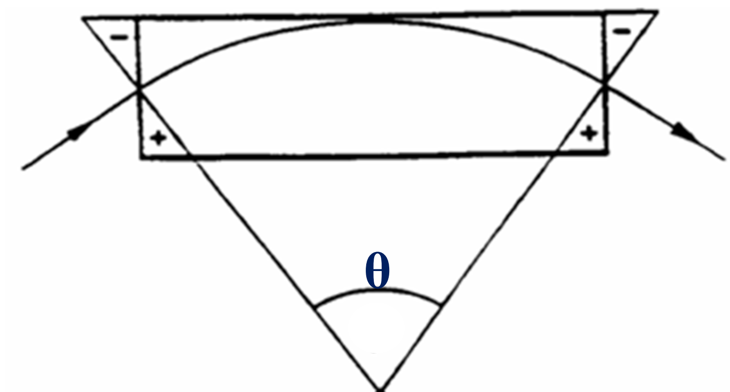
with the transfer matrices

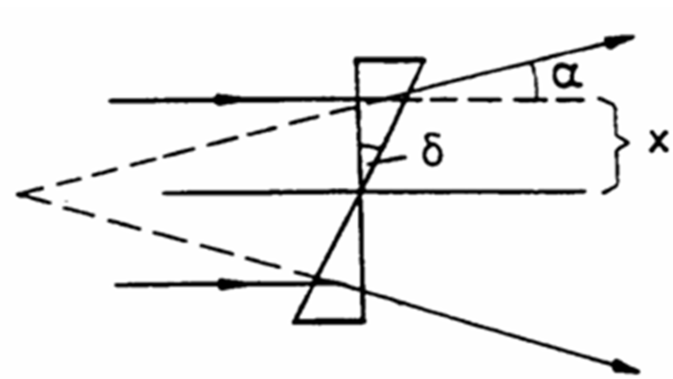
$$\mathcal{M}_{\mathrm{HQF}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad \mathcal{M}_{\mathrm{QD}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

- The total transfer matrix is

$$\boxed{\mathcal{M}_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) \\ \frac{L}{2f^2}(1 - \frac{L}{2f}) & 1 - \frac{L^2}{2f^2} \end{pmatrix}}$$

Rectangular Dipole





- Consider a rectangular dipole of length **L**. At each edge, the deflecting angle is

$$\alpha = \frac{\Delta L}{\rho} = \frac{\theta \tan\delta}{\rho} \qquad \frac{1}{f} = \frac{\tan\delta}{\rho}$$

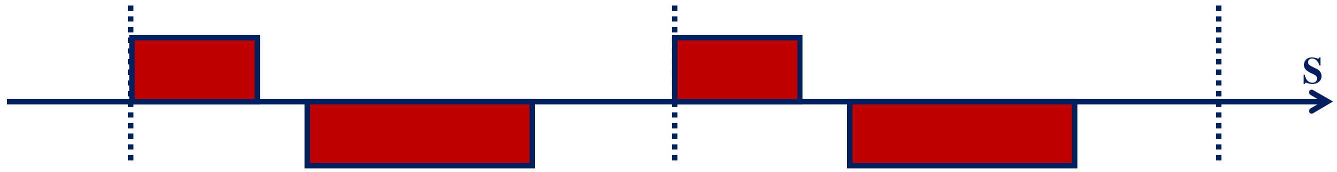
It acts as a thin defocusing lens with focal length

- The transfer matrix is $\mathcal{M}_{\text{rect}} = \mathcal{M}_{\text{edge}} \cdot \mathcal{M}_{\text{sector}} \cdot \mathcal{M}_{\text{edge}}$ with $\mathcal{M}_{\text{edge}} = \begin{pmatrix} 1 & 0 \\ \frac{\tan(\delta)}{\rho} & 1 \end{pmatrix}$
- For **θ<<1**, **δ=θ/2**.
- In deflecting plane (like **drift**) in non-deflecting plane (like **sector**)

$$\mathcal{M}_{x;\text{rect}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} \qquad \mathcal{M}_{y;\text{rect}} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

Symmetric Beamlines

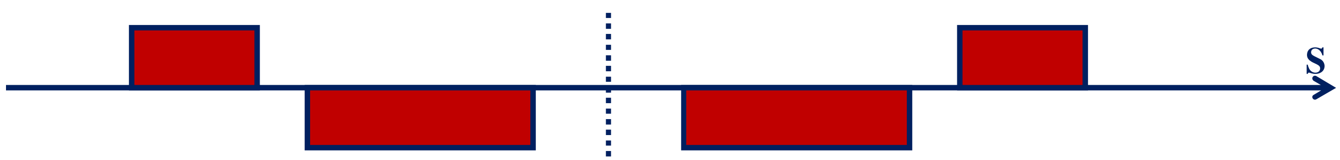
- System with normal symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M} \cdot \mathcal{M} = \mathcal{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

- System with mirror symmetry



$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \mathcal{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$

$$\mathcal{M}_{\text{tot}} = \mathcal{M}_r \cdot \mathcal{M} = \begin{pmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{pmatrix}$$