This solution is based on B. Rayhaun's homework.

$\boxed{1}$ Longitudinal dynamics of the ALS

For $E = 1.9\ GeV$, we will assume that the electrons are essentially going at the speed of light for some of the calculations. We note that

$$f_0 = f_{rf}/h = 1.523\ MHz$$

but also that

$$f_0 = c/C \implies C = 196\ m$$

The synchronous phase is given by

$$\varphi_s = \sin^{-1}\left(\frac{U_0}{qV}\right) = 7.18^\circ = .125\ radians$$

Since we are considering electrons that are highly relativistic, we'll assume $p_0 \approx 1.9\ GeV$ also. Then the synchrotron frequency is given by

$$\Omega^2 = \omega_0^2 \frac{q}{p_0} \frac{\alpha h V}{2\pi\beta_0 c} \cos(\varphi_s) \implies \Omega = 69229\ Hz$$

where we obtained $\omega_0 = 2\pi f_0$ calculated above.

The synchrotron tube is given by

$$v_s = \frac{\Omega}{\omega_0} = .0072$$

For the next part, we assume that $\sigma_p/p_0 = 10^{-3}$. Then, we get that

$$\sigma_{\Delta S} = \frac{c\alpha}{\Omega}\frac{\sigma_p}{p_0} = 6.9\ mm$$

Now, we derived φ_s using the voltage, but it actually doesn't depend on the voltage. It can be obtained with $\varphi_s = \omega_{rf} t_0$. So then, we know that $\sigma_{\Delta S} \propto 1/\sqrt{V}$ so that

$$\sigma_1/\sigma_2 = \sqrt{V_2}/\sqrt{V_1}.$$

Then we get that

$$V_2 = V_1 \left(\frac{\sigma_1}{\sigma_2}\right)^2 = 6.3\ MV$$

$\boxed{1}$ RLC circuits The impedance is given by

$$\frac{1}{Z} = \frac{1}{Z_L} + \frac{1}{Z_C} + \frac{1}{Z_R} = \frac{1}{j\omega L} + j\omega C + \frac{1}{R}$$

which we can simplify a little to get

$$Z = \left(\frac{1 - \omega^2 LC}{j\omega L} + \frac{1}{R} \right)^{-1}$$

since we're given that the impedance at the resonant frequency is purely resistive we want to eliminate the terms involving C and L. This happens precisely when the numberator of the first term is 0, or when

$$1 - \omega_0^2 LC = 0 \implies \omega_0 = \frac{1}{\sqrt{LC}}$$

so that we have derived the resonant frequency.

The total energy stored is

$$E = \frac{1}{2}\left[Li^2 + CV^2\right]$$

The AC voltage at resonance is $V = A\sin(\omega_0 t)$ so that $i = A\omega_0 C\cos(\omega_0 t)$. Then

$$E = \frac{1}{2}\left[LC^2 A^2 \omega_0^2 \cos^2(\omega_0 t) + CA^2 \sin^2(\omega_0 t)\right] = \frac{1}{2}CA^2$$

Now, the energy dissipated per radian is the energy dissipated per cycle divided by 2π, i.e.

$$E_d = \frac{1}{2\pi} RA^2 C^2 \omega_0^2 \int_0^{2\pi/\omega_0} \cos^2(\omega_0 t) dt = \frac{1}{2} RA^2 C^2 \omega_0$$

so that their ratio is

$$Q' = \frac{E}{E_d} = \frac{1}{RC\omega_0} = \frac{\sqrt{L}}{R\sqrt{C}}$$

so that, after invoking duality and swapping $R \leftrightarrow \frac{1}{R}$ and $L \leftrightarrow C$, we get

$$Q = \frac{\sqrt{C}R}{\sqrt{L}} = \frac{CR}{\sqrt{LC}} = \omega_0 RC$$

and we can get the other form by

$$\omega_0 RC = R\sqrt{C}/\sqrt{L} = R\sqrt{CL}/L = R/(L\omega_0)$$

which is the desired answer. Now, the impedance is given by

$$Z = \left(\frac{R - \frac{R\omega^2}{\omega_0^2} + j\omega L}{j\omega LR} \right)^{-1} = R\left(\frac{R}{j\omega L} - \frac{R\omega^2}{\omega_0^2 j\omega L} + 1 \right)^{-1} = R\frac{1}{-jQ\frac{\omega_0}{\omega} + jQ\frac{\omega}{\omega_0} + 1}$$

which is the desired answer. We can obtain the approximation by approximating $\omega + \omega_0 \approx 2\omega$ which gives us

$$Z \approx \frac{R}{1 + jQ\left(\frac{(\omega - \omega_0)(\omega + \omega_0)}{\omega\omega_0}\right)} = \frac{R}{1 + 2jQ\frac{\delta\omega}{\omega_0}}$$

where $\delta\omega := \omega - \omega_0$.

2 Cavity homework 2

a) We'll assume that v doesn't change much while it goes through the cavity and that the particle is essentially nonrelativistic. A particle at this speed will spend $\tau = L/v$ in the RF cavity. Thus we can obtain the energy it gets as

$$W = q \int_{cavity} E ds = qvE_0 \int_{-\tau/2}^{\tau/2} \cos(\omega t) dt = \frac{qvE_0}{\omega} \sin(\omega t)|_{-\tau/2}^{\tau/2} = \frac{2qvE_0}{\omega} \sin U$$

the energy obtained when the field is constant at peak value is

$$W' = qE_0 \int_{-\tau/2}^{\tau/2} v dt = qE_0 L$$

Taking the ratio of these two gives us

$$T = \sin(U)/U$$

as desired.

b) The only place the L appears is in $\sin(U)$ in the expression $\frac{2qvE_0}{\omega} \sin U$. sin achieves its max at $U = \pi/2$ which corresponds to

$$L = \frac{\pi c}{\omega}$$

in our case.