U.S. Particle Accelerator School
Education in Beam Physics and Accelerator Technology

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Transverse Dynamics, Single Particle

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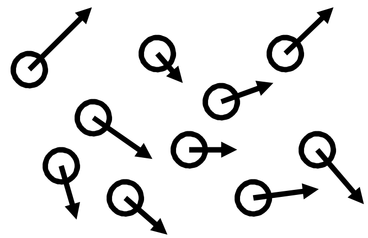
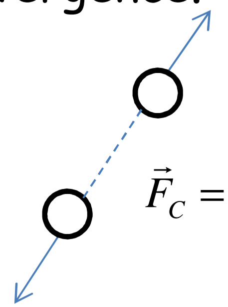
Magnetic Focusing

- Any beam of same-charge particles tend to disperse because of **repulsive Coulomb forces** and initial particles' angular divergence.





$$\vec{F}_C = C\frac{e^2}{r^3}\vec{r}$$

- **External transverse focusing** maintains the charge density high. For ultra-relativistic particles, magnetic focusing is more practical and efficient than electric. $\vec{F}_L = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$ *is the Lorentz force. To produce the same work of 1 MeV over 1 m, we need E = 1 MV/m or just B = 0.3 T.*

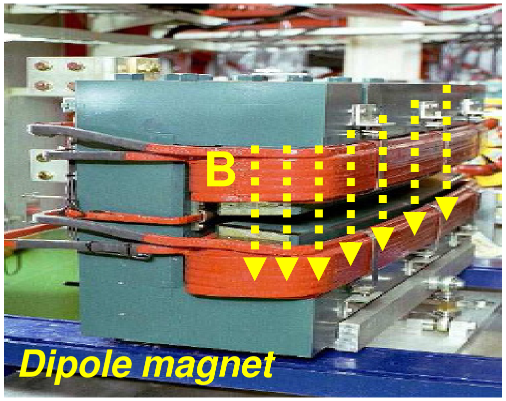
- An FEL beam delivery system is a sequence of RF and **magnetic** elements.
  - **Dipole** magnets [$B_y=B_0$] are used in spectrometer lines for beam dump and diagnostic, in magnetic compressors and transfer lines. They determine the beam direction.
  - **Quadrupole** magnets [$B_y=(dB_y/dx)\Delta x$] are in between RF structures, diagnostic stations, transfer lines and undulator. They determine the beam transverse size.
  - **Sextupole** magnets [$B_y=(d^2B_y/d^2x)\Delta x^2$] are rarely used in dispersive regions for linearization of the longitudinal phase space.

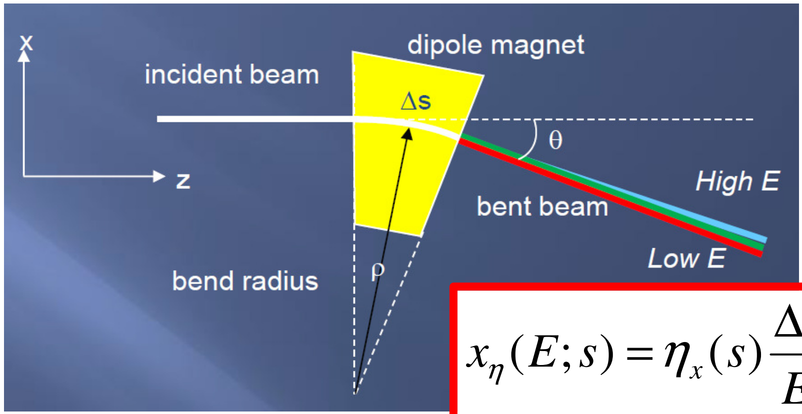
Dipole Magnet

- Particles with different **longitudinal momentum** follow different trajectories (*i.e.*, bending radius) according to:

$$p_z[GeV/c] = 0.2998 \cdot B_y[T] \cdot R[m]$$

- The lateral separation from the reference (*i.e.*, on-energy) trajectory per unit relative energy deviation is the **longitudinal momentum dispersion** function:





$$x_\eta(E;s) = \eta_x(s)\frac{\Delta E}{E_0}$$

- Together with the beam energy spread, η_x determines the ***chromatic beam size***. This can be regulated (or made null) along the beam line by controlling η_x:

$$\sqrt{\left\langle x_\eta^2(s)\right\rangle_N} = \left(\eta_x^2(s)\left\langle\frac{\Delta E}{E_0}\right\rangle_N^2\right)^{1/2} = \eta_x(s)\sigma_\delta \equiv \sigma_{x,\eta}(s)$$

EXERCISE: demonstrate the aforementioned relationship between p_z and B_y. *Hint*: use equation motion for the radial coordinate.

Quadrupole Magnet

- A quadrupole magnet implies a **transverse force** that is ***linear*** with the particle's **transverse displacement** from the quadrupole magnetic axis.

Linear gradient:

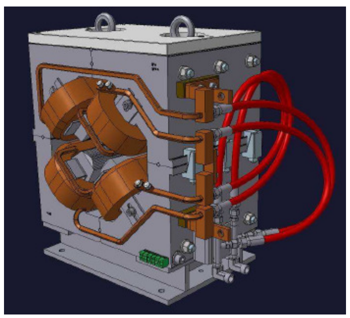
$$x''(s) = \frac{e}{p_z}\frac{dB_y}{dx}(s)x(s) \equiv kx$$

Normalized gradient:

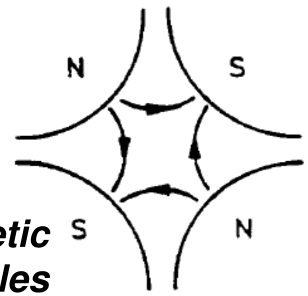
$$k[m^{-2}] = 0.2998\frac{g[T/m]}{p_z[GeV/c]}$$

Focusing length:

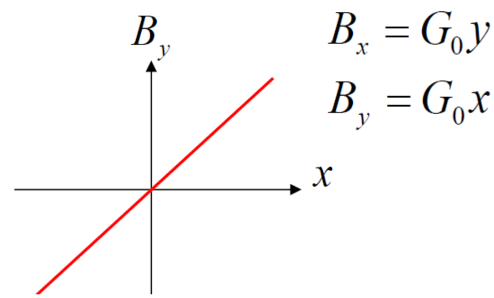
$$f[m] = \frac{1}{kl}$$



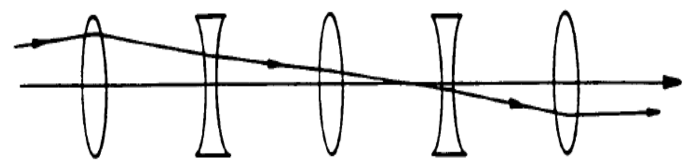
Normally-oriented quadrupole magnet



Magnetic poles



- **Alternating Strong Focusing** (alternating series of QF and QD) leads to **overall** ***focusing***, in ***both transverse planes***.

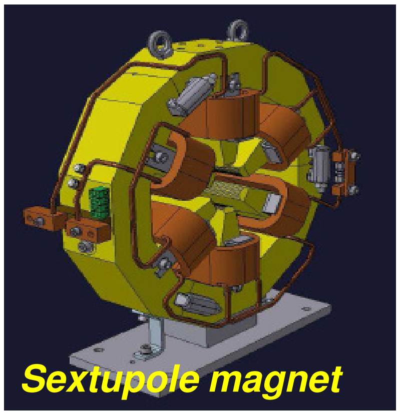


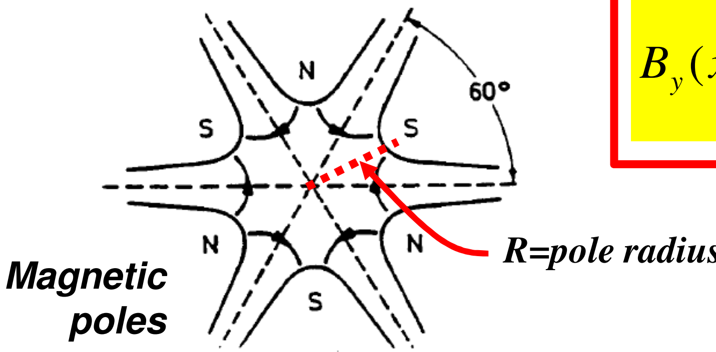
- If we consider the motion of the beam centroid into a **displaced quadrupole magnet**, we find that the beam is **kicked by**: $x' = klx$

EXERCISE: demonstrate the aforementioned relationship for the linear focusing. *Hint*: start from Lorentz force. Verify that a quadrupole focusing in one plane is *defocusing* in the other.

Multi-Pole Field Expansion

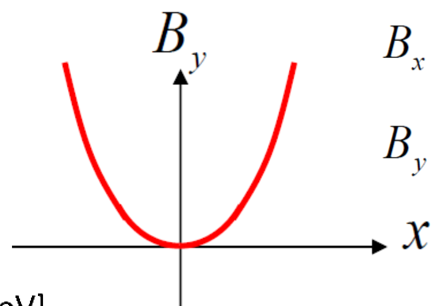
- Higher order magnets (*e.g.*, sextupoles) introduce **nonlinear focusing**, *i.e.* the restoring force goes like x^q, with $q \geq 2$. When used in dispersive regions, they **couple x_β and x_η**.





Multipolar field expansion:

$$B_y(x) = \sum_{0}^{n} b_n \left(\frac{x}{R}\right)^n_{y=0} \qquad b_n = \frac{1}{n!}\left(\frac{\partial^n B_y}{\partial x^n}\right)_{y=0} R^n$$

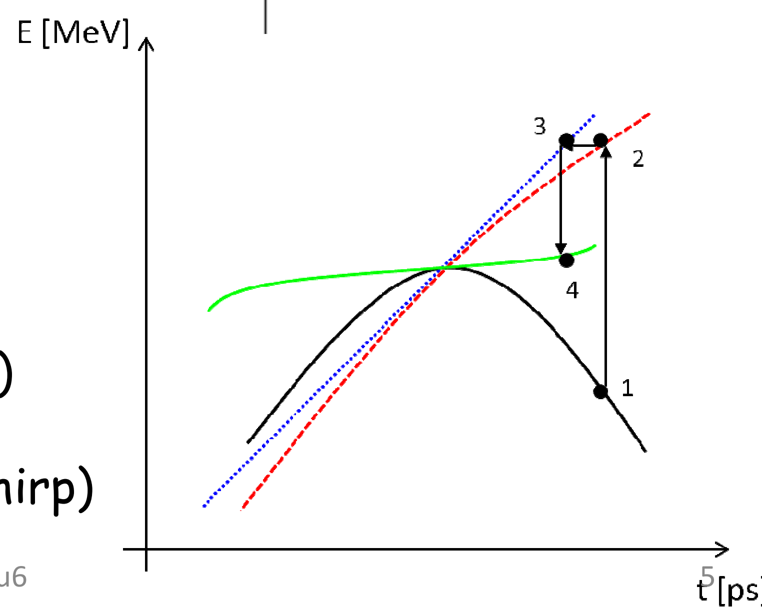


$$B_x = Cxy$$

$$B_y = \frac{1}{2} C\left(x^2 - y^2\right)$$

- *Sextupoles* used in *dispersive regions* and in the presence of *correlated energy spread*, can be used to manipulate (*e.g.*, linearize) the longitudinal phase space.

1. RF curvature
2. Off-crest acceleration (adds linear E-chirp)
3. Sextupole in dispersive region
4. Off-crest acceleration (removes linear E-chirp)



Hill's Equation

$$\gamma m_e\left(\ddot{r}-\dot{\theta}^2 r+\frac{\dot{\gamma}}{\gamma}\dot{r}\right)=-e\left(\vec{v}\times\vec{B}\right);$$

← r → x
expand B up to ***first order*** in x
d/dt → d/ds
consider an off-momentum $p_z = \gamma m_e v_z = p_{z,0}(1+\delta)$

$$x''(s)+\frac{\gamma'(s)}{\gamma(s)}x'(s)+\left[k(s)(1-\delta)-\frac{1}{R(s)^2}\right]x(s)=\frac{\delta}{R(s)}$$

x_β, solution of the homogeneous eq. describes the **betatron oscillations** (below, on-energy and with no acceleration)

x_η, solution of the complete eq. describes the **energy dispersion**, η_x.

SINGLE PARTICLE, LINEAR β-MOTION

$$x_\beta(s)=\sqrt{2J_x\beta_x(s)}\cos\Delta\mu_x$$

$$x_\beta'(s)=\frac{dx_\beta}{ds}=-\sqrt{\frac{2J_x}{\beta_x(s)}}\left[\alpha_x(s)\cos\Delta\mu_x+\sin\Delta\mu_x\right]$$

β-PHASE ADVANCE:

$$\Delta\mu_x(s)=\int_0^s\frac{1}{\beta_x(s')}ds'$$

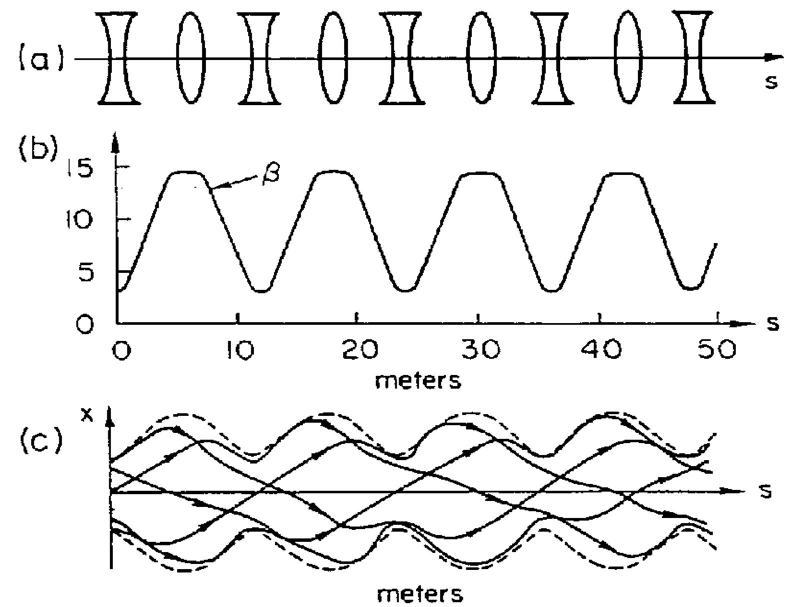
where: $\alpha_x=-\frac{1}{2}\frac{d\beta_x}{ds}$

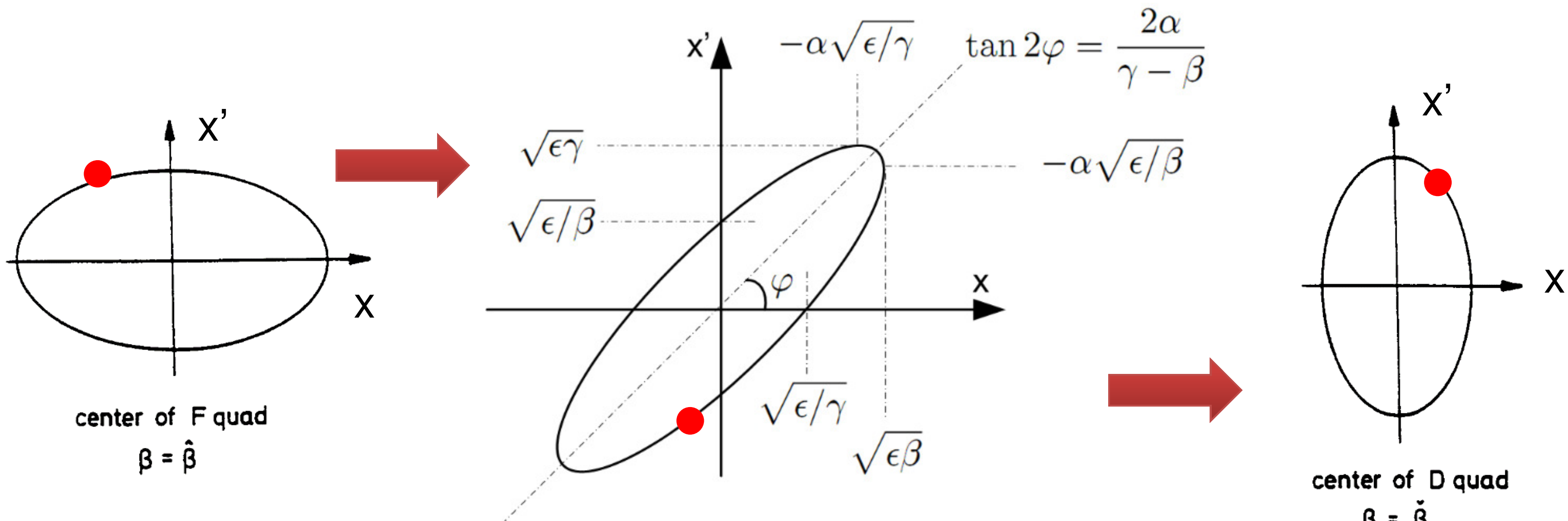
$$\gamma_x=\frac{1+\alpha_x^2}{\beta_x}$$

- β, α, γ are called Parameters of **Courant-Snyder** (also **Twiss** functions).
- Only 2 independent parameters over 3

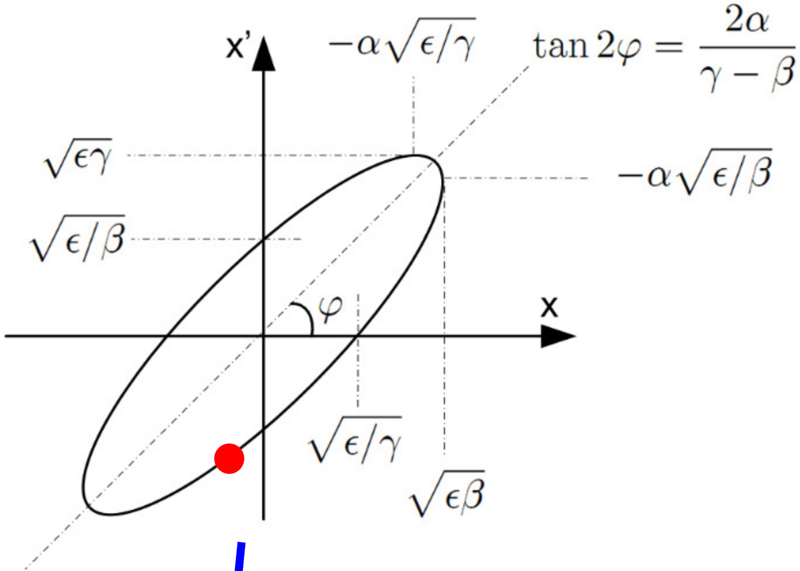
Single Particle, Phase Space Ellipse

- **(x_β, x'_β)** describe a **pseudo-harmonic oscillator**: motion is bounded, but the *oscillation amplitude depends on the s-coordinate* (or time).
- Like for an oscillator, the particle's trajectory maps an **ellipse** in the **phase space** (x,x').
- The ellipse's geometry is set by the **Twiss functions**. Thus, it **changes sizes and orientation at any s** (t).





Single Particle, Courant Snyder Invariant



Theorem: *the ellipse area is* ***constant*** *for a* ***linear motion****, and equal to :*

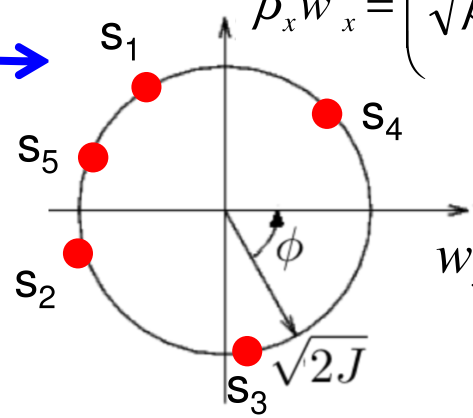
$$\varepsilon(s) = \left(\gamma_x x_\beta^2 + 2\alpha_x x_\beta x_\beta' + \beta_x x_\beta'^2\right)/\pi$$

Courant-Snyder Invariant

Verify: immediate for α=0, see diagram.
Verify: substitute x(s),x'(s) in ε(s)

The ellipse can be mapped to a circle, by using the so-called normalized **Floquet's coordinates**:

$$\beta_x w'_x = \left(\sqrt{\beta_x}\, x'_\beta + \frac{\alpha_x}{\sqrt{\beta_x}} x_\beta\right)$$



$$w_x = \frac{x_\beta}{\sqrt{\beta_x}}$$

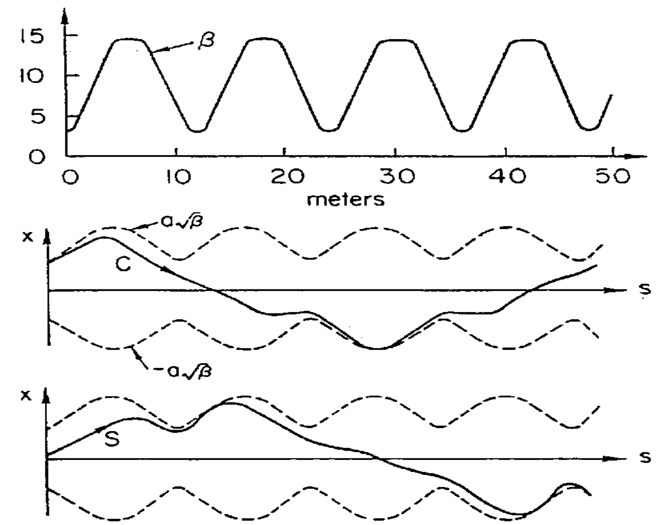
$$w_x^2 + \left(\beta_x w'_x\right)^2 = \varepsilon \equiv 2J$$

$$\phi(s) = \arctan\left(\frac{\beta_x w'_x}{w_x}\right)$$

Principal Trajectories

- The general solution of Hill's equation can equivalently be cast in the form of linear superposition of **two particulr solutions C(s) and S(s)**, whose initial conditions are <u>C(0)=1, S(0)=0, C'(0)=0, S'(0)=1</u>:

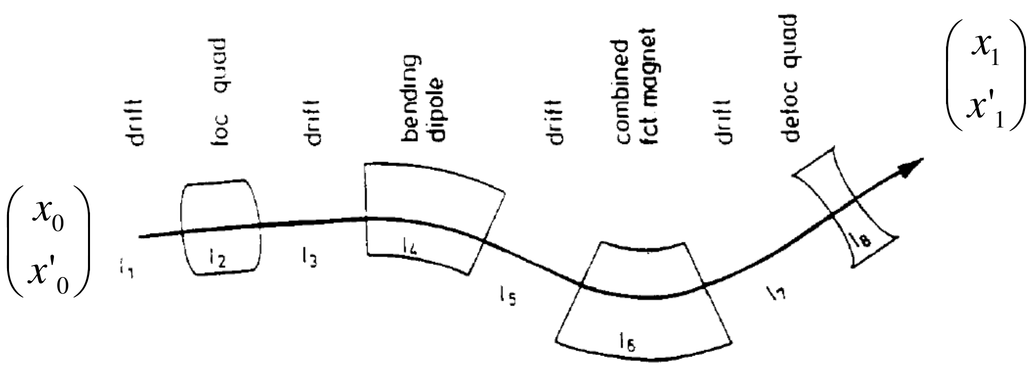
$$x(s) = x_0 C(s) + x'_0\, S(s),$$
$$x'(s) = x_0 C'(s) + x'_0\, S'(s)$$



- Equating those to the aforementioned x_β, x'_β we find:

$$C(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left(\cos\phi(s) + \alpha_0 \sin\phi(s)\right)$$
$$S(s) = \sqrt{\beta(s)\beta_0}\,\sin\phi(s)$$

- We then introduce matrix formalism to describe the evolution of a particle's coordinates. We introduce a matrix for each beamline element:



$$\begin{pmatrix} x_0 \\ x'_0 \\ \delta_0 \end{pmatrix} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \\ \delta_0 \end{pmatrix}$$

Beamline Matrices

1. C,S,C',S' depend only on the magnetic lattice, and NOT on initial beam parameters. For a generic magnetic element of length s, linear focusing strength k and curvature 1/R:

$$C(s)=\cos\left(s\sqrt{k+\frac{1}{R^2}}\right)\qquad S(s)=\frac{1}{\sqrt{k+\frac{1}{R^2}}}\sin\left(s\sqrt{k+\frac{1}{R^2}}\right)$$

QUAD

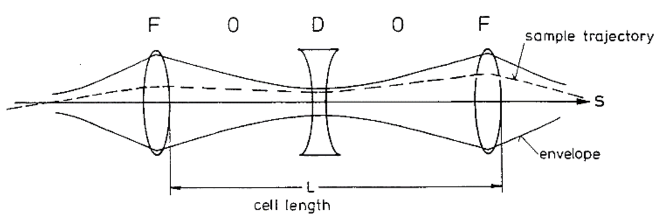
$$M_{Q,x}=\begin{pmatrix}\cos\left(l_q\sqrt{k}\right) & \frac{1}{\sqrt{k}}\sin\left(l_q\sqrt{k}\right) & 0\\ -\sqrt{k}\sin\left(l_q\sqrt{k}\right) & \sqrt{k}\cos\left(l_q\sqrt{k}\right) & 0\\ 0 & 0 & 1\end{pmatrix}$$

SBEND

$$M_{D,x}=\begin{pmatrix}\cos\theta & R\sin\theta & R(1-\cos\theta)\\ -\frac{1}{R}\sin\theta & \cos\theta & \sin\theta\\ 0 & 0 & 1\end{pmatrix}$$

2. Exercise: determine the transport matrix for a quadrupole magnet in *thin lens approximation*, that is $l_q \rightarrow 0$ but $f=kl_q = const.$

3. The matrix of a line is the result of a multiplication of individual matrices:



$$M=\begin{pmatrix}1 & 0\\ -\frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{L}{2}\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & \frac{L}{2}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1+\frac{L}{2f} & L\left(1+\frac{L}{4f}\right)\\ -\frac{L}{2f^2} & 1-\frac{L}{2f}-\frac{L^2}{4f^2}\end{pmatrix}$$

Exercise: Transport Matrices

Transport Matrix for Particle's Coordinates (in terms of <u>Twiss Functions)</u>

i) Impose equality of the the *C-S* invariant for $\mathbf{x}(s_1)=\mathbf{x}_1$ and $\mathbf{x}(s_2)=\mathbf{x}_2$.

ii) Use $\mathbf{x}_2=M(\mathbf{x}_1)$ in terms of Principal Trajectories and substitute into point 1.

iii) From the equality in ii), extract M_{TW} in terms of the Twiss functions:

$$M_{s_0\to s}=\begin{pmatrix} C(s) & S(s) \\ -C'(s) & S'(s)\end{pmatrix}=\begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}}(\cos\Delta\phi+\alpha_0\sin\Delta\phi) & \sqrt{\beta(s)\beta_0}\sin\Delta\phi \\ -\frac{(\alpha(s)-\alpha_0)\cos\Delta\phi+(1+\alpha(s)\alpha_0)\sin\Delta\phi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\frac{\beta_0}{\beta(s)}}[\cos\Delta\phi-\alpha(s)\sin\Delta\phi]\end{pmatrix}$$

Transport Matrix for Twiss Functions (in terms of <u>Principal Trajectories)</u>

i) Express $\mathbf{x}_2$ as fuction of $\mathbf{x}_1$ through Principal Trajectories, and write down the *C-S* invariant.

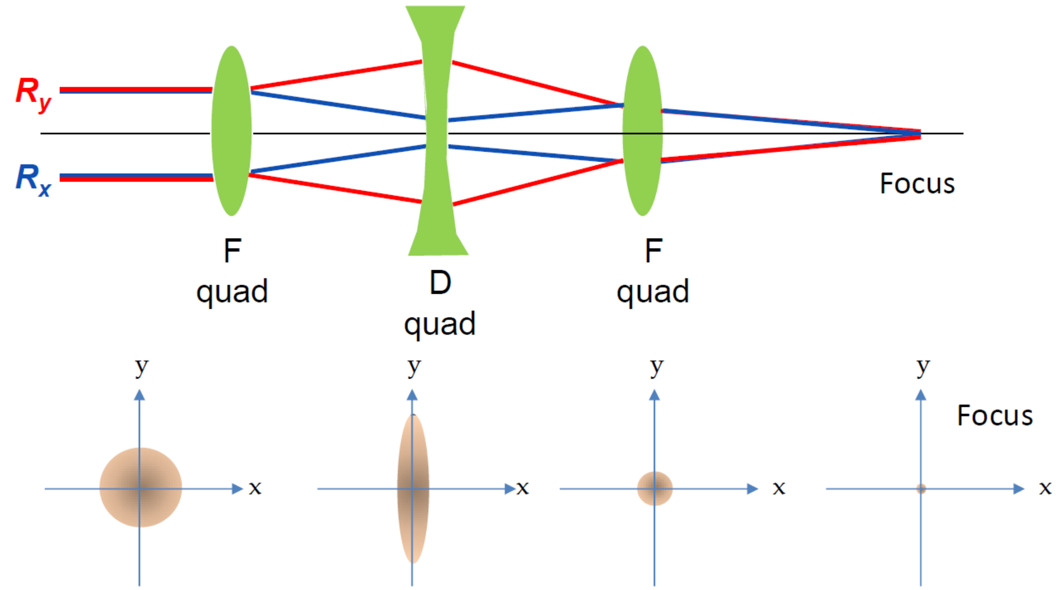
ii) Sort coefficients in i) for x^2, xx' and x'^2, and impose equality to a new *C-S* invariant.

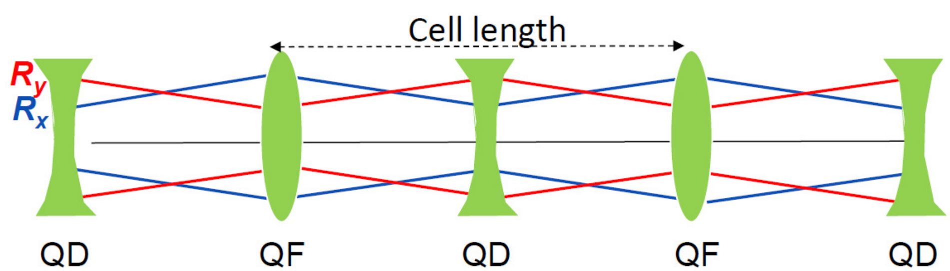
iii) Extract M_{PT} for the Twiss functions:

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}=\begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS'+SC' & -SS' \\ C'^2 & -2C'S' & S'^2\end{pmatrix}\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}$$

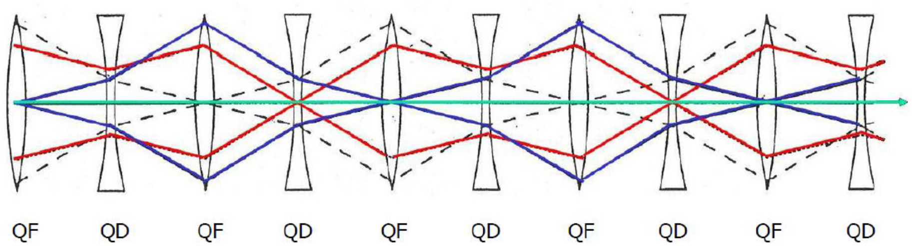
Beam Transport, Examples

Quadrupole Triplet





FODO lattice with 90° phase advance (one oscillation = 4 cells)

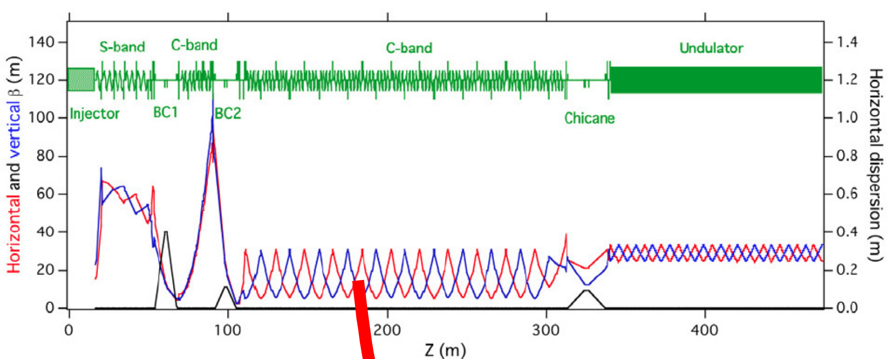


FODO (Cell=Focusing-Drift-Defocusing-Drift

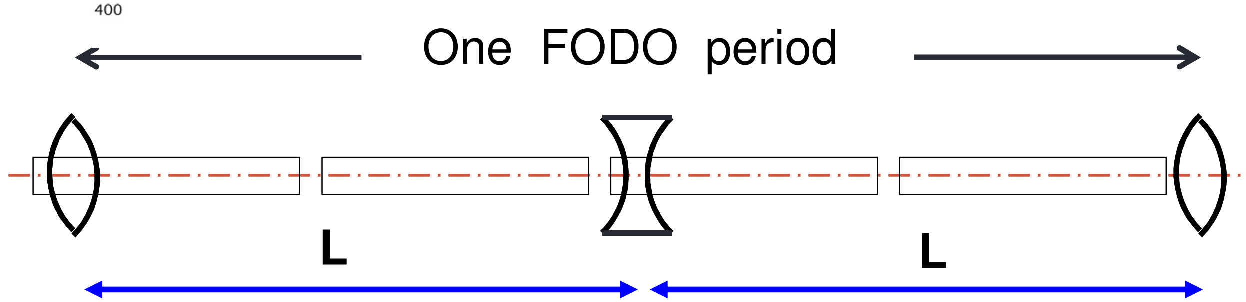
Stability

1. Consider M in terms of Twiss functions, and impose a periodic motion, i.e., same initial and final coordinates).
2. We find that |Tr(M)|=2|cosΔμ|.
3. Stability condition thus implies |Tr(M)|<2.

$$\left|\frac{1}{2}TrM\right| = \left|\cos\Delta\mu_{12}\right| < 1 \Rightarrow \frac{L^2}{2f_q^2} = \left(kl_q L\right)^2 < 2$$

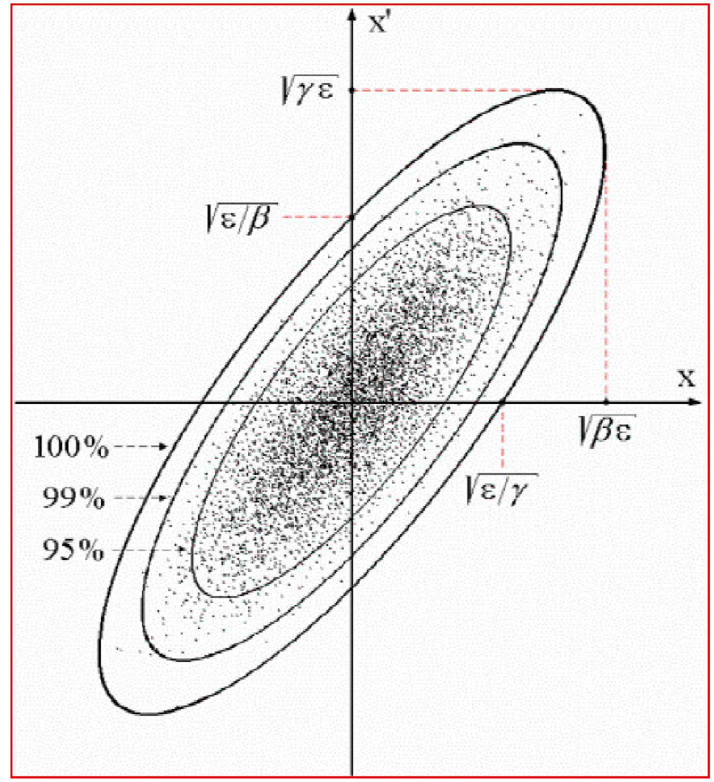


$$\frac{\beta_{max}}{\beta_{min}} = \frac{1+\sin\frac{\Delta\mu_{12}}{2}}{1-\sin\frac{\Delta\mu_{12}}{2}}$$



Beam Emittance

- We now consider the **ensemble of particles** at an *arbitrary point* of the line. For a **linear motion**, particles lye on **ellipses**.
- The beam is said to be **matched** to some design optics, if all particles' ellipses are described by the same Twiss functions, i.e. they are omothetic ellipses.



- We may also define a particles' distribution function ψ, so that:

$$\int \psi(\bar{x},s)d^6\bar{x} = 1 \qquad \bar{x} = (x, p_x, y, p_y, z, \delta)$$

$$<\bar{x}>_j (s) = \int x_j \psi(\bar{x},s)d^6\bar{x}$$ average coordinates, usually zero

- The 2nd **order momenta** of the distribution define the so-called **Σ-matrix** (or "beam matrix"):

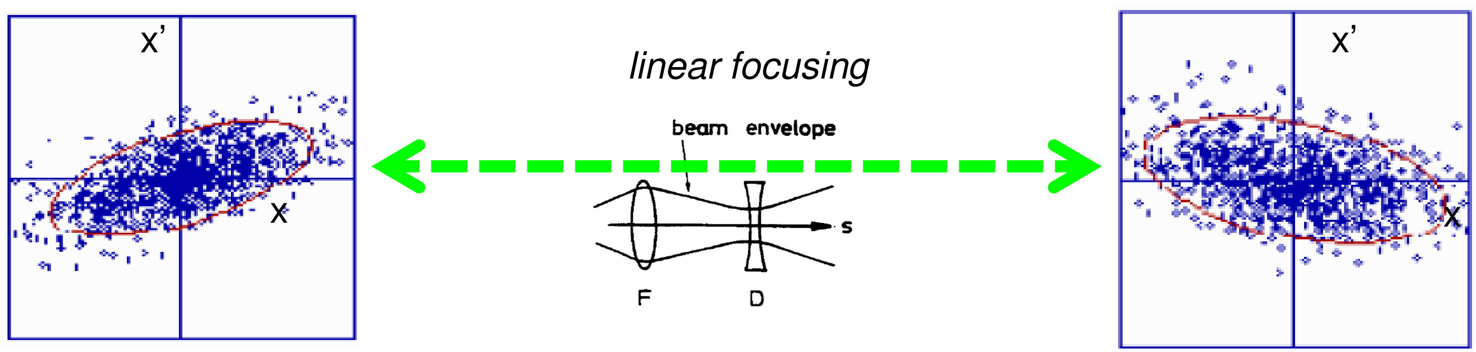
$$R_{ij}(s) = <(\bar{x}-<\bar{x}>)_i(\bar{x}-<\bar{x}>)_j> = \int (x_i - <\bar{x}>_i)(x_j - <\bar{x}>_j)\psi(\bar{x},s)d^6\bar{x}$$

Statistical or RMS Emittance

- **Statistical emittance**, $\varepsilon_x(P)$, is a measure of the *spread* in x and x′ of a given *fraction* P of beam particles.
- **Σ-matrix** states the equivalence of Twiss functions and RMS emittance:

$$\varepsilon_x = \sqrt{\det \varepsilon_x \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}} \equiv \sqrt{\det \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'^2 \rangle \end{pmatrix}}$$

- This is **as if** ψ were a **Gaussian**. Then, the beam evolution can be mapped through the Twiss functions, only.



Statistical emittance

$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

Beam size and divergence

- In the presence of dispersion:

$$\begin{cases} x = x_\beta + x_\eta = \sqrt{2J_x\beta_x} + \eta_x\delta \\ x' = x'_\beta + x'_\eta = \sqrt{2J_x\gamma_x} + \eta'_x\delta \end{cases}$$

STD ⇒

$$\begin{cases} \sigma_x = \sqrt{\varepsilon_x\beta_x + (\eta_x\sigma_\delta)^2} \\ \sigma'_x = \sqrt{\varepsilon_x\gamma_x + (\eta'_x\sigma_\delta)^2} \end{cases}$$

Transformation of Σ-Matrix

1. The rms ellipse is representative of the beam's particle distribution in the phase space.
2. The Σ-matrix characterizes the particle distribution, and its determinant is associated to the beam RMS emittance.
3. The transformation of Σ-matrix through a beamline represents the evolution of the beam ellipse, and in particular of its emittance.

- From the definition of the C-S invariant for a vector (x,x'), at location 0 and 1:

$$\vec{x}_0^T \Sigma_0^{-1} \vec{x}_0 = 1 = \vec{x}_1^T \Sigma_1^{-1} \vec{x}_1$$

Since: $\vec{x}_1 = M_{01}\vec{x}_0,$

$$\Sigma_0^{-1} = M_{01}^T \Sigma_1^{-1} M_{01},$$

$$\left(M_{01}^T\right)^{-1} \Sigma_0^{-1} \left(M_{01}\right)^{-1} = \Sigma_1^{-1},$$

And finally: $$\Sigma_1 = M_{01} \Sigma_0 M_{01}^T$$

This sets the rule for the evolution of the Σ-matrix through a beamline.

Preserving the Phase Space Area: *Det(M) = 1*

I. Principal Trajectories (PTs) are defined with initial conditions so that **det(M(0)) ≡ W(0) = 1**.

II. Each PT satisies Hill's eq. Now add a **frictional term** ∝ *C′*,*S′* and manipulate:

$$\begin{aligned} -S \cdot &\; C''+\varsigma C'+KC=0 \\ C \cdot &\; S''+\varsigma S'+KS=0 \end{aligned}$$

$$(CS''-SC'')+\varsigma(CS'-SC')+K(SC-CS)=0;$$

0

$$W'+\varsigma W=0$$

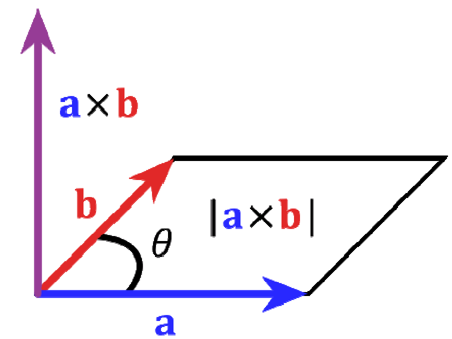
W(s)=1 ∀s ⇔ ζ=0.

III. Now consider the cross product *A = dx × dx′*.

It evolves according to the linear transformation:

$$d\vec{x} \cong \left(\frac{dx}{dx_0}dx_0, \frac{dx}{dx'_0}dx'_0\right) \equiv \left(C dx_0, S dx'_0\right)$$

$$d\vec{x}' \cong \left(\frac{dx'}{dx_0}dx_0, \frac{dx'}{dx'_0}dx'_0\right) \equiv \left(C' dx_0, S' dx'_0\right)$$



$$A = d\vec{x} \times d\vec{x}' = dx_0 dx'_0 \left(CS'-SC'\right) = A_0$$

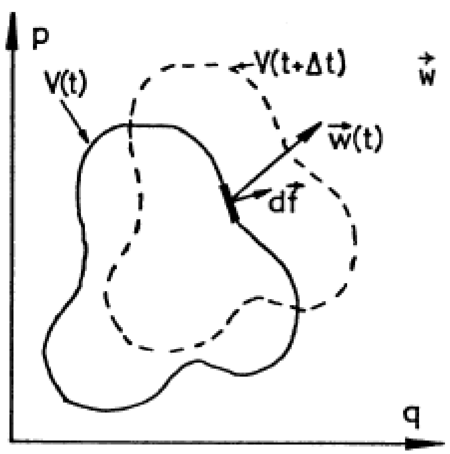
IV. We find ***A* = *W*· *A*$_0$**, that is a transport matrix with **unitary determinant preserves** the phase space **area** ($A=A_0$) in the **absence** of **frictional forces.**

Preserving the Phase Space Area: *Liouville's Theorem*

- **Liouville's theorem** states that in the absence of "frictional" forces (dissipative or diffusion terms, $\propto x'$ in Hill's eq.), the **area** of the beam **ellipse** is a **constant** of the motion.

$d\vec{f}$ ... vector of surface element

$\vec{w}(t)$ ... phase space velocity of surface element



$$\vec{w} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

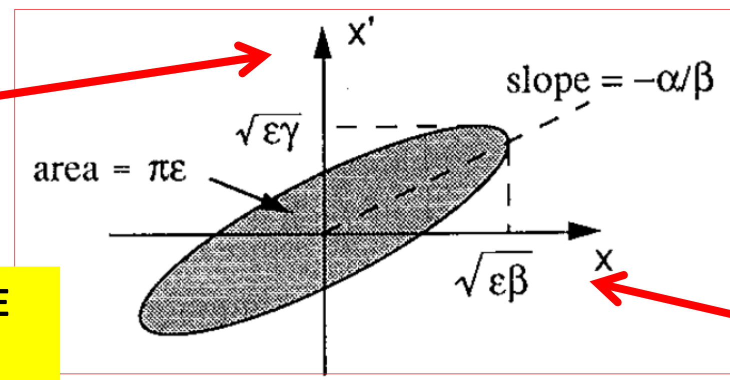
$$\frac{dV(t)}{dt} = \int \vec{w} \cdot d\vec{f} = \int (\nabla \cdot \vec{w})\, dv = \int \left(\frac{\partial}{\partial q} \dot{q} + \frac{\partial}{\partial p} \dot{p} \right) dv = 0$$

surface integral → volume integral (Gauss Theorem)

$$\left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q} \right) = 0$$

(Hamilton)

Beam angular divergence



BEAM PHASE SPACE AREA

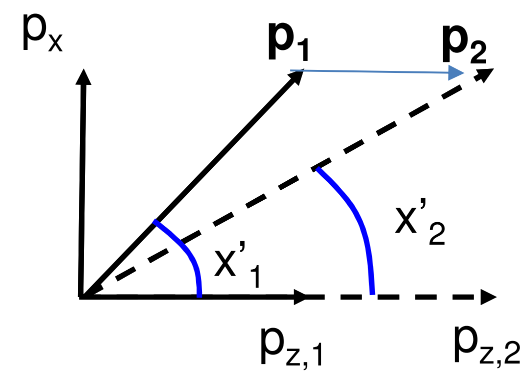
Beam size

- Liouville's theorem (area preservation) is still valid for a **nonlinear motion!**
- ***Any area*** is preserved, not only of ellipses!

Which Emittance ?

$$\varepsilon_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$

Geometric RMS emittance, invariant under linear focusing



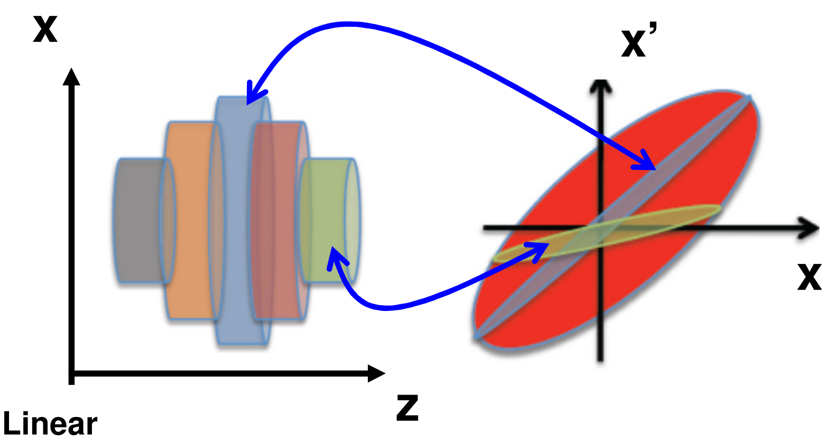
$$\varepsilon_{n,x} = \beta\gamma\varepsilon_x$$

Normalized RMS emittance, invariant under linear focusing and acceleration

$$\varepsilon^L_{n,x} = \iint dq_x dp_x$$

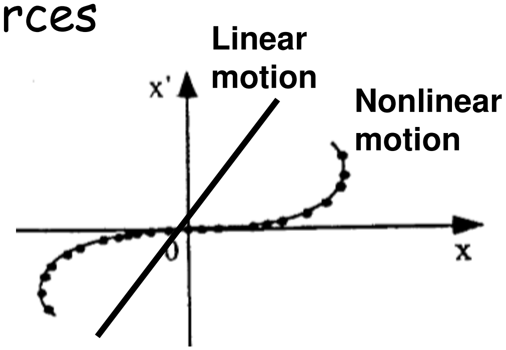
Normalized "Liouville's" emittance, invariant under linear, nonlinear focusing and acceleration

- ❑ When we refer to the whole particle distribution, ε is also said "**projected**". When we select a **longitudinal portion** of the beam, ε is named "**slice**" emittance.



- ❑ All "emittances" are degraded by frictional/dissipative/collision forces (Liouville's theorem falls short).

- ❑ The **RMS** emittance is **NOT** preserved under **NONLINEAR focusing**.

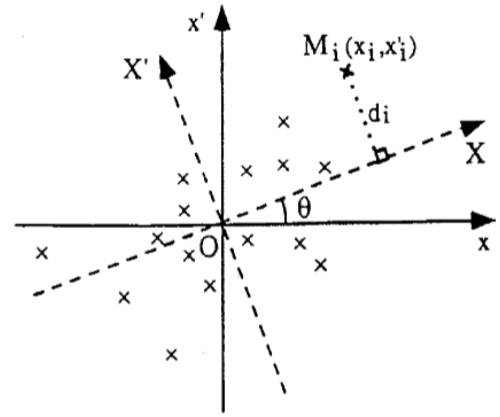


Hint: the phase space area of a line is always zero, while it is not for the spread of points along it.

Addendum on Hamiltonian Formalism

- The RMS emittance can alternatively be thought as the RMS area of triangles connecting the particles' representative points in phase space to the origin of coordinates (or barycenter):

$$\varepsilon_x = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2} = \ldots = \frac{1}{\sqrt{2}N}\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N}\left(x_i x'_j - x_j x'_i\right)} = \frac{\sqrt{2}}{N}\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} A_{ij}^2}$$



$$H(x,x') = \frac{x'^2}{2} + f(x)$$

$$\frac{dx'}{ds} = -\frac{\partial H}{\partial x}.$$ for a generic particle

$$\frac{d\overline{x'}}{ds} = -\frac{\overline{\partial H}}{\partial x}.$$ for the barycenter (O)

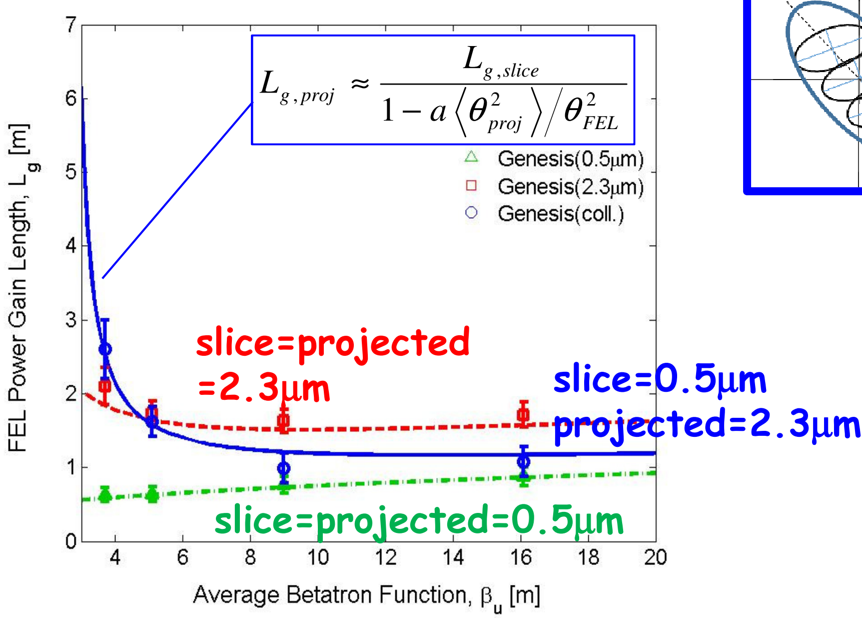
In general, **nonlinear motion** implies $\frac{\overline{\partial H}}{\partial x} \neq \frac{\partial H}{\partial x}(\bar{x})$ that is O moves with a different law than the representative points. In other words, **triangles M_iOM_j are NOT mapped into triangles**, thus their **area is not preserved**. We then expect the RMS emittance be degraded by nonlinear effects, such as "optical aberrations".

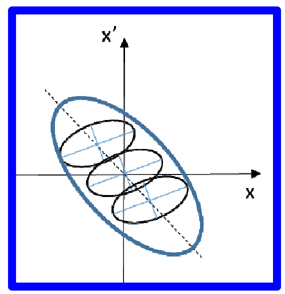
- It can be shown that canonical transformations of coordinates in a quadratic Hamiltonian system (like in an accelerator free of frictional forces) are represented by a group of **symplectic** matrices. These have **det = 1**, hence they ensure **preservation** of the **phase space area** in the Liouville's sense.

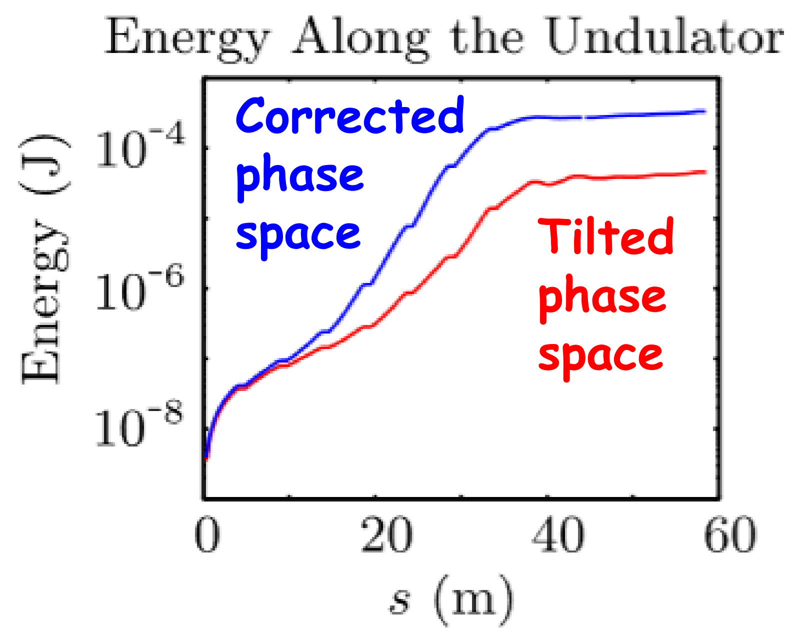
Is the Projected Emittance Relevant to FELs?

PRSTAB 17, 110702 (2014)
PRSTAB 18, 030701 (2015)

- 1-D & 3-D SASE FEL theory (baseline for *any* FEL scheme...) only deals with the slice emittance, whereas 3-D means non-zero slice emittance. However....
- Both theoretical and experimental evidences point out the importance of the **projected emittance** for the overall **FEL performance**.
  - A correlated energy spread may affect the FEL intensity, bandwidth and central wavelength (depending on the FEL scheme).
  - Correlations in the transverse phase space may reduce the FEL intensity and enlarge the FEL bandwidth.

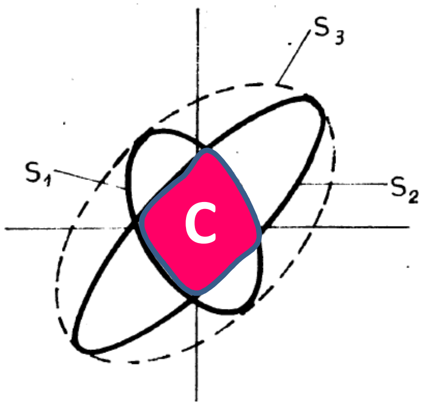






Optics Mismatch

- A beam is said to be **matched**, when its Twiss parameters (determined on the basis of its emittance, size and divergence) are equal to the user's defined design values. Since the Twiss parameters vary along a line, matching is a **local** condition.
- The actual beam may have the same emittance of the ideal (design) beam, but different Twiss parameters. To quantify the amount of «optics mismatch» of the actual vs. the design beam, we define:



- S1 (matched) and S2 (mismatched) have same area S, but different shape and orientation ($\beta_1 \neq \beta_2$, $\alpha_1 \neq \alpha_2$). Common area is:

$$C = S\frac{4}{\pi}\arctan\sqrt{\xi - \sqrt{\xi^2 - 1}},$$

$$\xi = \frac{1}{2}(\beta_1\gamma_2 - 2\alpha_1\alpha_2 + \beta_2\gamma_1) \geq 1$$

MISMATCH PARAMETER

- $C \to S$ when $\xi \to 1$ (matching), i.e. when the two ellipses overlap.

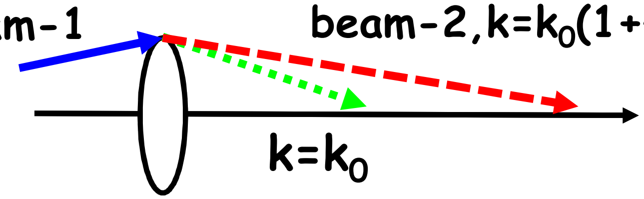
- Equivalently, we may define ξ (in literature, also named B_{mag}) as function of measurable quantities, i.e. emittance and beam sizes of the design and the perturbed beam:

$$\xi = \frac{1}{2}\frac{\varepsilon_1}{\varepsilon_2}Tr\left(\Sigma_2\Sigma_1^{-1}\right)$$

Coherent Error Kick: Quad *Gradient* Error

- Optics mismatch can be caused by a focusing error. Here, we consider a quadrupole gradient error **k = k$_0$+Δk = k$_0$(1+τ)**.
  - The following treatment applies to all errors that imply the same kick for all the beam's particles.
  - Because of <u>linearity</u> of the focusing force, we do <u>*not*</u> expect RMS emittance growth.

$$\tilde{Q} = \begin{pmatrix} 1 & 0 \\ -kl & 1 \end{pmatrix}$$

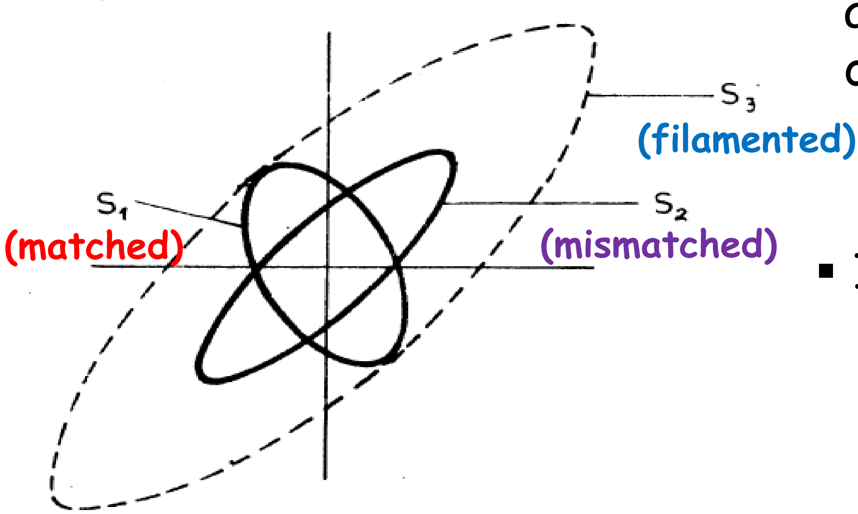


Emittance, $$\varepsilon_2^2 = \det \Sigma_2 = \det\left(\tilde{Q}\Sigma_1\tilde{Q}^T\right) = \det\begin{pmatrix} \varepsilon_1\beta_1 & -\varepsilon_1(\alpha_1-\beta_1 kl) \\ -\varepsilon_1(\alpha_1-\beta_1 kl) & \varepsilon_1\gamma_1 - 2\varepsilon_1\alpha_1(kl) + \varepsilon_1\beta_1(kl)^2 \end{pmatrix} = \varepsilon_1^2$$

Mismatch, $$\xi = \frac{1}{2}\frac{\varepsilon_1}{\varepsilon_2} Tr\left(\Sigma_2\Sigma_1^{-1}\right) = 1 + \frac{1}{2}\left(\beta_1 k_0 l \tau\right)^2$$

Filamentation of Phase Space

- We know that the RMS emittance can grow up because of **nonlinear focusing**. The latter implies that the particle's motion depends on higher orders of the particle's coordinates.
- **Optics mismatch** may bring particles to large oscillation amplitudes, thus sampling nonlinear magnetic field components.
- After **many «rotations»** in the phase space (i.e., large phase advance), particles tend to occupy a larger phase space area, namely the emittance has grown up.



- S1 (matched) and S2 (mismatched) have same area (S). After full filamentation, beam occupies S3, whose area is:

$$S_3 = S\left(\xi - \sqrt{\xi^2 - 1}\right) \equiv DS$$

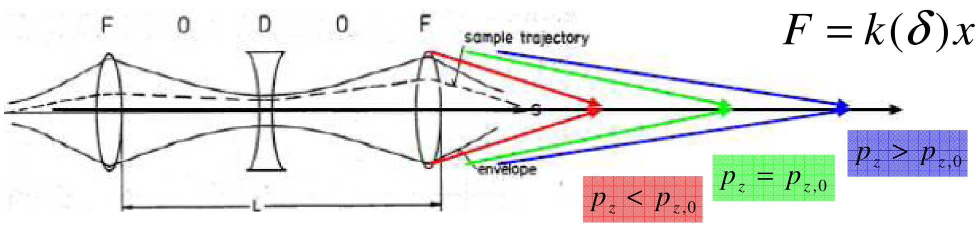
- It can be shown that, *after full filamentation*:

$$\varepsilon_{3,100\%} = D\varepsilon_{1,100\%}$$
$$\varepsilon_{3,RMS} = \xi\varepsilon_{1,RMS}$$

EXE: show that a quadrupole gradient error imply a fully filamented RMS emittance equal to: $\varepsilon_3 = \varepsilon_1\left[1 + \frac{1}{2}(\beta_1 k_0 l \tau)^2\right]$

Incoherent Error Kick: Quad *Chromatic* Error

- Optics mismatch can be caused by a focusing error. Here, we consider a quadrupole chromatic error **k=k$_0$(1+δ)**, and δ = single particle energy deviation.
  - The following treatment applies to all errors that imply a different kick error for differemt particles.
  - Because of <u>nonlinearity</u> of the focusing force, ***F* ~ *x*δ,** we <u>expect</u> RMS emittance growth.



$$F = k(\delta)x = \frac{eg}{p_z}x \cong \frac{eg}{p_{z,0}}x\left(1+\frac{\Delta p_z}{p_{z,0}}\right) = k_0 x + k_0\delta x$$

Emittance,

$$\varepsilon_2^2 = \det\Sigma_2 = \det\left(\tilde{Q}\Sigma_1\tilde{Q}^T\right) = \det\begin{pmatrix} \varepsilon_1\beta_1 & -\varepsilon_1\left(\alpha_1-\beta_1\langle k\rangle l\right) \\ -\varepsilon_1\left(\alpha_1-\beta_1\langle k\rangle l\right) & \begin{array}{l}\varepsilon_1\gamma_1-2\varepsilon_1\alpha_1\left(\langle k\rangle l\right)+\\ +\varepsilon_1\beta_1\left\langle (kl)^2\right\rangle\end{array}\end{pmatrix} = \varepsilon_1^2\left[1+(\beta_1 k_0 l\sigma_\delta)^2\right]$$

$$\varepsilon_2 \approx \varepsilon_1\left[1+\frac{1}{2}(\beta_1 k_0 l\sigma_\delta)^2\right], \quad when \quad \frac{\varepsilon_2}{\varepsilon_1}\approx 1$$

Mismatch,

$$\xi = \frac{1}{2}\frac{\varepsilon_1}{\varepsilon_2}Tr\left(\Sigma_2\Sigma_1^{-1}\right) = 1+\frac{1}{2}\left(\beta_1 k_0 l\langle\delta\rangle\right)^2 + O\left(\langle\delta\rangle^4,\sigma_\delta^4\right) \xrightarrow{when\ \langle\delta\rangle\approx 0} \xi \approx 1+\frac{1}{8}(\beta_1 k_0 l\sigma_\delta)^4$$

Optics Sensitivity to Focusing Errors

1. Assume **nonlinear** motion up to the **2nd order** in the particle coordinates (6-D).
2. Consider **small**, **independent** gradient-like and chromatic-like focusing error kicks, of the form $Q^2 = \langle \Delta x'^2 \rangle$.

- *Corollary 1:* *the largest value that the RMS emittance may assume,* ***after full filamentation****, because of each* ***individual kick*** *is:*

$$\left(\frac{\Delta\varepsilon}{\varepsilon_1}\right)_i \cong \frac{1}{2}\frac{\beta_i}{\varepsilon_1}Q_i^2$$

- *Corollary 2:* *the largest value that the RMS emittance may assume,* ***after full filamentation****, because of the* ***uncorrealted sum of error kicks*** *is:*

$$\frac{\Delta\varepsilon}{\varepsilon_1} \equiv \sqrt{\sum_{i=1}^{N}\chi_i^2} = \frac{1}{2}\sqrt{\sum_{i=1}^{N}\left(k_i l_i \beta_i \tau_i\right)^4} \le T$$

where:

$$\tau_i = \Delta k_i \ or \ \left(k_0\sigma_\delta\right)_i$$

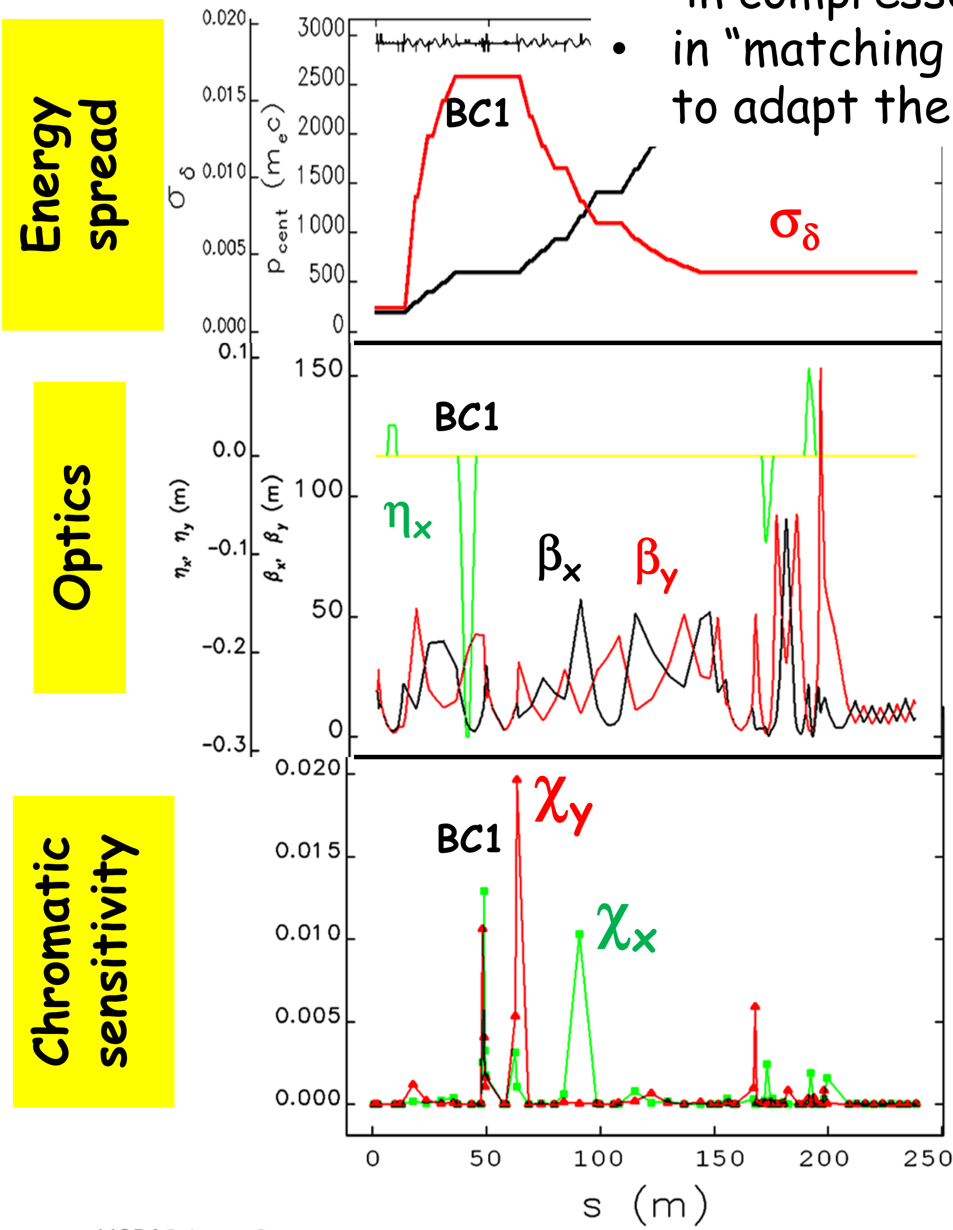
and it turns out:

$$\xi_i\left(\Delta k_i\right) = 1 + \chi$$

$$\xi_i\left(\sigma_{\delta,i}\right) \approx 1$$

- χ can be thought as the **optics sensitivity to focusing errors**. If *T*=5% is the *tolerance* on the *final emittance growth* induced by *N=100* error kicks, then *on average* χ (at each quad location) should be smaller than $T/\sqrt{N}$ = 0.5%.
  - *The same sensitivity applies identically to the local mismatch in the case of a coherent error kick.*

Optics Design



- ❑ ***Larger sensitivity*** to focusing errors is typically:
  - in compressors area, where σ_δ~1%;
  - in "matching stations", where strong k may be needed to adapt the beam to the design optics.

- ❑ ***Matching stations*** (series of 4-6 quads) are typically located:
  - at the ***injector exit***, because space-chage forces make the beam optics less predictable;
  - in front of ***diagnostic stations***, to improve the measurement resolution;
  - in front of ***magnetic compressors*** to counteract CSR effects;
  - in front of the ***undulator***, for optimum e-beam/photons overlap.

- ❑ Use codes for ***optimizing*** the quad strengths in order to minimize the sensitivty to focusing errors.
  - In general, we like few quads only, weak stengths, and small βs, low σ_δ beams. These guidelines are in open contradiction.

Magnetic Field Tolerances

- Every real magnet includes ***systematic*** and ***random field errors***, both due to the finite magnet dimension and mechanical tolerances. The formers are constrained by symmetries of the nominal field pattern. The latters may cover all orders of the field expansion.
- The magnets should be manufactured in a way that field components higher than the nominal should be small enough to avoid beam emittance dilution. We assume ***perfectly aligned magnets***.

Quadrupole component (n=1) in a Dipole magnet (n=0):

$$k_{1,0} = \frac{eg_{1,0}}{p_{z,0}} = \frac{\theta}{Rl}\left|\frac{b_1}{b_0}\right| \Rightarrow Q_{1,0} = \frac{\theta}{R}\eta\delta\left|\frac{b_1}{b_0}\right| \Rightarrow \left.\frac{\Delta\varepsilon}{\varepsilon}\right|_{1,0} \cong \frac{\beta}{2\varepsilon}\left(\frac{\theta}{R}\eta\sigma_\delta\frac{b_1}{b_0}\right)^2 \le 1\% \Rightarrow \left|\frac{b_1}{b_0}\right| \le \frac{1}{\theta}\frac{R}{\eta\sigma_\delta}\sqrt{\frac{\Delta\varepsilon}{\varepsilon}\frac{2\varepsilon}{\beta}}$$

Quadrupole-like strength in a dipole — **Quadrupole-like *chromatic* kick error** — **RMS emittance growth (tolerance)** — **Magnetic field tolerance (chromatic aberration)**

Sextupole component (n=2) in a Quadrupole magnet (n=1):

$$k_{2,1} = \frac{em_{2,1}}{p_{z,0}} = \frac{2k_1}{R}\left|\frac{b_2}{b_1}\right| \Rightarrow Q_{2,1} = \frac{2k_1 l}{R}x^2\left|\frac{b_2}{b_1}\right| \Rightarrow \left.\frac{\Delta\varepsilon}{\varepsilon}\right|_{2,1} \cong \frac{\beta}{2\varepsilon}\left(\frac{2k_1 l}{R}x^2\frac{b_2}{b_1}\right)^2 \le 1\% \Rightarrow \left|\frac{b_2}{b_1}\right| \le \frac{1}{k_1 l}\frac{R}{\varepsilon\beta}\sqrt{\frac{\Delta\varepsilon}{\varepsilon}\frac{2\varepsilon}{\beta}}$$

Sextupole-like strength in a dipole — **Sextupole-like kick error** — **RMS emittance growth (tolerance)** — **Magnetic field tolerance (geometric aberration)**

RF Focusing

Assume a TW-CG structure, transit time factor = 1. E_z has now explicit radial dependence. Maxwell's equations for t-dependent e.m. field:

$$\nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z} = 0$$

$$(\nabla\cdot\vec{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) = \frac{1}{c^2}\frac{\partial \bar{E}_z}{\partial t}$$

$$\Rightarrow \quad E_r \cong -\frac{r}{2}\frac{\partial E_z}{\partial z} \qquad B_\varphi \cong \frac{r}{2c^2}\frac{\partial E_z}{\partial t}$$

and use: $dE(z,t) = \frac{\partial E(z,t)}{\partial z}dz + \frac{\partial E(z,t)}{\partial t}dt$

$$\frac{\partial E_z}{\partial z} = \frac{dE_z}{dz} - \frac{\partial E_z}{\partial t}\frac{dt}{dz}$$

$$E_z = E_{z,0}\cos(\phi);\ \ \phi = k_z ct$$

In conclusion:

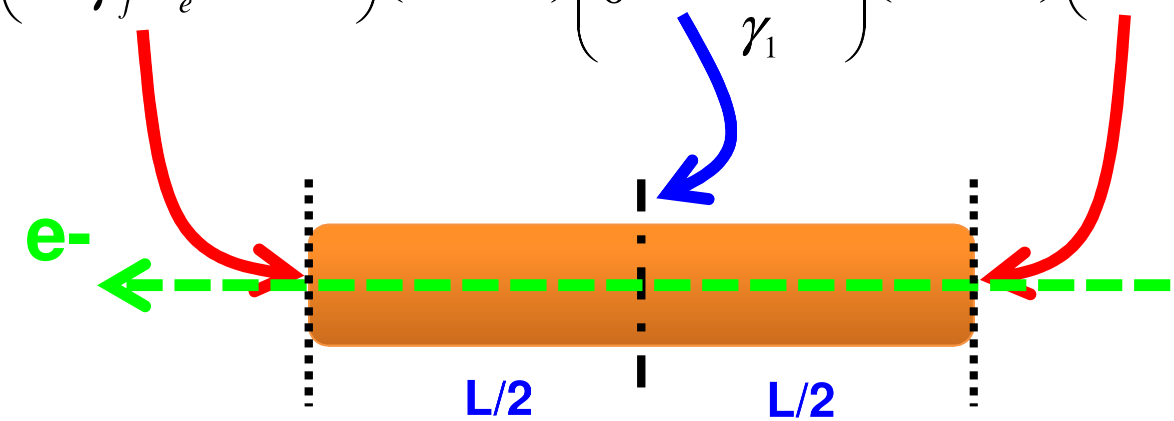
$$F_r = q(E_r - \dot{z}B_\varphi) = -\frac{q}{2}r\left[\frac{\partial E_z(z,t)}{\partial z} - \frac{\beta_z}{c}\frac{\partial E_z(z,t)}{\partial t}\right] = -\frac{q}{2}r\left[\frac{d}{dz} - \frac{k}{2\beta\gamma^2}\frac{\partial}{\partial\phi}\right]E_z(z,\phi)$$

1. Neglect $\sim\gamma^2$ and keep **$E_z=E_{z,0}$ through a gap l_g**: $F_r = -\frac{qE_{z,0}r}{2l_g} \approx -\frac{(qE_{z,0})^2}{2\beta\gamma m_e c^2}r$
2. For **$E_z=E_{z,0}\cos\phi$** at the structure's **edges**: $\Delta r' = \frac{\Delta p_r}{p_z} \cong \frac{F_r(\phi)dt}{p_z} \cong \mp\frac{qE_{z,0}\cos(\phi)}{2\beta_{i,f}^2\gamma_{i,f}m_e c^2}r$
3. In a **cell-to-cell** focusing model: $F_{r,eff} = \frac{\eta(\phi)}{4}\frac{(qE_{z,0})^2}{2\beta_i\gamma_i m_e c^2}r,$ $\quad \eta(\phi)\approx 0, TW$; $\eta(\phi)\approx 1, SW$
4. Term $\sim\gamma^2$ provides **RF phase focusing** : $F_r(\phi) = -\frac{qk_z r}{2\beta\gamma^2}E_{z,0}\sin(\phi)$

RF Transport Matrix

- Cell-to-cell (also «ponderomotive» or «body-focus») and edge focusing describe the fringe field effect inside and at the edge of the structure, respectively.
- In the following, we will consider TW structures, at energies > 100 MeV.
- Transport matrix for acceleration with pseudo-canonical coordinates (x, x') is *not simplectic* $\Rightarrow$ automatically includes *adiabatic damping of geometric emittance*.

$$\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{qE_{z,0}\cos(\phi)}{2\gamma_f m_e c^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{\gamma_0}{\gamma'}\ln\frac{\gamma_1}{\gamma_0} \\ 0 & \frac{\gamma_0}{\gamma_1} \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{qE_{z,0}\cos(\phi)}{2\gamma_i m_e c^2} & 1 \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



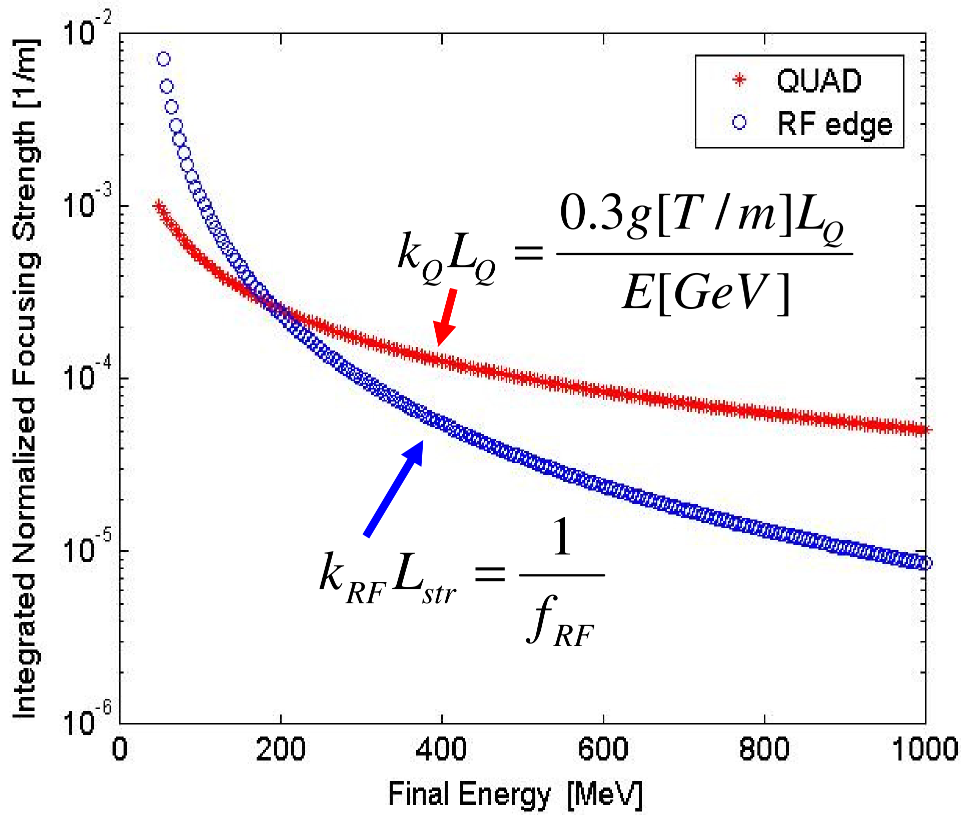
$$\gamma' = \frac{\gamma_1 - \gamma_0}{L} \cong$$
$$\cong \frac{qE_{z,0}\cos(\phi)}{m_e c^2}$$

RF *vs.* Magnetic Focusing

In the limit L → 0, $\frac{1}{f_{RF}} = M_{21} \approx -\frac{1}{|f_{edge}|}\left[\frac{1}{|f_{edge}|}\frac{\gamma_0}{\gamma'}\ln\left(\frac{\gamma_1}{\gamma_0}\right) - \frac{\Delta\gamma}{\gamma_1}\right] < 0$

***Focusing** at the **entrance dominates** over defocusing at the exit*

*The overall focusing is **damped** by acceleration.*



$E_1 - E_0$ = 50 MeV / structure
L_{str} = 3 m
g = 1.7 T/m
L_Q = 0.1 m

Coupler Cell RF Kick

- Geometric asymmetries of the input/output **coupler cells** may contribute with transverse electric field kicks that affect the beam trajectory and size, with dipole, quadrupole and higher order E_z dependence on the particle offset.

1. **Coupler acc. field with ampl. & phase y-gradient, dipole approximation**

$$E_z(y,t) = \left(E_{z,0} + \Delta E_{z,0}\frac{y}{2a}\right)\cos\left(\phi_s + \Delta\phi_c\frac{y}{2a} + \omega_{rf}\Delta t\right)$$

2. **Panofsky – Wenzel theorem**

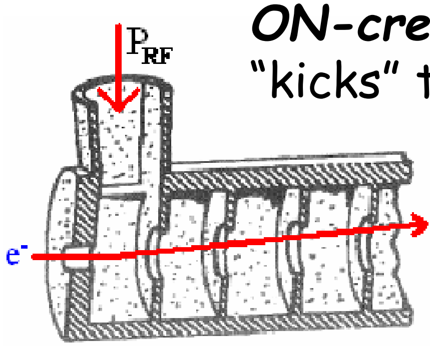
$$\Delta y' = \frac{\Delta p_y}{p_z} = -i\frac{e}{k_{rf}p_z c}\int_0^{l_{cell}}\nabla_\perp E_z dz = \ldots$$

expand for ωΔt<<1

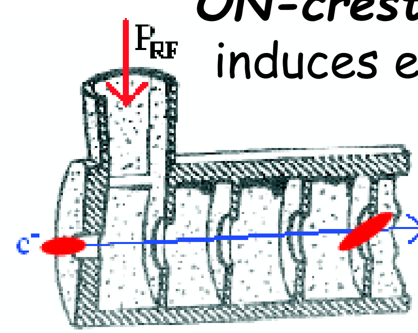
$$\ldots \cong \frac{el_{cell}}{2ak_{rf}p_z c}\left[E_{z,0}\Delta\phi_c\cos\phi_s - \Delta E_{z,0}\sin\phi_s\right] + k_{rf}\Delta z\left[E_{z,0}\Delta\phi_c\sin\phi_s + \Delta E_{z,0}\cos\phi_s\right]$$

centroid kick **head-tail kick**

ON-crest, phase error "kicks" the beam centroid.



ON-crest, amplitude error induces emittance growth.



Impact on the Beam Motion

- The **input coupler** effect typically **dominates** because:
  - beam is at a lower energy,
  - The accelerating field at the entrance is not attenuated yet.

- **Trajectory (mi)steering** can be compensated with steering magnets in proximity of the accelerating structure.
  - However, a beam passing off-axis in the structure can excite transverse wakefields (see next lectures). Use *feed-forward* steering scheme or put steerers *on* the structure.

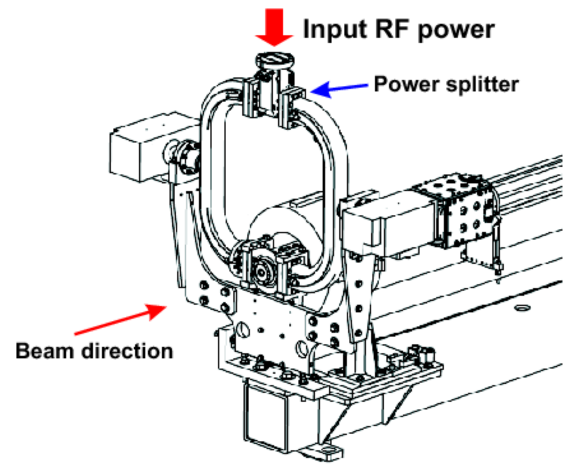
- For on-crest acceleration (typical in injector), the **head-tail** induced **emittance growth** is (from eq. in the previous slide + Σ-matrix) :

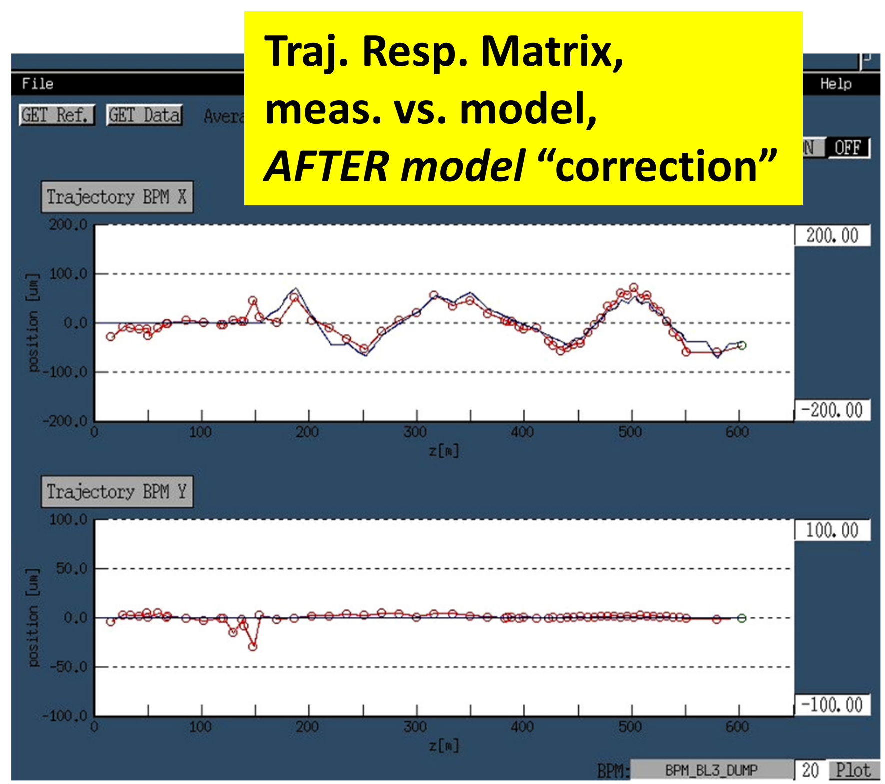
$$\varepsilon_y = \sqrt{\langle y^2\rangle\langle y'^2\rangle - \langle yy'\rangle^2} \approx \sqrt{\sigma_{y,0}^2\left(\sigma_{y',0}^2 + \langle \Delta y'^2\rangle\right)} \overset{on-crest}{\cong} \sqrt{\varepsilon_{y,0}^2 + \frac{\sigma_{y,0}^2}{4a^2}\left(\frac{e\Delta E_{z,0} l_{cell}}{p_z c}\right)^2 \sigma_z^2}$$

Spurious RF Focusing

Data sets courtesy of T. Hara

- Special coupler designs ("racetrack" cell shaping, symmetric RF waveguide, cell tuning) are usually adopted to get rid of dipolar and/or quadrupolar field component.
- Residual effects have to be taken into account as a "correction factor" in the modeling (matrix) of RF focusing.





RF Focusing in ELEGANT

- **TWLA**: 2π/3 CG, edge focusing (optional), numerical integration.

- **RFCA**: π SW, edge focusing (optional), body-focus (optional), matrix (single-kick approx. by default), *N_KICKS*, *PHASE_REFERENCE*.
  - Also good for TW-CG, with body-focus turned off.
  - "N_KICKS = XX" is equivalent to a split structure. Used for numerical integration of wakes (*e.g.*, geometric, LSC, etc.) in a long structure.
  - For the one-structure model, just use: N_KICKS=0, PHASE_REFERENCE = 0.

- **RFCA *split in units*** (*e.g.*, for dynamics inside a long structure).
  - Each unit length has to be integer multiple of λ_{RF}.
  - Proper focusing for a TW-like structure is given by setting: N_KICKS = 1, END1_FOCUS = 1 and END2_FOCUS = 1 in each unit (inner focusing is cancelled out and only that at the edges remains).
  - Set PHASE_REFERENCE=*n*, with *n* integer and unique for each unit (otherwise the units will be individually phased, which could cause unphysical result).

!! **Warning!!** In old Elegant versions, Twiss functions are computed correctly only for N_KICKS = 0 !!