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Resonator Figures of Merit

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USPAS – Applied Electromagnetism Lecture 3

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Pillbox Cavity – All the Details

Lecture Plan

- Last time, we got to the basic field description of a pillbox cavity.
- This is the workhorse geometry, for reasons that will rapidly become evident.
- We're going to fully characterize this geometry, all the parameters that we'll need for later, and then move on to more complicated cavity geometries.

Standing Waveguide Modes

- $E_z = E_0 J_m\left(j_{m,n}\frac{\rho}{R}\right)\cos\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}$
- $E_\rho = -E_0\frac{l\pi R}{j_{m,n}L}J'_m\left(j_{m,n}\frac{\rho}{R}\right)\sin\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}$
- $E_\phi = -E_0\frac{iml\pi R^2}{\rho j_{m,n}^2 L}J_m\left(j_{m,n}\frac{\rho}{R}\right)\sin\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}$
- $B_\rho = E_0\frac{m\omega R^2}{c^2\rho j_{m,n}^2 L}J_m\left(j_{m,n}\frac{\rho}{R}\right)\cos\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}$
- $B_\phi = E_0\frac{i\omega R}{c^2 j_{m,n}}J'_m\left(j_{m,n}\frac{\rho}{R}\right)\cos\left(\frac{l\pi z}{L}\right)e^{-i\omega t - im\phi}$
- Note the change in the dispersion curve! No longer continuous with all frequencies allowed.
- $\omega_{m,n,l} = \sqrt{\left[\left(\frac{cl\pi}{L}\right)^2 + \left(\frac{c\,j_{m,n}}{R}\right)^2\right]}$

Pillbox Cavity

- You can repeat all this for TE modes, but we want longitudinal electric fields for acceleration!
- Pick the lowest frequency, simplest mode: TM_{010}
- $B_\rho = E_\rho = E_\phi = 0$ and $j_{m,n} = 2.405$
- $E_z = E_0 J_0\left(\frac{2.405\rho}{R}\right)e^{-i\omega t}$
- $H_\phi = \frac{E_0}{\eta} J_1\left(\frac{2.405\rho}{R}\right)e^{-i\omega t}e^{\frac{i3\pi}{2}}$ with $\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 376.7\ \Omega$ is the impedance of free space.
- $\omega_{010} = \frac{2.405c}{R}$ Note: only depends on radius, not length!

Now we…. Wellll…… First thing's first. RF Losses!

- Now it's unavoidable, how is power dissipated in a metallic surface?
- We proved that the skin depth was related to the conductivity and frequency: $\delta^{-1} = \sqrt{\pi f \mu_0 \sigma}$
- This came from solving for the fields in a metallic layer as it screened the imposed fields, and we did it with the Electric Field: $E_z = E_0 e^{-\tau_n x}$ where $\tau_n = \sqrt{i\omega\sigma\mu_0}$ (the real part of this gives the skin depth)
- We want the surface resistance, which is the real part of the surface impedance.

Surface Resistance – Normal Conducting Materials

- First, need the total current: $I = \int_0^\infty j_z(x)dx = \int_0^\infty j_0\, e^{-\tau_n x} dx = j_0/\tau_n$
- So, Impedance $Z_s = \frac{E_0}{I} = \frac{\tau_n}{\sigma} = \frac{\sqrt{i\omega\mu_0\sigma}}{\sigma} = R_s + i\mathrm{X}_s$
- Turn the crank: $R_s = \sqrt{\frac{\pi\mu_0 f}{\sigma}} = \frac{1}{\sigma\delta}$
- Two things to note:
  - Highly conducting materials, low R_s ($\sim m\Omega$), good!
  - $R_s \propto f^{\frac{1}{2}}$ Increases with frequency, but not quickly.

Surface Resistance – Superconducting Materials!

- Some materials, when cooled below a certain ‘transition’ temperature lose their DC resistance.
- Technically they are even better than a ‘perfect conductor’ because upon transition, they expel magnetic field instead of trap it.
- Most common superconducting material for cavities (but not only!) is niobium (9.2 K)
- However, no free lunch. While DC resistance is zero, RF resistance is merely very, very small (electrons still have mass, after all)

Surface Resistance – Superconductivity!

- The physics of this is very different than normal metals:
  - Surface resistance is now determined by a far more complex physical process, modeled by BCS theory:
  - $R_{BCS} = \frac{2^{-4} C_{RRR}}{T} \left(\frac{f}{1.5}\right)^2 e^{-\frac{17.67}{T}}$
    - f is in GHz
    - T is in Kelvin
    - C_{RRR} varies from 1 to 1.5 depending on material purity
  - Even worse! High magnetic fields (the thing we'll be applying to the cavity) break the superconducting state.
  - If the superconductivity is broken in one place, it reverts to a normal conducting metal, and the dissipated power there will almost certainly rapidly heat the rest of the cavity above the transition temperature.

Superconducting Practicalities

- Runaway is called a quench, and it's a bad thing.
- Peak surface magnetic field matters quite a bit for superconducting applications, often totally dominating design
- The real surface resistance, what's achievable, is actually a combination of effects:
- $R_S = R_{BCS} + R_{res}$ where R_{res} is a combination of many factors
  - Impurities on the cavity surface
  - Adsorbed gasses
  - Ambient magnetic field trapped during cooldown
  - Many more
- Modern processing techniques can achieve $R_s = 10n\Omega$ reliably in most applications, and sometimes $< 1n\Omega$ in certain circumstances (real cavities, though!).

Moar superconducting…

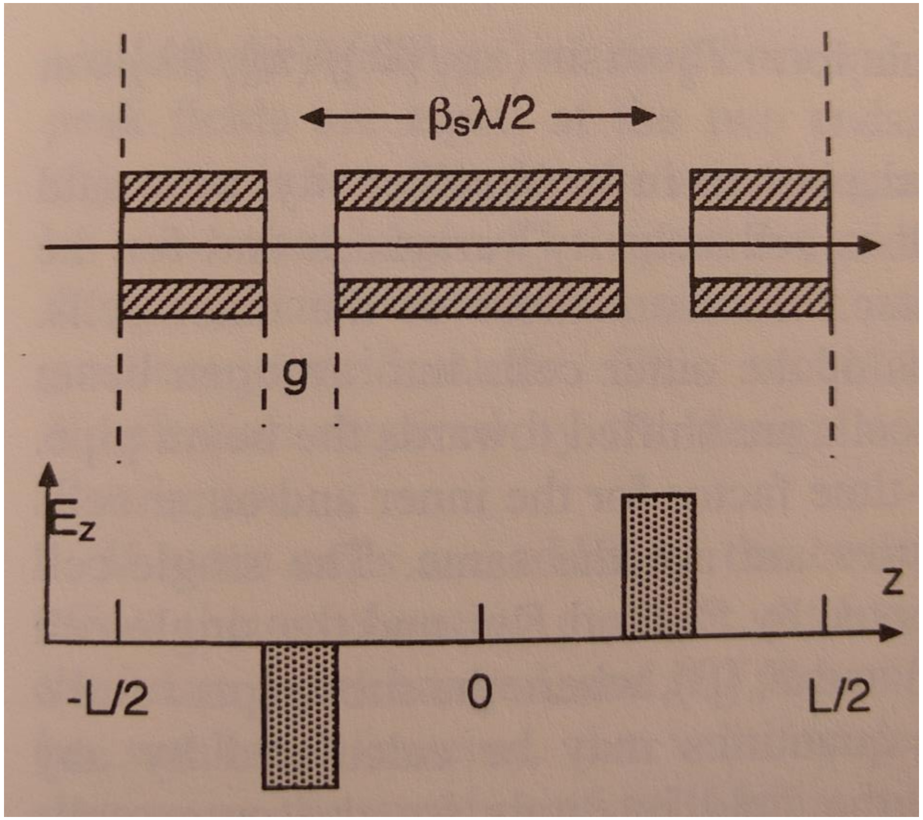
- Last take away points:
  - $R_{s,SRF} \propto f^2$ Pushes applications to lower frequency
  - Complex dependence on temperature, but lower is almost universally better (from a performance point of view, not cost!)
  - Achieving the best performance is very labor/infrastructure/cost intensive. Just ask LCLS-II! Or ILC! Or XFEL! Or CEBAF!
- I'll spare you the math, but the equivalent skin depth for this application is about 350Å.
- Also, remember your Carnot: $\eta_c = \frac{T_C}{T_H - T_C}$, and operating at $4K$, we get $\eta_c = 0.013$. We save six orders of magnitude on R_s but lose three because of the temperature. We gain efficiency, but pay for it in complexity.
- Full comparison of the materials later.

Pillbox, for real this time.

- What quantities do we care about?
  - Accelerating Voltage
  - Stored Energy
  - Peak Surface Fields
  - Efficiency of storing energy
  - Efficiency of transferring energy to the beam
- Peak Fields are obviously defined.
- Let's tackle the others in detail.

Accelerating Voltage

- Got a good taste of this in the homework
- Generalizing to two gaps, 180 degrees out of phase:
- $T = \frac{\sin\left(\frac{\pi g}{\beta\lambda}\right)}{\frac{\pi g}{\beta\lambda}} \sin\left(\frac{\pi\beta_s}{2\beta}\right)$
- Similar, but with an extra factor of synchronization between the gaps
- Model as $T = T_g S\left(N, \frac{\beta_s}{\beta}\right)$



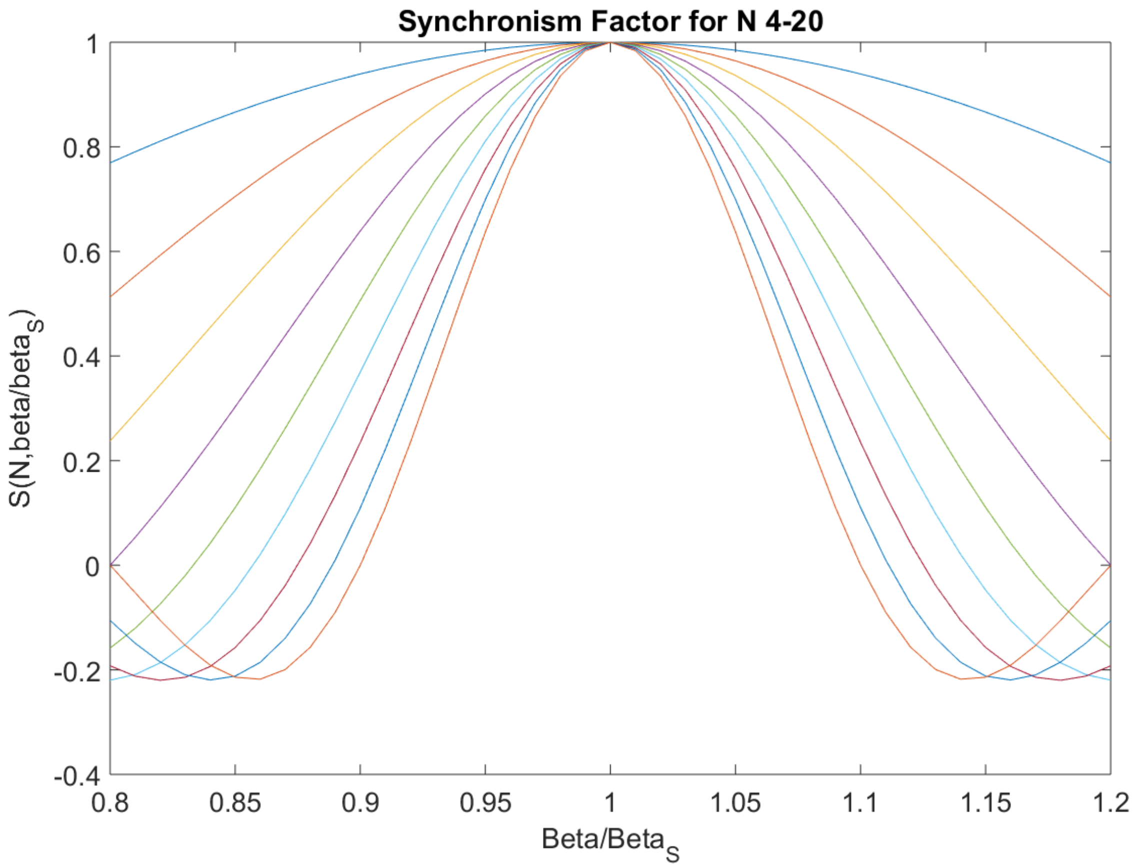
Gap Synchronism

Plotted is the synchronism factor for 20% error in β for gaps ranging from 4 to 20.

Larger number of gaps have smaller velocity acceptance.

Machine parameters drive design here, heavy ion v electrons, for instance.

For wide range of β, multiple cavity types may be needed.



Effective Length – A Warning

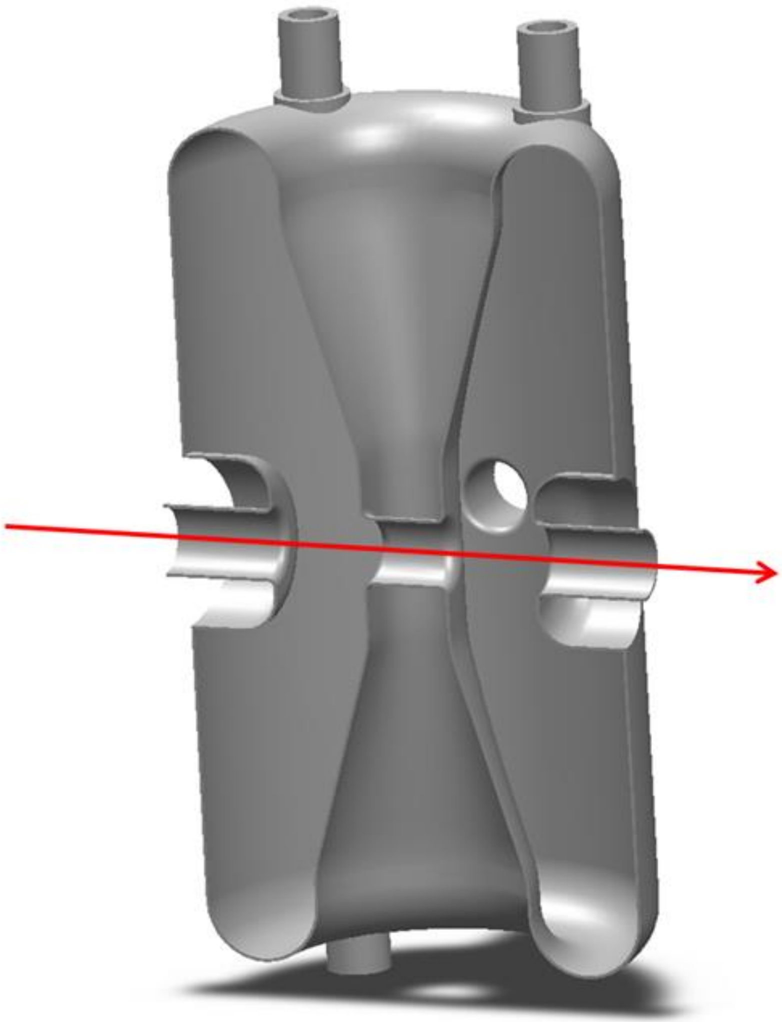
One Last Comment:

An often quoted figure of merit is the Accelerating Electric Field: $E_{acc} = \frac{V_{acc}}{L}$

While pillbox-style cavities are relative easy to determine the length, more complex geometries are more open to interpretation.

V_{acc} is unambiguous.

Pillbox: $E_{acc} = \frac{V_{acc}}{L} = \frac{2E_0}{\pi}$



Stored Energy

- We stated earlier: $u = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)$
- So it follows that U = $\int_V \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) dV$
- While this is generally true, we can chose a time where this calculation is easier. Choose time such that the electric fields are zero and magnetic fields are maximized.
- So, U = $\int_V \frac{1}{2}\left(\frac{1}{\mu_0}\vec{B}^2\right) dV$
- Generally, this is done for you in simulation. For a pillbox, this can be done analytically.
- $U = E_0^2 \pi L \epsilon_0 \int_0^R \rho J_1^2\left(\frac{2.405\rho}{R}\right) d\rho = \frac{\pi \epsilon_0 E_0^2}{2} J_1^2(2.405) L R^2$

Peak Surface Fields

- We want to calculate the peak surface fields.
- $E_{pk} = E_0$ is easy.
- Maximizing magnetic field on the end wall:
- $B_{pk} = \frac{E_0}{c} J_1(1.84) = \frac{E_0}{c} 0.583$ or where $\rho = 0.77R$
- But what we also want are normalized quantities.
- $\frac{B_{pk}}{\sqrt{U}}, \frac{E_{pk}}{\sqrt{U}}$ and, by extension, $\frac{V_{acc}}{\sqrt{U}}$
- These quantities can be scaled nicely, and are less prone to change during optimization of unrelated features.
- Speaking of, that last one seems quite useful…

Shunt Impedance

- Remember, we want a quantity that can be used to judge the efficiency of transferring the stored energy to the beam.
- The (effective) shunt impedance is defined as:
- $\frac{R}{Q} \stackrel{\text{def}}{=} \frac{V_{acc}^2}{\omega U}$ which is the ratio of the accelerating voltage squared and the reactive power in the cavity (in the equivalent circuit).
- This is a purely geometric factor that is very useful in describing the accelerating efficiency of a cavity geometry.
- Other definitions of this may not include the TTF, or may have a factor of two for historical reasons, so watch out.
- Note that this does not scale with frequency. You can directly scale a geometry to a different frequency, and this will stay the same. Very useful.

Shunt Impedance 2

- $\frac{R}{Q} = 150\ [\Omega] \frac{L}{R} = 196\beta[\Omega]$
- Linear with optimum particle velocity! Higher frequencies are better.
- Makes sense, U scales like L, but so does V_{acc}.

Quality Factor

- A standard metric for how efficiently a resonator stores energy is the quality factor.
- This is a quantity related to the number of cycles it would take to dissipate a given amount of stored energy.
- $Q_0 = \frac{\omega U}{P_d}$ But this means that we need a definition of P_d
- Fortunately, we've done the ground work:
- $P_d = \frac{1}{2} R_s \int_S \left|\vec{H}\right|^2 dA$ Integrated over the cavity walls
- Note the implicit assumption, that surface resistance is uniform over the entire cavity! Probably not the greatest assumption for superconductors, but not much else you can do without significant effort.

Temperature Mapping

Geometry Factor

- R_s is quite variable, especially for superconducting cavities.
- The quality factor that doesn't depend on R_s would be of great usefulness.
- The R_s dependence comes from the dissipated power.
- $Q_0 = \frac{\omega U}{P_d} = \frac{\omega U}{\frac{P_d}{R_s} R_s}, G = R_s Q_0 = \frac{\omega U}{\frac{P_d}{R_s}}$
- This, while adding dimensions to the quality, depends strictly on geometry and not material.
- Again, doesn't scale with frequency (make sure to gather all the scaling of U and P_d)

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Pillbox Quality Factor

- $P_d = \frac{R_s E_0^2}{\eta^2}\left\{2\pi \int_0^R \rho J_1^2\left(\frac{2.405\rho}{R}\right) d\rho + \pi R L J_1^2(2.405)\right\}$
- Outer wall + end wall
- $P_d = \frac{\pi R_s E_0^2}{\eta^2} J_1^2(2.405) R(R+L)$
- Giving:
- $G = \frac{\omega_0 \mu_0 L R^2}{2(R^2+RL)} = \eta \frac{2.405L}{2(R+L)} = \frac{453\frac{L}{R}}{1+\frac{L}{R}}[\Omega]$ With an optimum L ...
- $\frac{L}{R} = \frac{\beta\pi}{2.405}$, $G = 257\beta[\Omega]$
- A highly useful result, indicating that pillbox cavities are more efficient at higher optimum particle velocities.

Cryogenic Efficiency

- A quantity that is often used to compare efficiency of superconducting cavities is $\frac{R}{Q} * G = \frac{V_{acc}^2}{\frac{P_d}{R_S}}$
- Calculates directly cost of voltage to dissipated power.
- Cryogenic refrigeration is at a premium, so this can be an excellent comparison between very different cavity geometries.

Pillbox Scaling

Clearly better at high beta, best at $\beta = 1$.

Mechanical concerns also come into play:

Aspect ratio:

$$\frac{L}{R} = \frac{\beta\pi}{2.405}$$

This gets pretty sub-optimal at low beta, thin pancake cavities have poor mechanical properties.

- $G = 257\beta[\Omega]$
- $\frac{R}{Q} = 196\beta[\Omega]$
- $E_{pk} = E_0$
- $cB_{pk} = 0.583E_0$
- $U = \frac{\pi\epsilon_0 E_0^2}{2} J_1^2(2.405)LR^2$
- $P_d = \frac{\pi R_s E_0^2}{\eta^2} J_1^2(2.405)R(R+L)$
- $TTF = \frac{2}{\pi}$

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Material Comparison

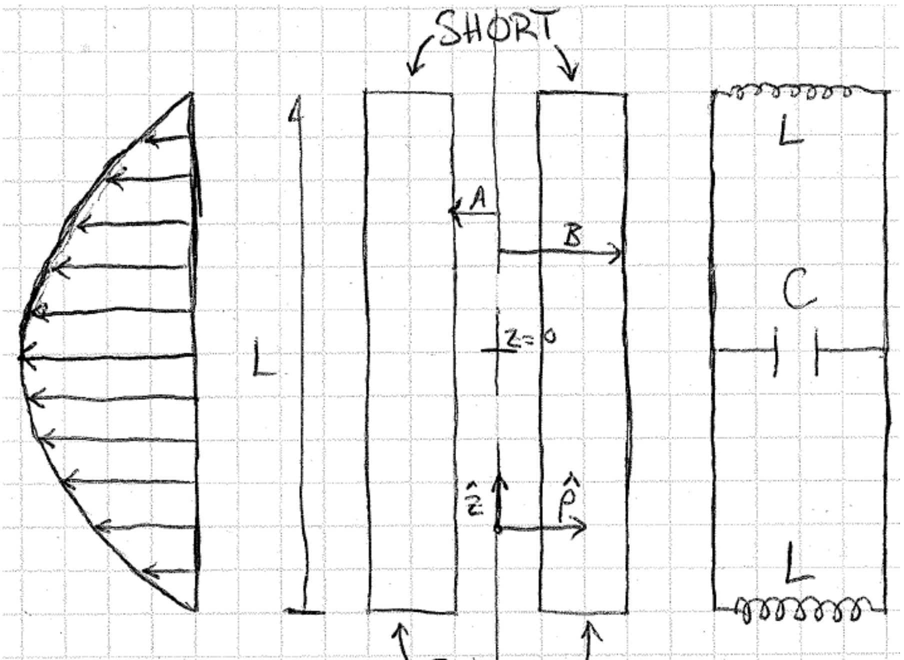
- Superconducting Cavity
  - Peak Surface Fields dominate design
  - ~220 mT is theoretical max, 120 mT is doing very well in practice
  - Pushes for high Q
  - Technologically Challenging
  - Processing requirements put significant constraints on complex cavity geometries
  - $R_s \propto f^2, P_d \propto f, Q \propto f^{-2}$
- Normal Conducting Cavity
  - Limited by dissipated power
  - Limits duty cycle or gradient
  - Pushes for highest $\frac{R}{Q}$
  - Local power density also a concern (local heating), maxes at ~20 W/cm^2
  - Electrical breakdown limited peak electric fields
  - Cheaper material (copper!)
  - Cooling design can be quite complex (non-uniform)
  - $R_s \propto f^{\frac{1}{2}}, P_d \propto f^{-\frac{1}{2}}, Q \propto f^{-\frac{1}{2}}$

Coaxial Resonators

Coaxial Waveguide

- A fundamentally different transmission line is coaxial geometry
- In contrast to circular/rectangular waveguide, there is a second conducting surface that's disconnected (in a waveguide) from the outer conductor.
- Assume that we have a cylindrical outer conductor, radius b and co-radial inner conductor, radius a. Both are aligned on the $\hat{z}$ axis.
- Solving the Helmholtz Equation and putting shorting plates at $\pm\frac{L}{2}$ we get similar solutions:
- $E_\rho = \frac{E_0 a}{\rho}\cos\left(\frac{p\pi z}{2L}\right)e^{i\omega t}$, $B_\phi = -i\frac{E_0 a}{\rho c}\sin\left(\frac{p\pi z}{2L}\right)e^{i\omega t}$
- $\omega = pc\pi/2L$

Half Wave Resonator

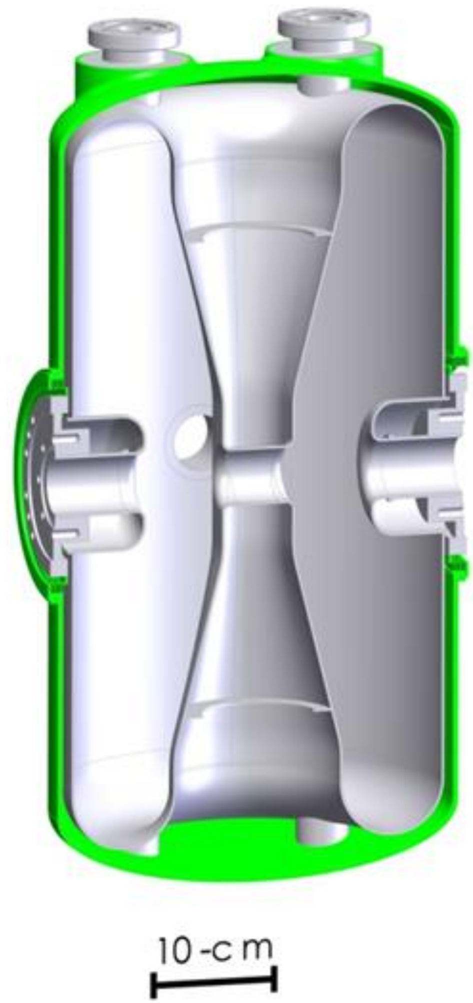


$$L = \frac{\lambda}{2} = \frac{c}{2f}$$

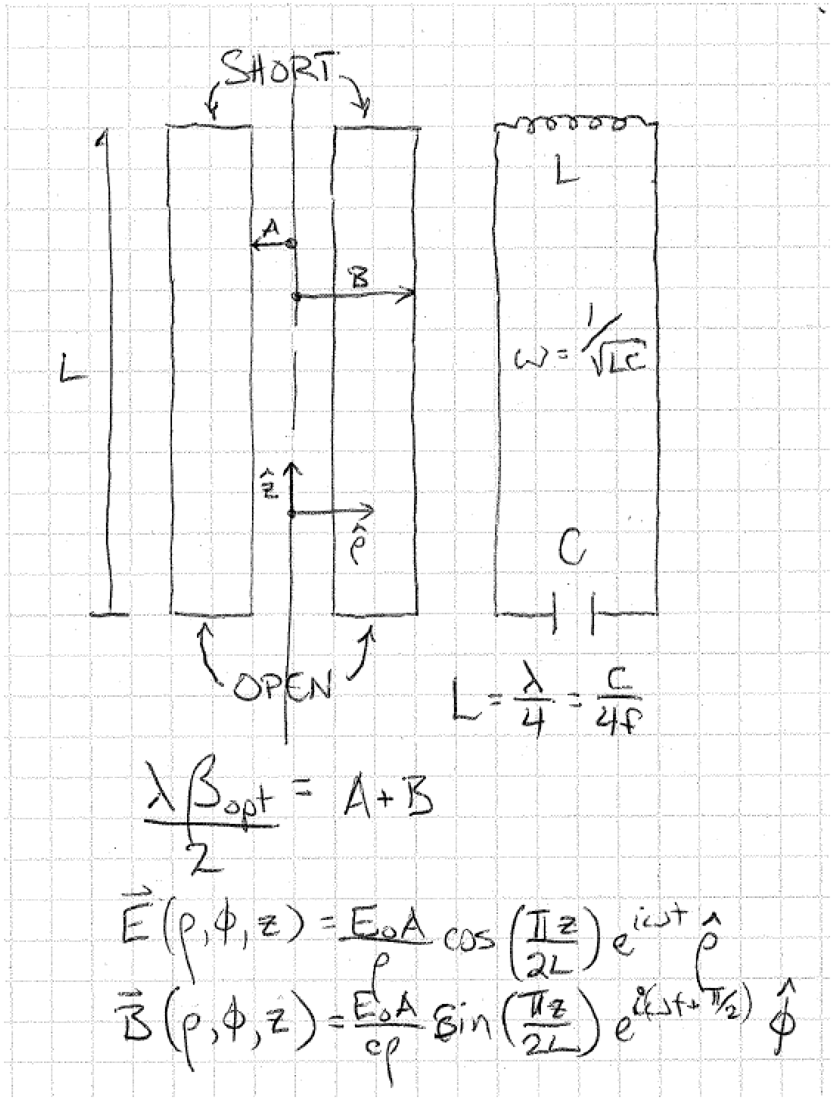
$$\frac{\lambda \beta_{opt}}{2} = A + B$$

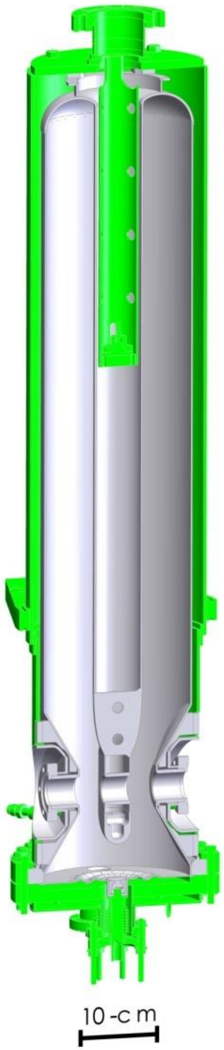
$$\vec{E}(\rho,\phi,z,t) = \frac{E_0 A}{\rho} \cos\left(\frac{\pi z}{2L}\right) e^{i\omega t}\,\hat{\rho}$$

$$\vec{B}(\rho,\phi,z,t) = \frac{E_0 A}{c\rho} \sin\left(\frac{\pi z}{2L}\right) e^{i(\omega t + \pi/2)}\,\hat{\phi}$$



Quarter-Wave Resonator





Short

"Open"

Coaxial Cavity Discussion

- Decouples beam line/accelerating gap size/geometry from the transverse dimension.
- Allows very low frequency resonators with small gaps in a mechanically robust geometry, very low beta resonators.
- Complicated fabrication and processing
- Quarter Wave Resonators are significantly different from ideal because the ‘open’ boundary condition isn’t physical.
- Lack of rotational symmetry can lead to transverse accelerating fields, especially with QWRs.