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RF Devices – Waveguides

J.P. Holzbauer, Ph.D.

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Circular Waveguide Modes

Assumptions

- Metallic circular tube with Radius: R
- Axis of tube points in $+\hat{z}$
- If we assume that the wave is propagating in $+\hat{z}$, we get:
- $\vec{E}(\vec{x}, t) = \vec{E}_0(\rho, \phi)e^{ikz-i\omega t}$
- $\vec{H}(\vec{x}, t) = \vec{H}_0(\rho, \phi)e^{ikz-i\omega t}$
- This is just the generalized plane wave solution to Helmholtz
- Now, we assume $B_z = 0$ (gotta pick something)
- Also, we need the fields to be azimuthally consistent, so let's assume they have a ϕ dependence of the form $e^{im\phi}$ where m is an integer.
- $E_z(\rho, \phi, z, t) = f(\rho)e^{ikz-i\omega t-im\phi}$

Helmholtz Solutions

- $E_z(\rho, \phi, z, t) = f(\rho)e^{ikz - i\omega t - im\phi}$
- $\frac{1}{c^2} \frac{\partial^2 \mathcal{F}}{\partial t^2} = \nabla^2 \mathcal{F} \Rightarrow \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{df}{d\rho} - \frac{m^2}{\rho^2} f + \left(\frac{\omega^2}{c^2} - k^2 \right) f = 0$



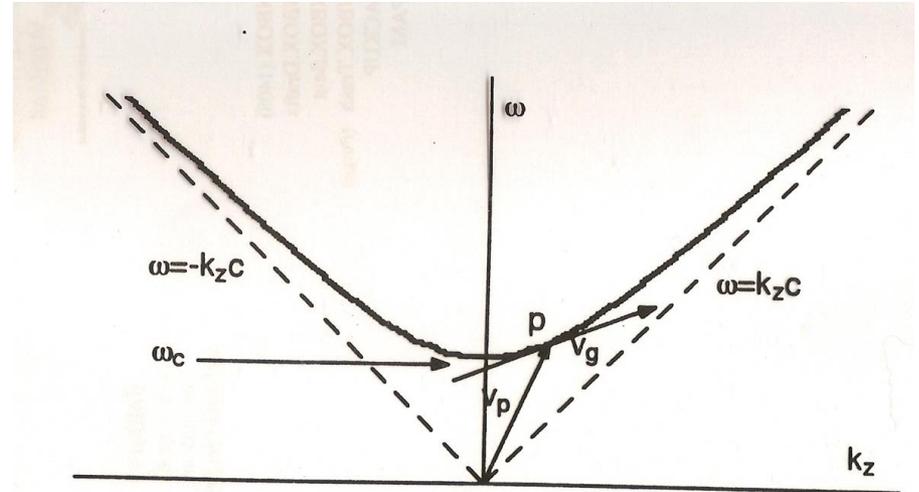
Math!

- $f(\rho) = E_0 J_m(k_{\perp} \rho)$ Bessel Functions! Wee.
- One last thing, we have to satisfy our electric field boundary condition: $\hat{n} \times \hat{E} = 0$
- Thus: $E_z(\rho = R) = 0$
- Gives the condition that $k_{\perp} R = j_{m,n}$ where $j_{m,n}$ is the n_{th} Bessel Zero of order m .

Dispersion Curve

- The relationship between the wavenumber k and frequency ω is called the dispersion curve. For a circular waveguide, we get the following equation:

- $k_{m,n} = \pm \left(\frac{\omega^2}{c^2} - \frac{j_{m,n}^2}{R^2} \right)^{1/2}$
- Remember that $v_P = \omega/k$ and $v_G = d\omega/dk$
- Phase advances at v_p , energy advances at v_G



There is a frequency with no real solutions for k ! Called Cutoff.

$$\omega_c = \frac{j_{m,n} c}{R_c}$$

Practical Effects of Cutoff

- Cutoff is not a hard limit, i.e. frequencies above cutoff propagate freely and with no problems and frequencies below cutoff die immediately.
- From the dispersion curve, it's clear that the group velocity drops to zero at cutoff. Waves just above cutoff propagate very slowly, and this makes their transmission very sensitive to mechanical errors or perturbations (fractionally larger change in impedance close to cutoff). Very dangerous design situation.
- What does it mean to be below cutoff? Imaginary wave number!
- Imaginary k means that the wave propagates like e^{-kz} , exponentially decaying. Can be slowly so if k is small.

Dispersion – Other Comments

- Every mode (different $j_{m,n}$) has a different cutoff. Higher order modes (higher m and n) will have a higher cutoff frequency. The most important implication of this is that there is a frequency range where only ONE mode is allowed. Which is nice.
- Circular Waveguide modes have a rotational degeneracy, which is a huge pain to work with. The lowest mode is a dipole mode, which with the slightest provocation, will rotate as it propagates. (Which is why you only use circular waveguides in very specific applications)

Phase Velocity Synchronization

- To be useful for acceleration, the phase velocity must be synchronized with the particles ($v_P \leq c$)
- For Circular Waveguide modes:

- $$\frac{v_P}{c} = \frac{\omega}{c \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{j m, n}{R}\right)^2}} = \frac{\omega}{\sqrt{\omega^2 - \left(\frac{j m, n c}{R}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{j m, n c}{\omega R}\right)^2}} > 1$$

- Not great. Why doesn't this break physics?
- This means that we can't synchronize no matter what.
- What can we do to slow down the phase velocity?
- Note:

- $$\frac{v_G}{c} = \sqrt{1 - \left(\frac{j m, n c}{\omega R}\right)^2}$$
 which is always less than 1. Phew.

Transverse Magnetic Modes (TM_{m,n})

- Note: Use $\vec{\nabla} \times \vec{E} = i\omega\vec{B}$ to get the magnetic field.
- $E_z = E_0 J_m \left(j_{m,n} \frac{\rho}{R} \right) e^{ik_{m,n}z - i\omega t - im\phi}$
- $E_\rho = E_0 \frac{ik_{m,n}R}{j_{m,n}} J'_m \left(j_{m,n} \frac{\rho}{R} \right) e^{ik_{m,n}z - i\omega t - im\phi}$
- $E_\phi = -E_0 \frac{mk_{m,n}R^2}{\rho j_{m,n}^2} J_m \left(j_{m,n} \frac{\rho}{R} \right) e^{ik_{m,n}z - i\omega t - im\phi}$
- $B_\rho = E_0 \frac{m\omega R^2}{c^2 j_{m,n}} J_m \left(j_{m,n} \frac{\rho}{R} \right) e^{ik_{m,n}z - i\omega t - im\phi}$
- $B_\phi = E_0 \frac{i\omega R}{c^2 j_{m,n}} J'_m \left(j_{m,n} \frac{\rho}{R} \right) e^{ik_{m,n}z - i\omega t - im\phi}$
- Standard notation with m,n indicating the order.
- If you need TE modes, set $E_z = 0$ and go crazy.

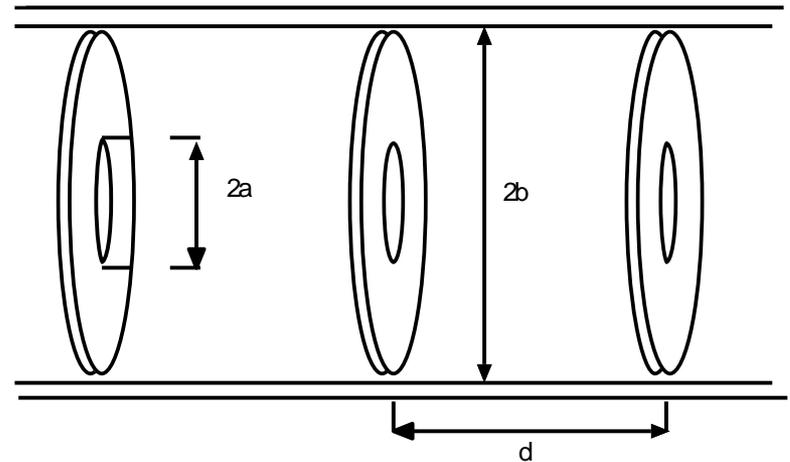
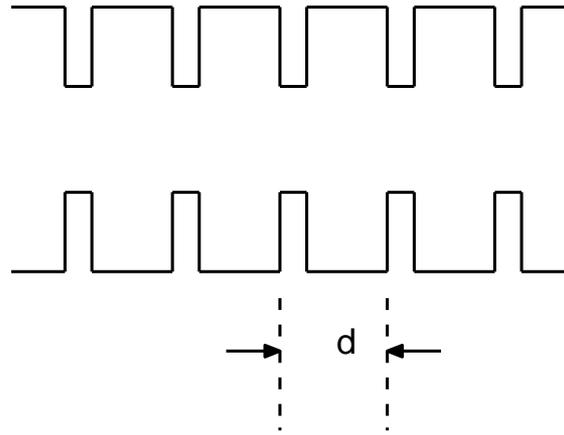
Traveling Wave Structures

Disk Loaded Circular Waveguide

Adding circular disks with apertures to the waveguide adds reflection terms at each disk.

The sum of these reflection terms gives an overall wave with $v_p < c$.

We need to calculate the dispersion relationship for this sort of structure so we can design a usable accelerator.



Floquet Theorem

- Floquet Theorem: Fields for a propagating wave at two different cross sections separated by one period differ by only a phase factor.
- $\vec{E}(r, z, t) = \vec{E}_d(r, z)e^{i\omega t - k_0 z}$ where \vec{E}_d is periodic with period d .
- $\vec{E}(r, z + d, t) = \vec{E}_d(r, z + d)e^{i\omega t - k_0(z+d)}$
- $\vec{E}_d(r, z) = \vec{E}_d(r, z + d)$ from definition of \vec{E}_d
- So: $\vec{E}(r, z + d, t) = \vec{E}_d(r, z + d)e^{i\omega t - k_0(z+d)} = \vec{E}(r, z, t)e^{-ik_0 d}$
- $\vec{E}(r, z + d, t) = \vec{E}(r, z, t)e^{-i\phi}$ where $\phi = k_0 d$
- So, we have a definition of the phase advance per cell.
- If the phase advance is pure imaginary, it's a stopband mode
- Real structures have some small imaginary part (losses)

Traveling Wave “Space Harmonics”

- Fourier expand solution: $\vec{E}_d(r, z) = \sum_{n=-\infty}^{\infty} A_n(r) e^{-i \frac{2\pi n z}{d}}$
- Apply Wave Equation!
- $\vec{E}_d(r, z) = \sum_{n=-\infty}^{\infty} E_0 I_0(K_n r) e^{-i \frac{2\pi n z}{d}}$
- Where $K_n^2 = \left(\frac{\omega}{c}\right)^2 - \left(k_0 + \frac{2\pi n}{d}\right)^2$
- So, for each frequency, there are many space harmonics, and you can choose the structure parameters to have synchronous acceleration.
- We want to set the phase velocity of the $n = 0$ wave to be equal to the particle velocity β_0

Synchronous Acceleration

- General Solution: $\vec{E}_z(r, z) = \sum_{n=-\infty}^{\infty} E_n I_0(K_n r) e^{i\omega t - ik_n z}$
- With $K_n^2 = \left(\frac{\omega}{c}\right)^2 - \left(k_0 + \frac{2\pi n}{d}\right)^2$ and $k_n = k_0 + \frac{2\pi n}{d}$
- So, $n = 0$
- Want $v_P = \frac{\omega}{k_0} = \beta_0 c$ with $\beta_0 < 1$
- Thus $k_0 = \frac{\omega}{\beta_0 c} = \frac{2\pi}{\beta_0 \lambda}$
- So, we get a Radial Wave Number K_0^2 of:
- $K_n^2 = \left(\frac{\omega}{c}\right)^2 - (k_0)^2 = \left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{2\pi}{\beta_0 \lambda}\right)^2 = \left(\frac{2\pi}{\beta_0 \lambda}\right)^2 (\beta_0^2 - 1)$
- $K_n^2 = -\left(\frac{2\pi}{\gamma_0 \beta_0 \lambda}\right)^2$

Energy Gain

- We want to field at the particle at position z , time t .
- $t = \frac{z}{\beta_0 c} + t_0$ if the particle is moving at speed $\beta_0 c$
- Also, for the synchronous particle $\frac{\omega}{\beta_0 c} - k_0 = 0$
- So, the field seen by the synchronous particle is:
- $E_{z0} = E_0 I_0(K_0 r) e^{i\phi}$ where the angle is related to the definition of t_0 , $\phi = \omega t_0$
- This gives an energy gain of a properly phased particle of:
- $\Delta W = q \int_0^d E_0 I_0(K_0 r) dz = q E_0 d I_0(K_0 r)$
- Particle sees continuous acceleration for the length of the structure.
- Other n modes are non-synchronous and will integrate out

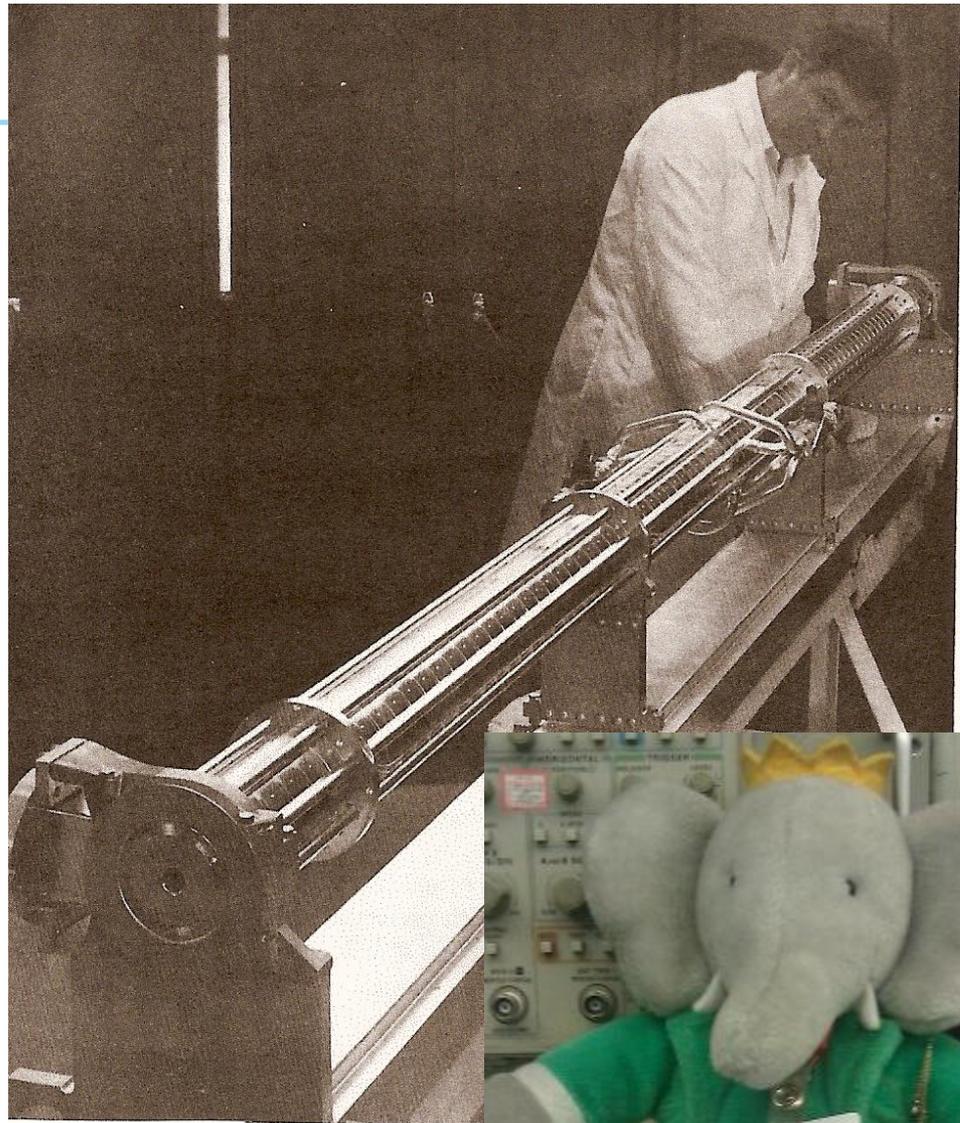
Separate Interpretation

- This treatment can also be repeated by treating each ‘cavity’ as an individual resonant circuit with some coupling between neighboring ‘cavities’.
- Different models, with and without losses, inductive or capacitive coupling, give different but instructive answers.
- These models give analytical answers for optimal iris sizes and other design parameters, but are very involved calculations.
- See T. Wangler’s RF Linear Accelerators for more information if this is of interest.

SLAC Disk-Loaded Linac

SLAC had a high-energy physics collider (BaBar experiment)

- Stanford Linear Collider
 - SLAC Large Detector (lame)
- PEP, then PEP-II with BaBar
- Eventually: LCLS
- Finally to be replaced with a superconducting LCLS-II
- 2856 MHz klystron driven copper disk loaded structures
- 960 tanks with 87 cells each



Reality! Let's do it.

- We derived an interesting example of an infinite periodic structure with no losses. Still pretty far from reality.
- We need to add losses both to the walls and the beam.
- We also need to consider what happens when the structure ends. The SLAC tanks were only ~3m long!
- Are we going to throw power away? Make the tanks long?
- Also, these cavities are copper. SLAC pulsed the linac with a ~5% duty cycle to keep cooling reasonable.
- An aside: Why 2.865 GHz? We'll get to that tomorrow.

Group Velocity

- When you turn the RF power on, it enters the tank and start flowing. It's going to move at the group velocity, giving a fill time of $t_F = \frac{L}{v_G}$
- Higher group velocity also means higher power needed to maintain a given stored energy density $P_W = v_G U$
- Low Group Velocity: Higher stored energy density means high fields, but also high Ohmic losses in the walls.
- High Group Velocity: Low losses, but also low gradients, poor acceleration.
- Generally a design compromise based on many other details.
- Larger disk aperture = Higher Group Velocity

Disk Geometry

- Choosing phase velocity (speed of light for SLC) and optimizing group velocity leaves disk separation and thickness.
- Disk separation has a significant impact on how much power is available in the synchronous wave as opposed to non-synchronous waves. Closer is better/more efficient.
- The closer they are together, the more there are in a given tank, meaning higher losses to Ohmic heating.
- Disk thickness doesn't have a large impact on RF performance, mostly mechanical concerns. Thicker allows better heat dissipation and strength. Too thick reduced coupling between neighboring cells.

Constant Impedance Structure

- If you assume that each cell is identical, you get an attenuation of power as you go down the tank:
- $P_W(z) = P_0 e^{-2\alpha_0 z}$ with $\alpha_0 = \frac{\omega}{2Qv_G}$ and $Q = \frac{\omega U}{-\frac{dP_W}{dz}}$
- Recalculating energy gain gives:
- $\Delta W = \frac{q\sqrt{2r_L P_0 L}(1-e^{-\tau_0})}{\sqrt{\tau_0}}$ where $\tau_0 = \alpha_0 L$
- Maximum at $\tau_0 \cong 1.26$, gives optimum $v_G = \frac{\omega L}{1.52Q}$
- Low group velocity increases attenuation, lose too much in walls.
- High group velocity gives low gradient and much energy lost to the load at the end of the tank.

Constant Impedance Structure (SLAC)

- What if we allow the structure to vary along the tank?
- Keep a constant gradient, for instance by allowing the group velocity to change.
- Now, $\tau_0 \stackrel{\text{def}}{=} \int_0^L \alpha_0(z) dz$
- Turn the crank, and we get $\Delta W = q\sqrt{r_L P_0 L(1 - e^{-2\tau_0})}$
- Optimum τ_0 is now infinite, but now the filling time becomes important: $\tau_F = \tau_0 \frac{2Q}{\omega}$
- Longer filling time means more losses before the tank is usable for beam, more losses.
- Benefits: Uniform dissipated power, lower E_{pk}
- Cons: More complex construction (tapered apertures)

From Traveling Waves to Standing Waves

Standing Wave Structures

- You can build an entirely different class of accelerating structures if you counter-propagate two traveling waves with $\pm k_{m,n}$
- Then you get a space terms:
- $\frac{1}{2} (e^{ik_{m,n}z} + e^{-ik_{m,n}z}) = \cos(k_{m,n}z)$
- $\frac{1}{2} (k_{m,n}e^{ik_{m,n}z} - k_{m,n}e^{-ik_{m,n}z}) = ik_{m,n} \sin(k_{m,n}z)$
- Rolling the factor of 2 into E_0 makes sense here.
- Assuming metal walls at 0 and L gives the requirement that $E_\phi(z = L) = E_\phi(z = 0) = E_\rho(z = L) = E_\rho(z = 0) = 0$
- Which leads to: $\sin(k_{m,n}L) = 0 \therefore k_{m,n}L = l\pi$
- Where l is the longitudinal order (TM_{mn l} modes!)

Standing Waveguide Modes

- $E_z = E_0 J_m \left(j_{m,n} \frac{\rho}{R} \right) \cos \left(\frac{l\pi z}{L} \right) e^{-i\omega t - im\phi}$
- $E_\rho = -E_0 \frac{l\pi R}{j_{m,n} L} J'_m \left(j_{m,n} \frac{\rho}{R} \right) \sin \left(\frac{l\pi z}{L} \right) e^{-i\omega t - im\phi}$
- $E_\phi = -E_0 \frac{iml\pi R^2}{\rho j_{m,n}^2 L} J_m \left(j_{m,n} \frac{\rho}{R} \right) \sin \left(\frac{l\pi z}{L} \right) e^{-i\omega t - im\phi}$
- $B_\rho = E_0 \frac{m\omega R^2}{c^2 \rho j_{m,n}^2 L} J_m \left(j_{m,n} \frac{\rho}{R} \right) \cos \left(\frac{l\pi z}{L} \right) e^{-i\omega t - im\phi}$
- $B_\phi = E_0 \frac{i\omega R}{c^2 j_{m,n}} J'_m \left(j_{m,n} \frac{\rho}{R} \right) \cos \left(\frac{l\pi z}{L} \right) e^{-i\omega t - im\phi}$
- Note the change in the dispersion curve! No longer continuous with all frequencies allowed.

- $$\omega_{m,n,l} = \sqrt{\left[\left(\frac{cl\pi}{L} \right)^2 + \left(\frac{cj_{m,n}}{R} \right)^2 \right]}$$

Pillbox Cavity

- You can repeat all this for TE modes, but we want longitudinal electric fields for acceleration!
- Pick the lowest frequency, simplest mode: TM_{010}
- $B_\rho = E_\rho = E_\phi = 0$ and $j_{m,n} = 2.405$
- $E_z = E_0 J_0 \left(\frac{2.405\rho}{R} \right) e^{-i\omega t}$
- $H_\phi = \frac{E_0}{\eta} J_1 \left(\frac{2.405\rho}{R} \right) e^{-i\omega t} e^{\frac{i3\pi}{2}}$ with $\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 376.7 \Omega$ is the impedance of free space.
- $\omega_{010} = \frac{2.405c}{R}$ Note: only depends on radius, not length!

Summary

- We went through an overview of how bound RF waves could be used to build accelerators, and we saw a bit of the process of optimization that's used to arrive at a final design.
- We got to a pillbox cavity, the most simple work-horse cavity topology that's used today.
- For homework, you'll explore simple cavity geometries and fields.
- Tomorrow is a deep dive into how we characterize and optimize these cavities and into the zoo of topologies actually used in the real world (read: lots of pictures!).