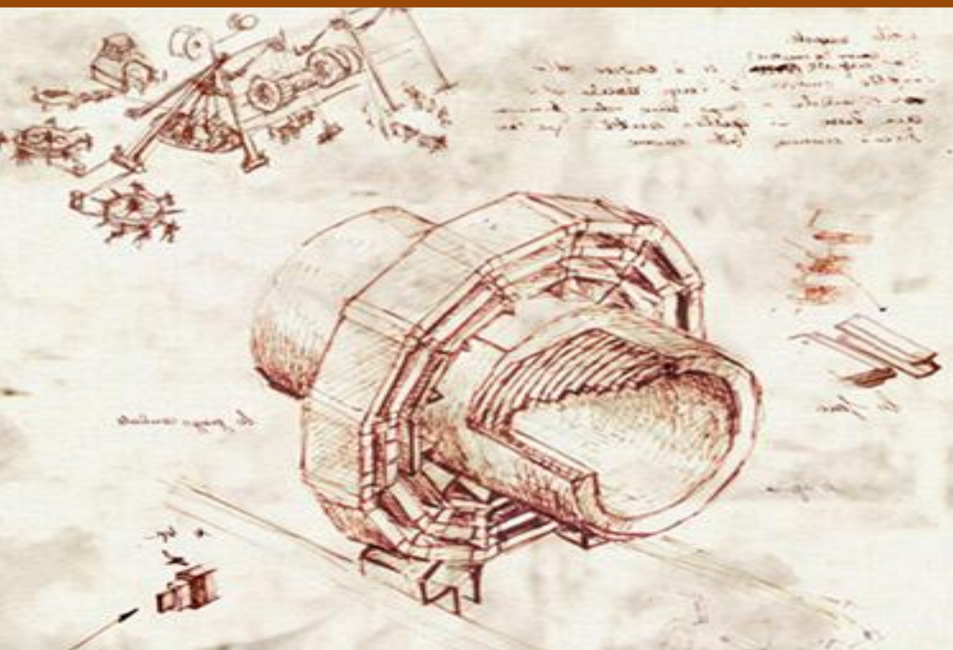
John Adams Institute for Accelerator Science

Imperial College London | Royal Holloway University of London | University of Oxford

Unifying physics of accelerators, lasers and plasma



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Lecture 2: Transverse dynamics

USPAS 2016

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Basics of accelerators

- **Plan of the lecture**
  - **Basics of beam dynamics (transverse)**

Equations and units

SI

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \oiint_{\partial\Omega} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho dV$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \oiint_{\partial\Omega} \mathbf{B} \cdot d\mathbf{S} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \oint_{\partial\Sigma} \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \iint_{\Sigma} \mathbf{B} \cdot d\mathbf{S}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \qquad \oint_{\partial\Sigma} \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \iint_{\Sigma} \mathbf{J} \cdot d\mathbf{S} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_{\Sigma} \mathbf{E} \cdot d\mathbf{S}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Gauss

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \left(4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right)$$

Microscopic Maxwell equations and Lorentz force in SI and Gaussian-cgs units

The SI units are the standard, but Gaussian units are more natural for electromagnetism. Advice: deriving the formula, instead of writing for example *e* or *h*, express the end result via more natural quantities ($m_e c^2$, r_e, λ_e, α, etc.)

SI

$$r_e = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{m_e c^2}$$

$$\alpha = \frac{e^2}{(4\pi\varepsilon_0)\hbar c}$$

$$r_e \approx 2.82 \cdot 10^{-15} m$$

$$\alpha \approx 1/137$$

$$\lambda\!\!\!^{-}_e = r_e / \alpha \approx 3.86 \cdot 10^{-13} \mathrm{m}$$

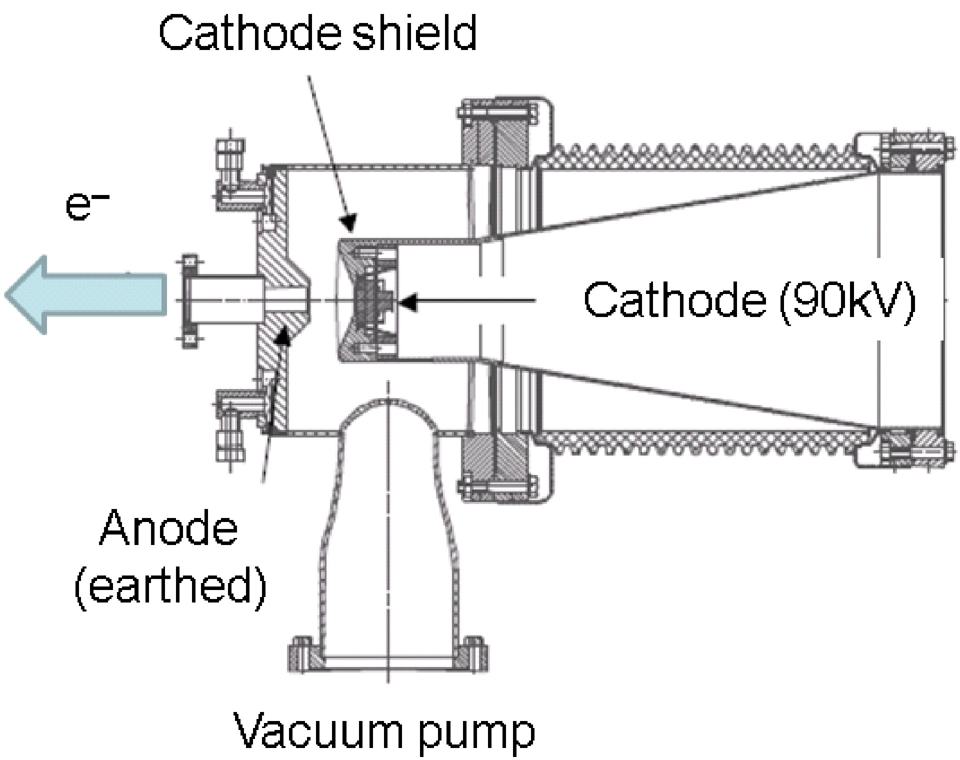
Gauss

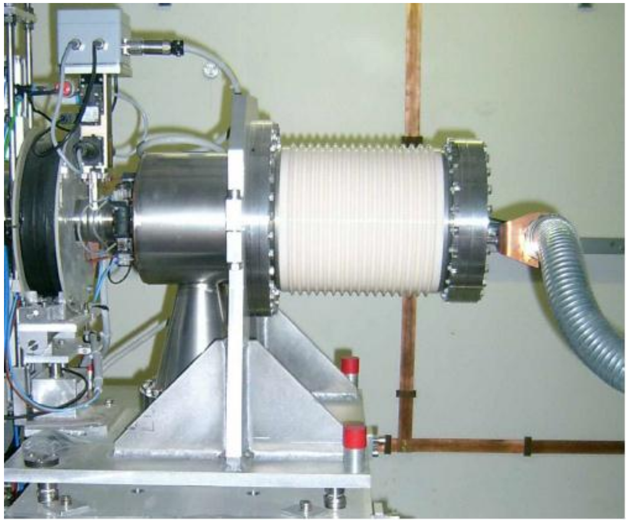
$$r_e = \frac{e^2}{m_e c^2}$$

$$\alpha = \frac{e^2}{\hbar c}$$

Accelerator starts from … - thermionic gun

Electrons are generated by thermionic emission from the cathode and accelerated across a high voltage gap to the anode. A grid between anode and cathode can be pulsed to generate a train of pulses suitable for RF acceleration

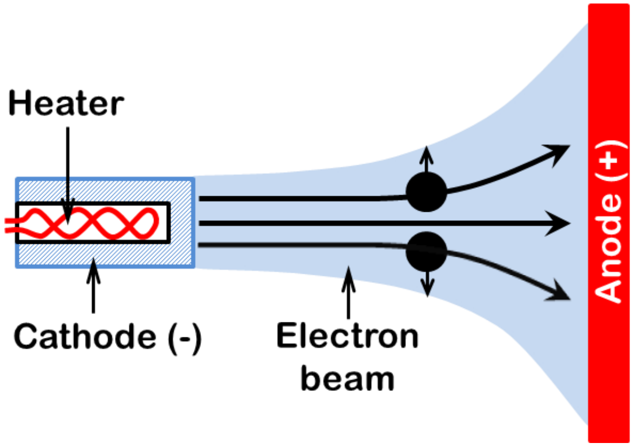




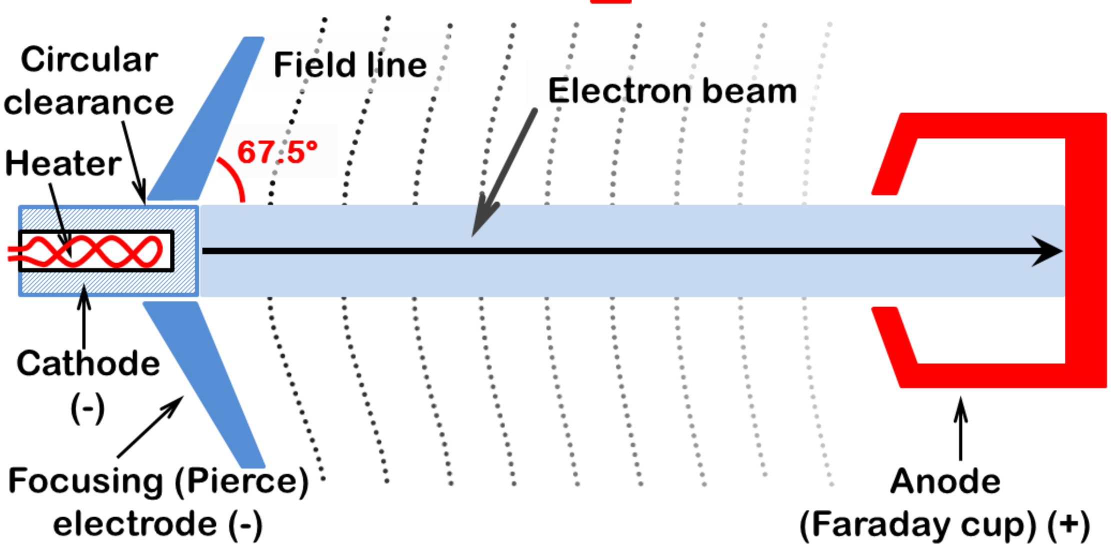
Cathode assembly BaO/CeO-impregnated tungsten disc is heated and electrons are emitted

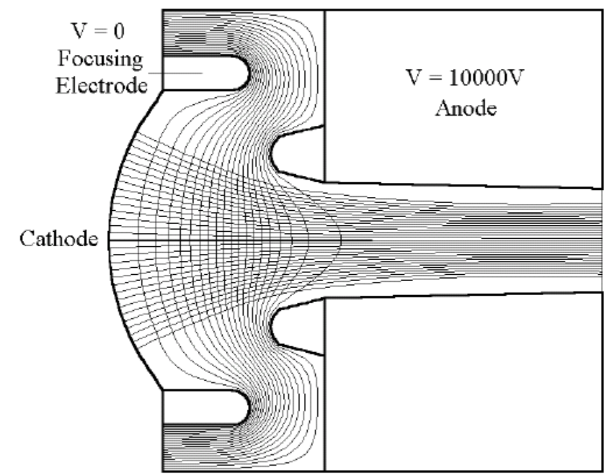
Thermionic gun – space charge and electrode shape

Electrons generated by thermionic emission tend to repel therefore an advance e.m. design is envisaged to control the beam dynamics and reduce the emittance of the beam.



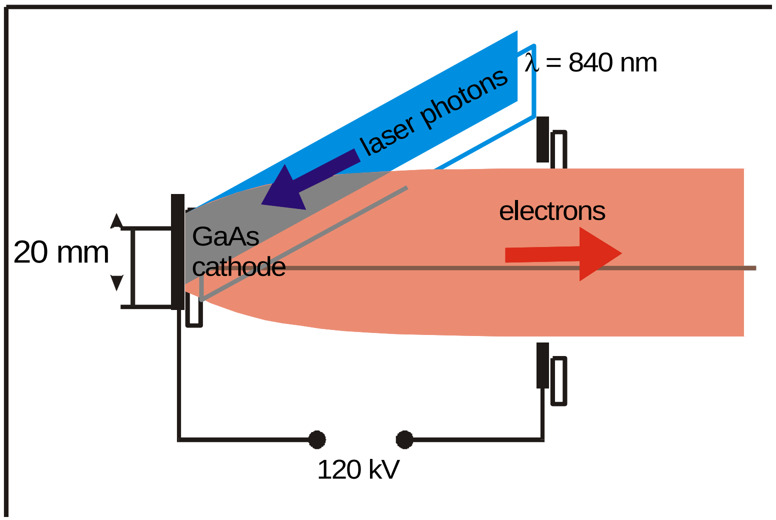
This requires solving Laplace equation $\Delta\varphi=0$ for the potential of the e.m. field in the given geometry





Polarized e- sources

- Sometime we need large number of bunches of polarized electrons
- electron sources:
  - laser-driven photo injector
  - circularly polarized photons on GaAs cathode
  - ε_n ~ 50 µm rad
    factor ~10 in x plane
    factor ~500 in y plane
    too large in case we plan to use it for colliders
  - dominated by space charge
  - RF bunching system needed to generate bunch structure for the linac, and DR to reduce emittance



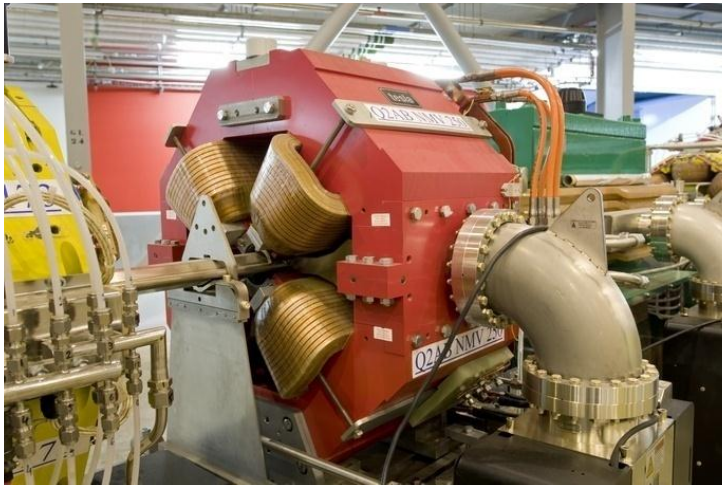
Linear accelerators

Circular accelerators

Main components of a storage ring

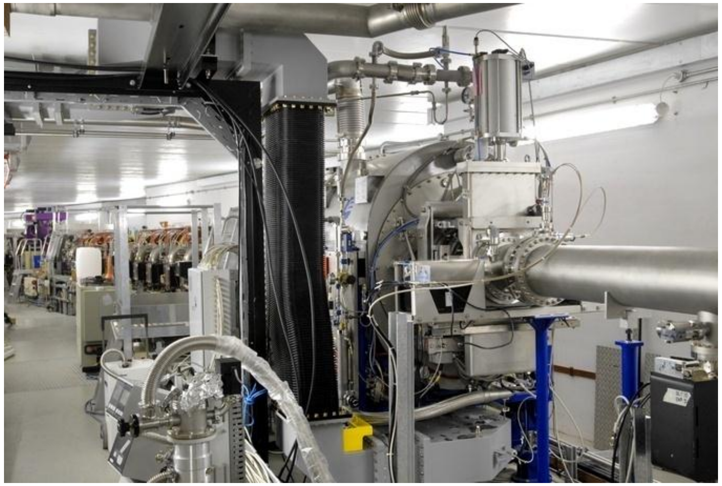
Dipole magnets to bend the electrons

Quadrupole magnets to focus the electrons



Sextupole magnets to focus off-energy electrons (mainly)

RF cavities to accelerate or replace E losses due to synchrotron radiation



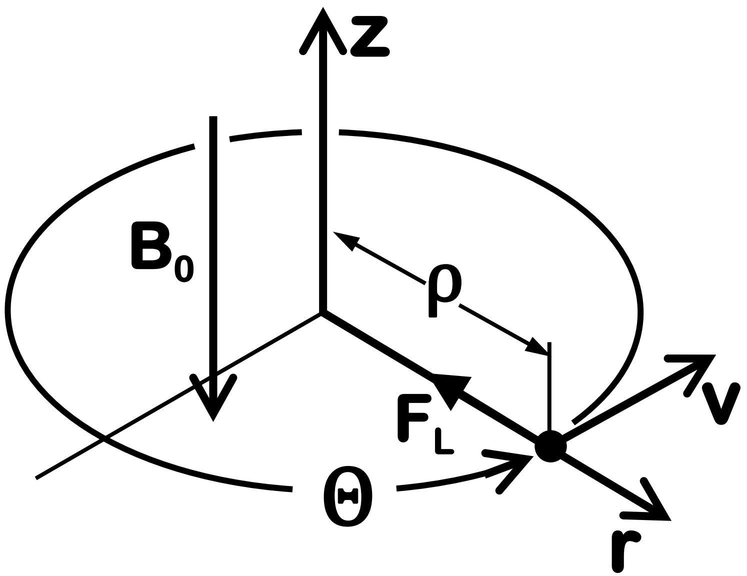
Diamond storage ring

Motion of charged particles in e.m. fields (I)

$$\frac{d\overline{p}}{dt} = q(\overline{E} + \overline{v} \times \overline{B}) \qquad \frac{d\varepsilon}{dt} = \overline{F} \cdot \overline{v} \qquad \overline{p} = m_0 \gamma \overline{v} \qquad \varepsilon = m_0 \gamma c^2$$

In uniform magnetic field $m_0 \frac{d\gamma\overline{v}}{dt} = q\overline{v} \times \overline{B}$ **=>** $m_0 \gamma \dot{v}_x = q v_y B$ **&** $m_0 \gamma \dot{v}_y = -q v_x B$

Thus $\ddot{v}_x = \frac{qB}{m_0\gamma} \dot{v}_y = -\left(\frac{qB}{m_0\gamma}\right)^2 v_x$ **solution** $v_x = v_0 \cos(\omega t)$ **with** $\omega = \frac{qB}{m_0\gamma}$



or $x = \frac{v_0}{\omega} \sin(\omega t)$

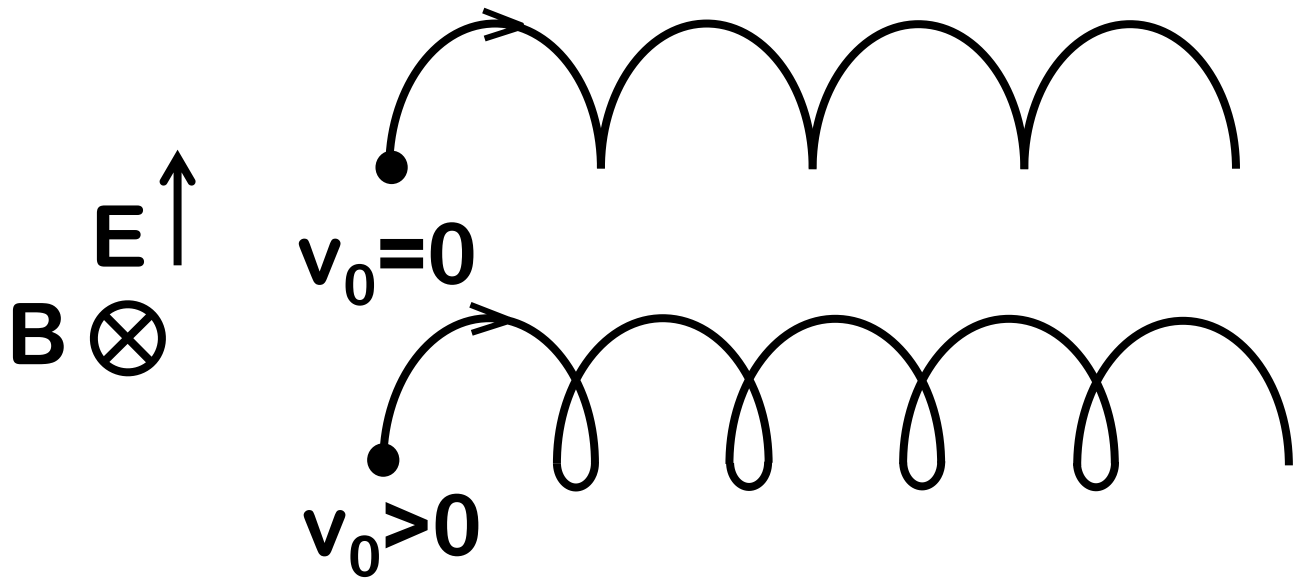
With radius $\rho = \frac{v_0}{\omega} = \frac{v_0 m_0 \gamma}{qB}$

SI: $\rho = \frac{p}{qB}$ **Gaussian:** $\rho = \frac{pc}{qB}$

While we on this topic – drift in ExB fields

Consider uniform E and B that are perpendicular –often met in plasma & beams

Qualitative picture:

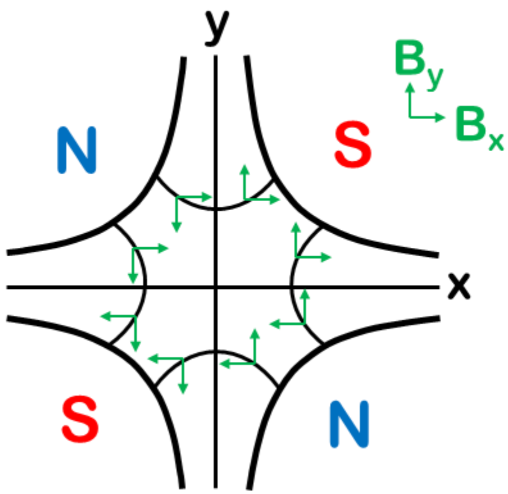


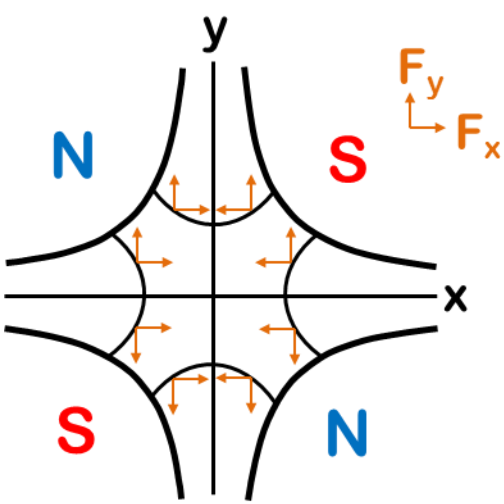
It is easy to find that equations predict drift with constant velocity

$$v_d = \frac{\overline{E} \times \overline{B}}{B^2}$$

Gaussian: $$v_d = c\frac{\overline{E} \times \overline{B}}{B^2}$$

Motion of charged particles in e.m. fields (II)





$(z \equiv y \quad \text{in here})$

Motion in quadrupole magnet

$B_x = Gz \qquad B_z = Gx$

$$m_0 \frac{d\gamma \bar{v}}{dt} = q\bar{v} \times \bar{B}$$

use Cartesian coordinate

$$\ddot{x} = -\frac{q}{m_0 \gamma} G\dot{s}x \qquad \ddot{z} = \frac{q}{m_0 \gamma} G\dot{s}z$$

$$\ddot{s} = \frac{q}{m_0 \gamma} G(\dot{x}x - \dot{z}z)$$

Change the independent variable from time to path length and for small deviations from the axis these reduce to

$$x'' - Kx = 0$$

$$z'' + Kz = 0$$

$$K = \frac{e}{p} \frac{\partial B_z}{\partial x} = \frac{e}{p} G$$

Linear betatron equations of motion

In the magnetic fields of *dipoles magnets* and *quadrupole magnets* the coordinates of the charged particle *w.r.t. the reference orbit and using the curvilinear abscissa s* are given by the Hill's equations

$$\frac{d^2y}{ds^2} + K_y(s)y = 0$$

$$K_x(s) = \frac{1}{\rho^2(s)} - \frac{1}{B\rho}\frac{\partial B_z(s)}{\partial x}$$

$$K_z(s) = \frac{1}{B\rho}\frac{\partial B_z(s)}{\partial x}$$

weak focussing of a dipole

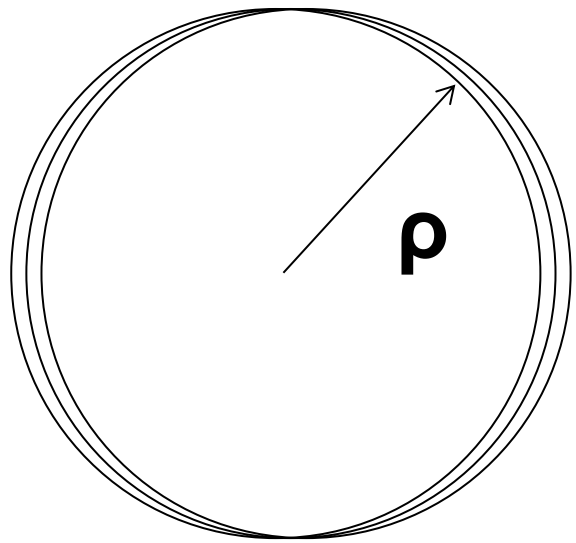
quadrupole focussing

No periodicity is assumed but for a circular machine K_x, K_z and ρ are periodic

These are linear equations (in y = x, z). They can be integrated.

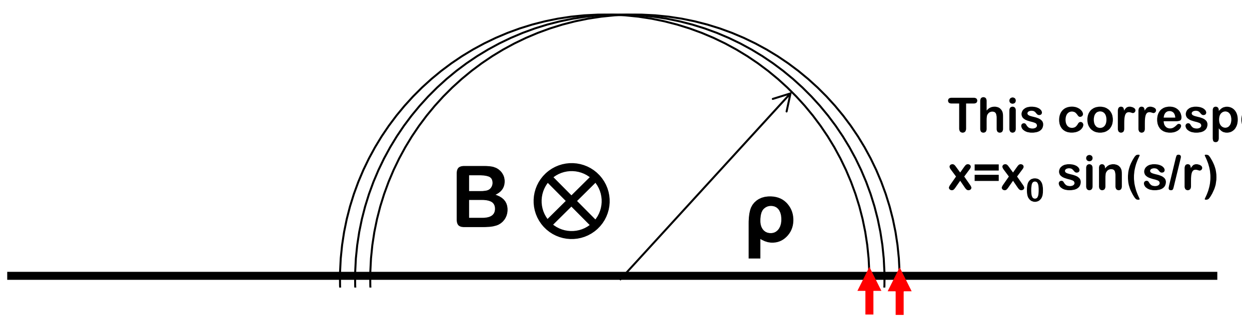
Origin of weak focusing in a dipole

Consider shifted circles



They cross

This is "focusing" with wavelength of motion 2pr



This correspond to $x=x_0$ sin(s/r)

Or to equation

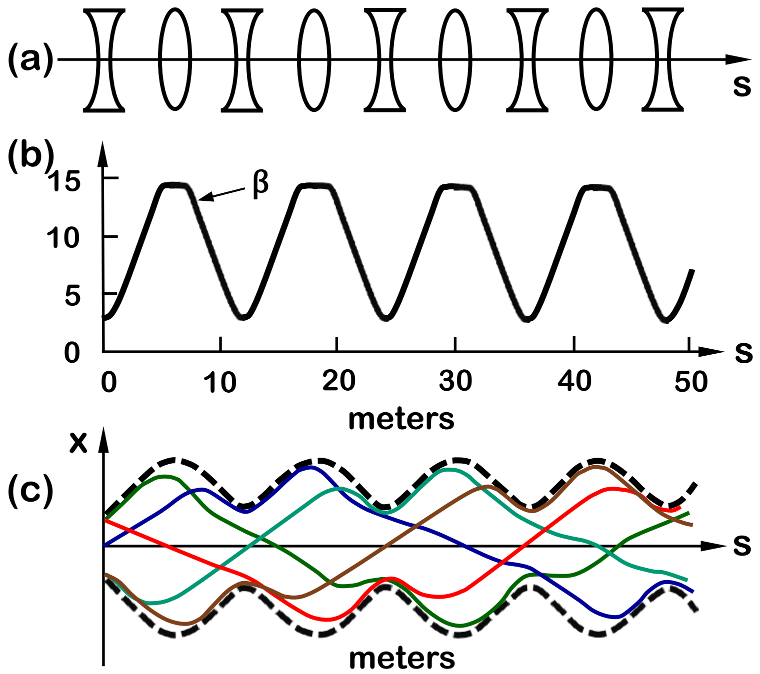
$$\frac{d^2x}{ds^2} + \frac{x}{\rho^2} = 0$$

Pseudo-harmonic oscillations

The solution can be found in the form

$$y(s) = \sqrt{\varepsilon_y \beta_y(s)} \cos\left[\varphi_y(s) - \varphi\right] \qquad \varphi_y(s) = \int_{s_0}^{s} \frac{ds'}{\beta_y(s')}$$

which are ***pseudo-harmonic oscillations***



The beta functions (in x and z) are proportional to the square of the envelope of the oscillations

The functions φ (in x and z) describe the phase of the oscillations

Differential equation for the beta functions

Use this form

$$y(s) = \sqrt{\varepsilon_y \beta_y(s)} \cos\left[\varphi_y(s) - \varphi\right]$$

Prepare to substitute to Hill's equation:

$$y'(s) = \frac{\beta'(s)}{2}\sqrt{\frac{\varepsilon}{\beta(s)}}\cos(\varphi(s)-\varphi) - \varphi'(s)\sqrt{\varepsilon\beta(s)}\sin(\varphi(s)-\varphi)$$

$$y''(s) = \left[\frac{\beta''(s)}{2\sqrt{\beta(s)}} - \frac{\beta'^2(s)}{4\beta^{3/2}(s)} - \sqrt{\beta(s)}\varphi'^2(s)\right]\sqrt{\varepsilon}\cos(\varphi(s)-\varphi) - \left[\varphi''(s)\sqrt{\beta(s)} + \frac{\beta'(s)\varphi'(s)}{\sqrt{\beta(s)}}\right]\sqrt{\varepsilon}\sin(\varphi(s)-\varphi)$$

Substitute to Hill's equation and equating to zero the coefficients of sin and cos we have

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + k(s)\beta^2 = 1 \qquad \varphi'_y(s) = \frac{1}{\beta_y(s)}$$

(We call –b'/2 = a)

Principal trajectories

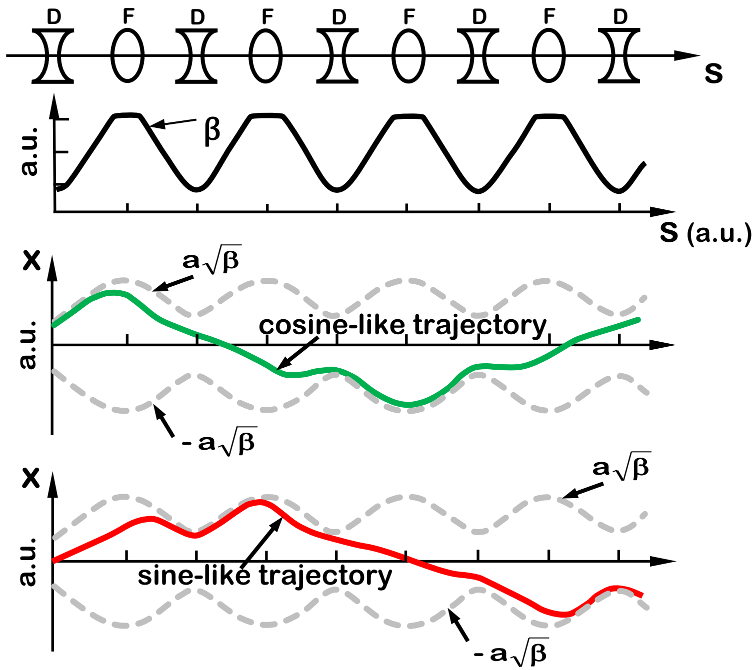
The solutions of the Hill's equation can be cast equivalently in the form of *principal trajectories*. These are two particular solutions of the homogeneous Hill's equation

$$y''+k(s)y=0$$

which satisfy the initial conditions

$C(s_0) = 1; C'(s_0) = 0$; cosine-like solution

$S(s_0) = 0; S'(s_0) = 1$; sine-like solution



The general solution can be written as a linear combination of the principal trajectories

$$y(s)=y_0 C(s)+y'_0 S(s)$$

Principal trajectories vs pseudo harmonic oscillations

We can express amplitude and phase functions

$$\begin{cases} y(s) = \sqrt{\varepsilon\beta(s)}\cos\left(\varphi(s)-\phi\right) \\ y'(s) = -\sqrt{\dfrac{\varepsilon}{\beta(s)}}\left[\sin\left(\varphi(s)-\phi\right)+\alpha(s)\cos\left(\varphi(s)-\phi\right)\right] \end{cases}$$

in terms of the principal trajectories ⟶ $y(s) = y_0 C(s) + y'_0 S(s)$

Simple algebraic manipulations yield

$$C(s) = \sqrt{\frac{\beta(s)}{\beta_0}}(\cos\phi(s) + \alpha_0 \sin\phi(s)) \qquad S(s) = \sqrt{\beta(s)\beta_0}\sin\phi(s)$$

and viceversa

$$\varphi(s) = \operatorname{arctg}\frac{S(s)}{\beta_0 S(s) - \alpha_0 C(s)} \qquad \beta(s) = \frac{1}{\beta_0}\left\{\frac{S^2(s) + \left[\beta_0 S(s) - \alpha_0 C(s)\right]^2}{\beta_0 S(s) - \alpha_0 C(s)}\right\}$$

or more simply

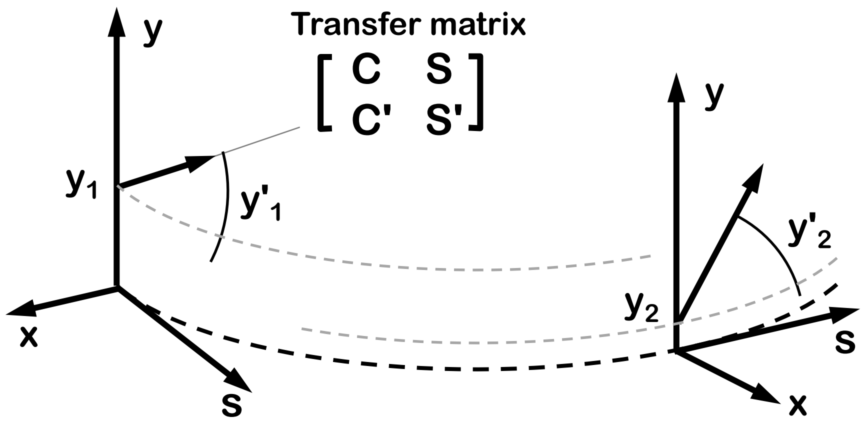
$$\beta(s) = \frac{1}{\beta_0}\left[\frac{S(s)}{\sin\varphi(s)}\right]^2 \qquad \alpha(s) = \frac{-S'(s)\sqrt{\dfrac{\beta(s)}{\beta_0}} + \cos\varphi(s)}{\sin\varphi(s)}$$

Principal trajectories

As a consequence of the linearity of Hill's equations, we can describe the evolution of the trajectories in a transfer line or in a circular ring by means of linear transformations

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

C(s) and S(s) depend only on the magnetic lattice not on the particular initial conditions



$$M_{1\to2} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}$$

This allows the possibility of using the matrix formalism to describe the evolution of the coordinates of a charged particles in a magnetic lattice

Matrices of most common elements

Transfer lines or circular accelerators are made of a series of drifts and quadrupoles for the transverse focusing and accelerating section for acceleration.

Each of these element can be associated to a particular transfer matrix

Matrix of a drift space

$$M = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

Matrix of a focussing quadurpole

$$M = \begin{pmatrix} \cos(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}L) \\ -\sqrt{|K|}\sin(\sqrt{|K|}L) & \cos(\sqrt{|K|}L) \end{pmatrix}$$

Thin lens approximation
$L \to 0$, with KL finite

$$M = \begin{pmatrix} 1 & 0 \\ -|K|L & 1 \end{pmatrix}$$

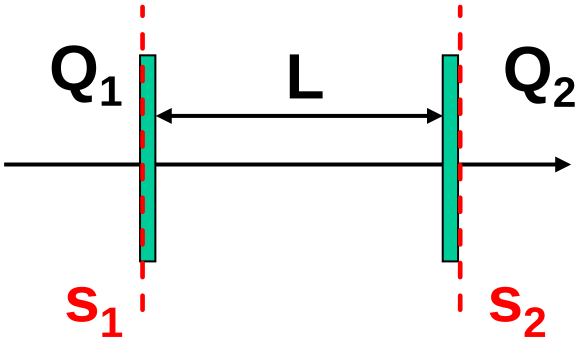
Matrix of a defocussing quadurpole

$$M = \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh(\sqrt{K}L) \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 0 \\ KL & 1 \end{pmatrix}$$

Matrix formalism for transfer lines

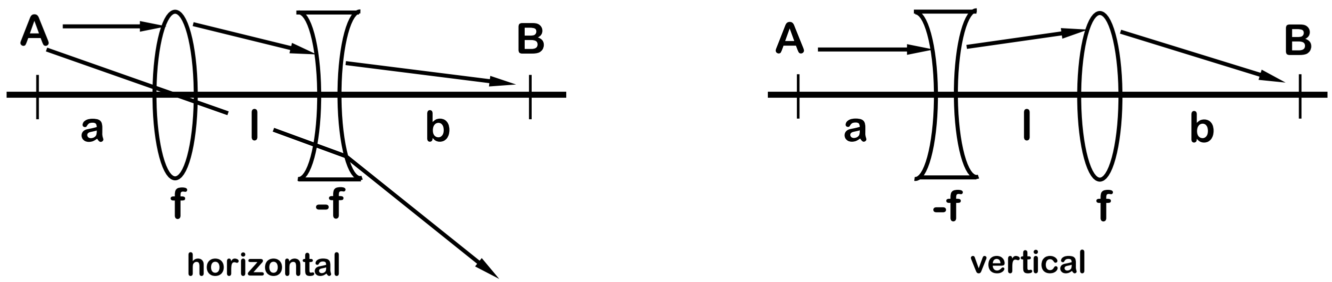
For each element of the transfer line we can compute, once and for all, the corresponding matrix. The propagation along the line will be the piece-wise composition of the propagation through all the various elements



$$M_{1\to 2} = \begin{pmatrix} 1 & 0 \\ -1/f_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-L/f_1 & L \\ -1/f^* & 1-L/f_2 \end{pmatrix} \qquad \frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$$

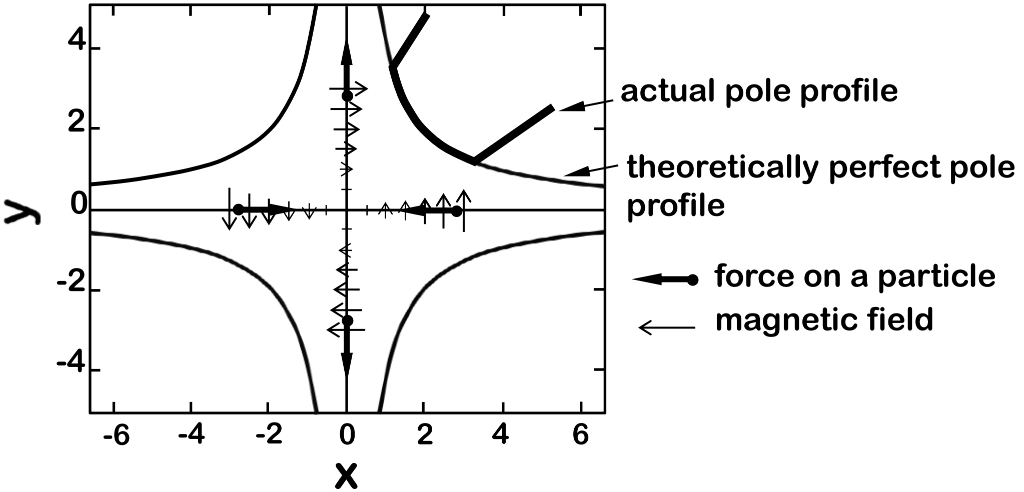
Matrix formalism and analogy with geometric optics

Particle trajectories can be described with a matrix formalism analogous to that describing the propagation of rays in an optical system



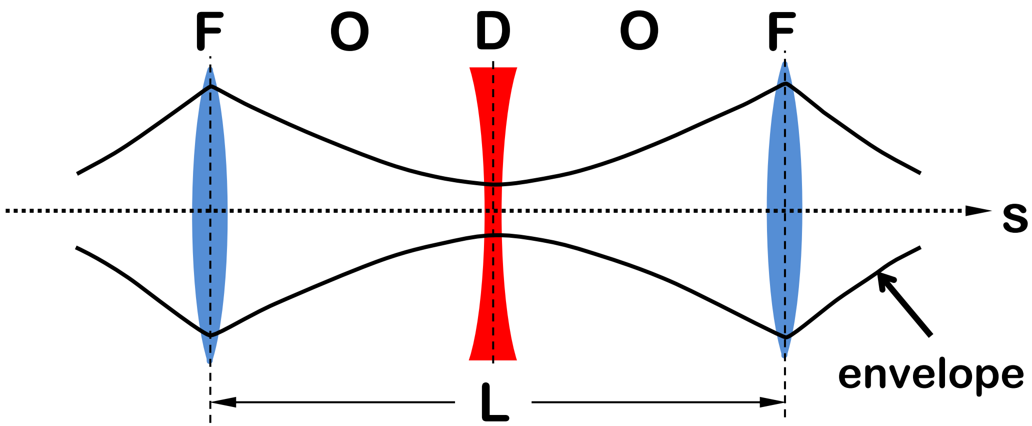
The magnetic quadrupoles play the role of focusing and defocussing lenses, however notice that, unlike an optical lens, a magnetic quadrupole is focusing in one plane and defocussing in the other plane

Magnetic field of a quadrupole and Lorentz force



An example: the FODO lattice (I)

Consider an alternating sequence of focusing (F) and defocusing (D) quadrupoles separated by a drift (O)



The transfer matrix of the basic FODO cell reads

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\frac{L}{2f} & L\left(1+\frac{L}{4f}\right) \\ -\frac{L}{2f^2} & 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{pmatrix}$$

Matrix elements from principal trajectories and optics functions

In terms of the amplitude and phase function the transfer matrix will read

$$M_{s_0\to s} = \begin{pmatrix} C(s) & S(s) \\ -C'(s) & S'(s) \end{pmatrix} = \begin{pmatrix} \sqrt{\dfrac{\beta(s)}{\beta_0}}(\cos\Delta\phi + \alpha_0 \sin\Delta\phi) & \sqrt{\beta(s)\beta_0}\sin\Delta\phi \\ -\dfrac{(\alpha(s)-\alpha_0)\cos\Delta\phi + (1+\alpha(s)\alpha_0)\sin\Delta\phi}{\sqrt{\beta(s)\beta_0}} & \sqrt{\dfrac{\beta_0}{\beta(s)}}\left[\cos\Delta\phi - \alpha(s)\sin\Delta\phi\right] \end{pmatrix}$$

where β_0 , α_0 and the phase ϕ_0 are computed at the beginning of the segment of transfer line

We still have not assumed any periodicity in the transfer line.

If we consider a periodic machine the transfer matrix over a whole turn reduces to (put $\mu = \Delta\phi$ the phase advance in one turn)

$$M_{s_0\to s_0} = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix} \qquad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0}$$

This is the ***Twiss parameterization of the one turn map***

Stability of motion with the matrix formalism

Consider a circular accelerator with transfer matrix over one turn equal to M (one turn map). Using the Twiss parameterization for M

$$M = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu & \cos\mu - \alpha_0 \sin\mu \end{pmatrix} = \cos\mu \cdot I + \sin\mu \cdot J \qquad J = \begin{pmatrix} \alpha_0 & \beta_0 \\ -\gamma_0 & -\alpha_0 \end{pmatrix}$$

After n turns, the transformation of the particle coordinates will be given by the successive application of the one turn matrix n times

$$\overline{x}_1 = M\overline{x}_0 \qquad \overline{x}_n = M^n \overline{x}_0$$

In order for the phase advance μ to be real and hence for the motion to be a *stable oscillation*, the one turn map must satisfy the condition

$$|\cos\mu| = \frac{1}{2}|trM| < 1$$

It can be proven that

$$M^n = \begin{pmatrix} \cos n\mu + \alpha_0 \sin n\mu & \beta_0 \sin n\mu \\ -\gamma_0 \sin n\mu & \cos n\mu - \alpha_0 \sin n\mu \end{pmatrix}$$

Example: the FODO lattice (II)

Using the Twiss parameterization of the matrix or the FODO cell we have

$$M = \begin{pmatrix} 1+\frac{L}{2f} & L\left(1+\frac{L}{4f}\right) \\ -\frac{L}{2f^2} & 1-\frac{L}{2f}-\frac{L^2}{4f^2} \end{pmatrix} = \begin{pmatrix} \cos\mu+\alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu \end{pmatrix}$$

hence

$$\cos\mu = \frac{1}{2}\mathrm{tr}M = 1-\frac{L^2}{8f^2}$$

The stability requires

$$|\cos\mu| = \left|1-\frac{L^2}{8f^2}\right| < 1 \longrightarrow f > \frac{L}{4}$$

In a similar way we can compute the optics functions at the beginning of the FODO cell.

Optics functions in a transfer line

While in a circular machine the optics functions are uniquely determined by the periodicity conditions, in a transfer line the optics functions are not uniquely given, but depend on their initial value at the entrance of the system.

We can express the optics function in terms of the principal trajectories as

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS'+SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}$$

This expression allows the computation of the propagation of the optics function along the transfer lines, in terms of the matrices of the transfer line of each single element, i.e. also the optics functions can be propagated piecewise from

$$M_{1\to 2} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}$$

Examples

In a drift space

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}=\begin{pmatrix}1 & -2s & s^2\\ 0 & 1 & -s\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}$$

$$M=\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}$$

The β function evolves like a parabola as a function of the drift length.

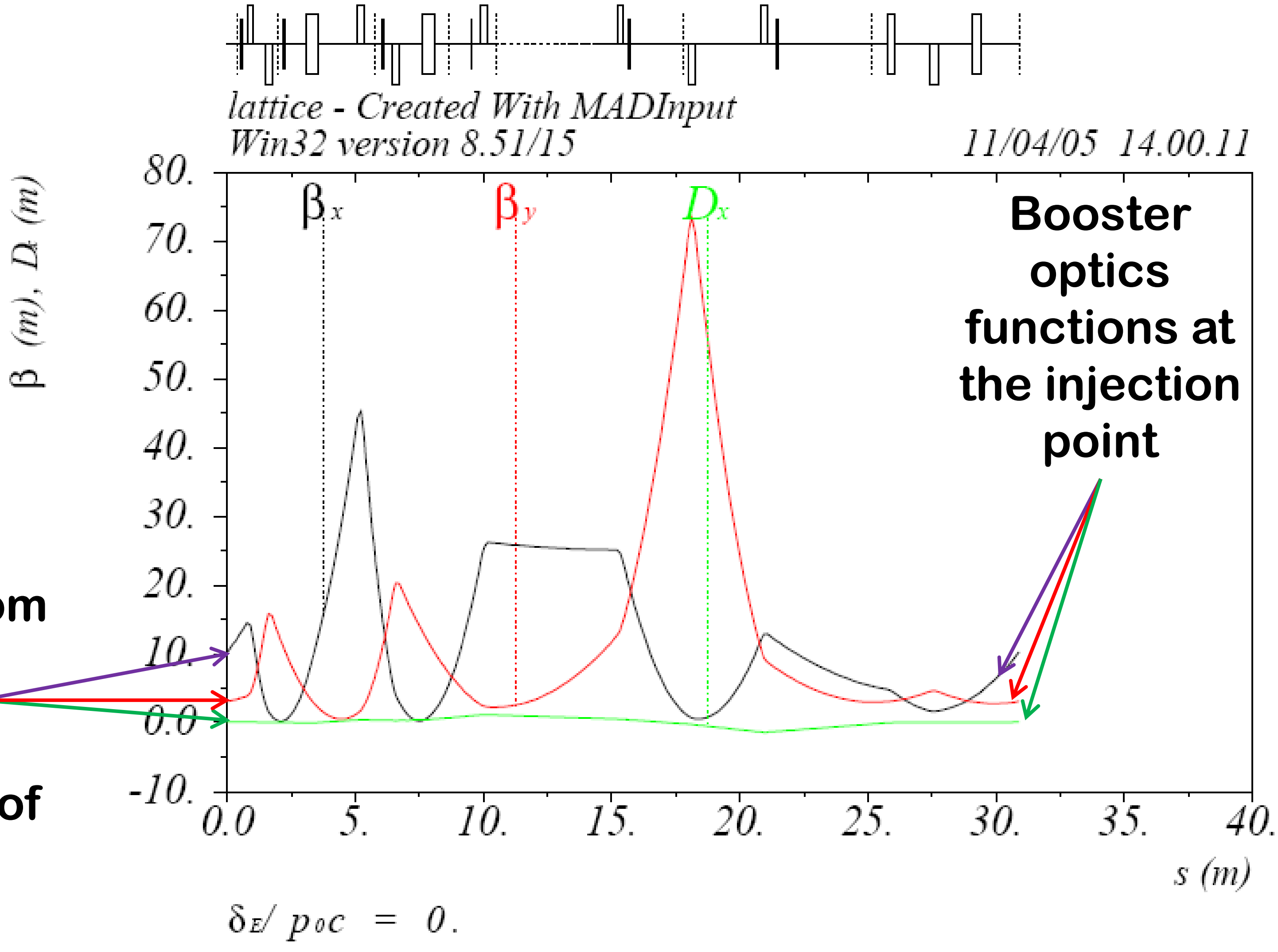
In a thin focusing quadrupole of focal length f = 1/KL

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ KL & 1 & 0\\ (KL)^2 & 2KL & 1\end{pmatrix}\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix}$$

$$M=\begin{pmatrix}1 & 0\\ KL & 1\end{pmatrix}$$

The γ function evolve like a parabola in terms of the inverse of focal length

Diamond LINAC to booster transfer line

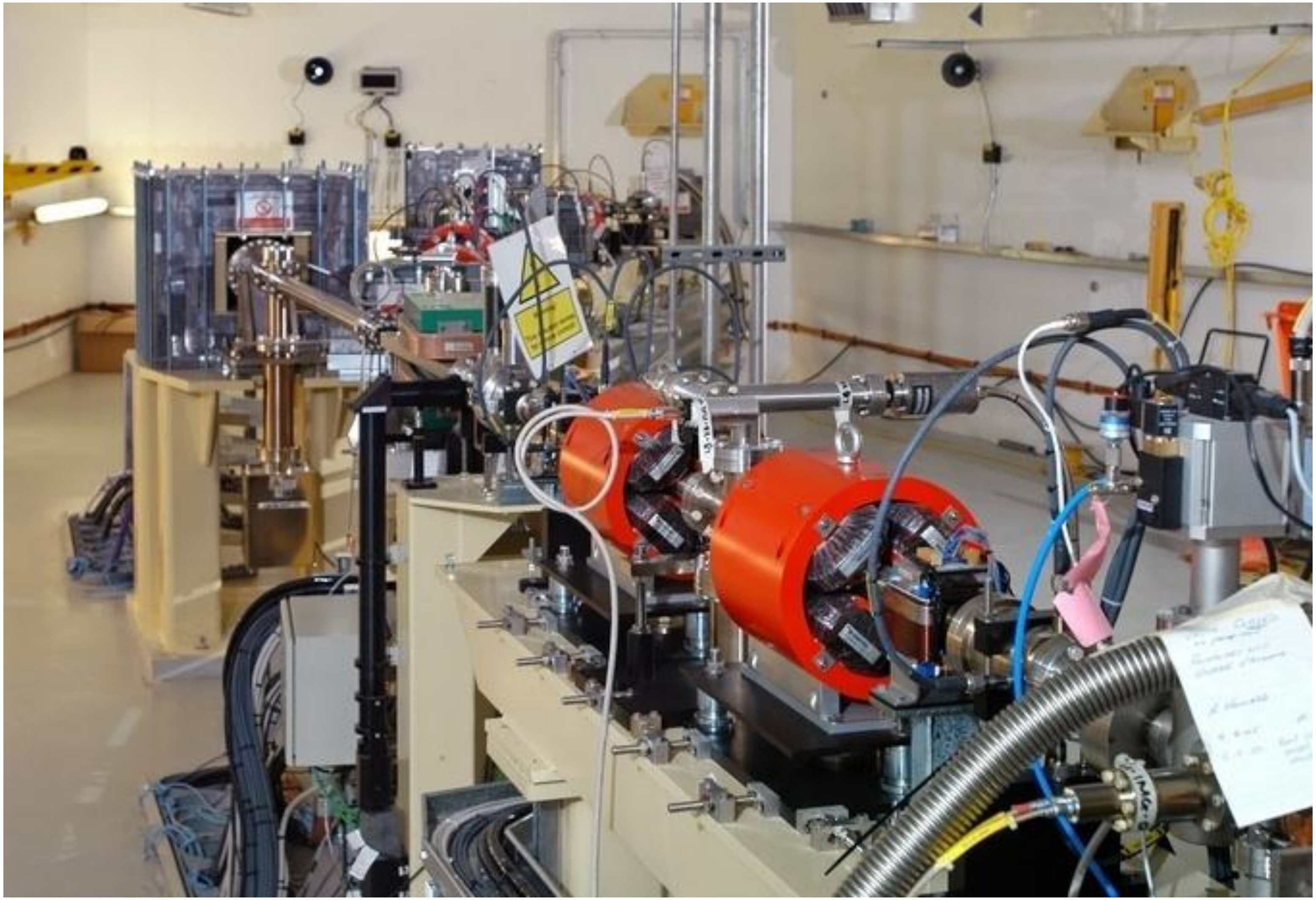


Optics functions from the LINAC

(Twiss parameters of the beam)

Booster optics functions at the injection point

Transfer line example: Diamond LTB



Betatron motion in phase space (recap)

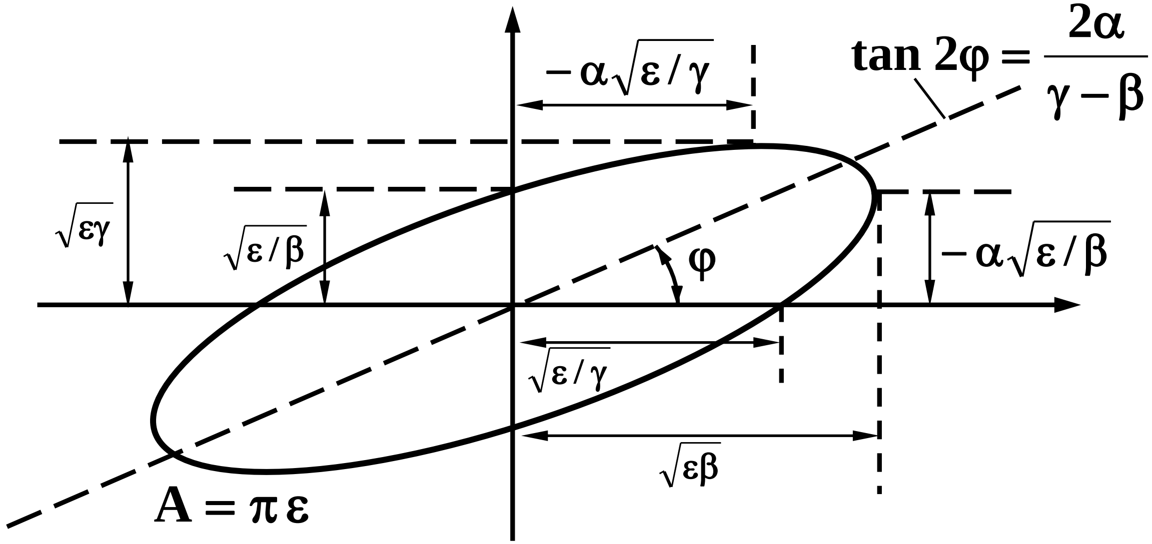
The solution of the Hill's equations

$$\frac{d^2y}{ds^2} + K_y(s)y = 0$$

$$y(s) = \sqrt{\varepsilon\beta(s)}\cos\left(\varphi(s) - \phi\right)$$

$$y'(s) = -\sqrt{\frac{\varepsilon}{\beta(s)}}\left[\sin\left(\varphi(s)-\phi\right) + \alpha(s)\cos\left(\varphi(s)-\phi\right)\right]$$

describe an ellipse in phase space (y, y')



area of the ellipse in phase space (y, y') is $A(s) = (\beta y'^2 + 2\alpha y y' + \gamma^2 y^2)/\pi$

Courant-Snyder invariant (I)

Hill's equations have an invariant

$$\frac{d^2 y}{ds^2} + K_y(s)y = 0 \longrightarrow \mathrm{A(s)} = \beta\, \mathrm{y'}^2 + 2\alpha \mathrm{yy'} + \gamma^2 \mathrm{y}^2 = \mathrm{const.}$$

This invariant is the area of the ellipse in phase space (y, y') multiplied by π.

This can be easily proven by substituting the solutions **y, y'**

$$\mathrm{y(s)} = \sqrt{\varepsilon\beta(\mathrm{s})}\cos\left(\varphi(\mathrm{s}) - \phi\right)$$

$$\mathrm{y'(s)} = -\sqrt{\frac{\varepsilon}{\beta(\mathrm{s})}}\left[\sin\left(\varphi(\mathrm{s}) - \phi\right) + \alpha(\mathrm{s})\cos\left(\varphi(\mathrm{s}) - \phi\right)\right]$$

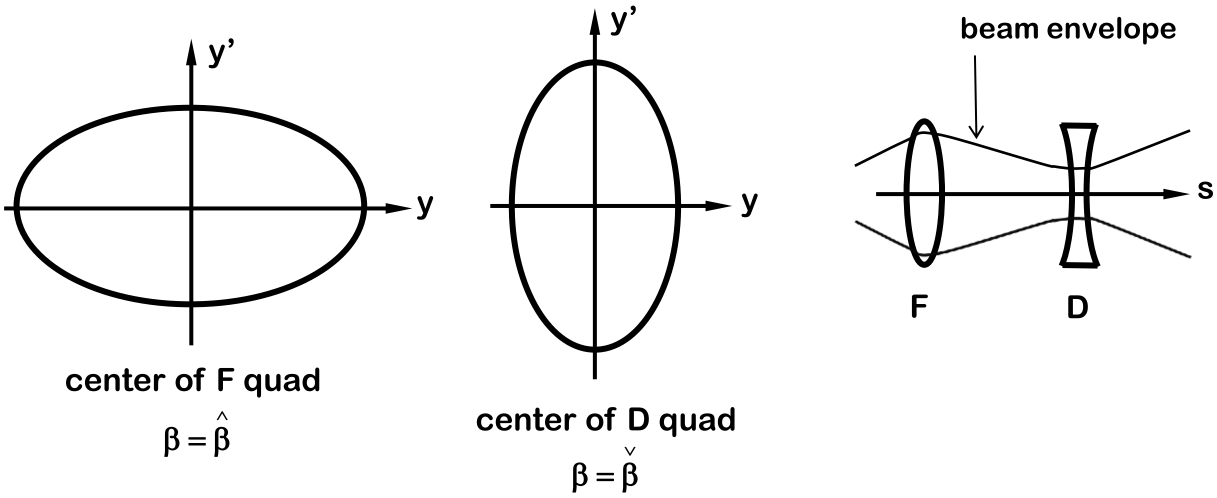
into A(s). You will get the constant ε

A(s) is called *Courant-Snyder invariant*

Courant-Snyder invariant (II)

Whatever the magnetic lattice, the area of the ellipse stays constant (if the Hill's equations hold)

At each different sections s, the ellipse of the trajectories may change orientation shape and size but the area is an invariant.



This is true for the motion of a single particle !

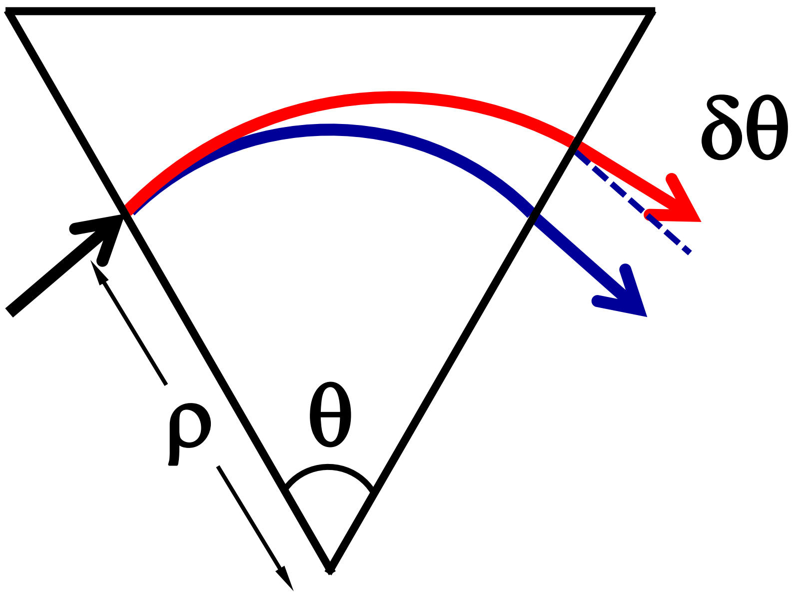
Dispersion

Another basic function used in the characterization of charge particle motion linear accelerator is the so called dispersion function.

The dispersion function describes the orbit of an off energy particle.

The Hill's equation for an off energy particle reads

$$\frac{d^2x}{ds^2} - \left(k_1(s) - \frac{1}{\rho^2(s)} \right) x = \frac{1}{\rho(s)} \frac{\Delta p}{p_0}$$



obtained remembering that the radius of curvature of the trajectory for an off energy particle is

$$\frac{1}{\rho(s)} = \frac{eB}{p} = \frac{eB}{p_0\left(1+\frac{\Delta p}{p_0}\right)} = \frac{1}{\rho_0(s)}\left(1-\frac{\Delta p}{p_0}\right)$$

Dispersion

The solution for the closed orbit can be searched as

x = x0 + dp/p * D

Where D is the dispersion function and x0 describes betatron oscillation around the dispersive orbit

D can be expressed in terms of the principal trajectories as

$$D(s) = S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt$$

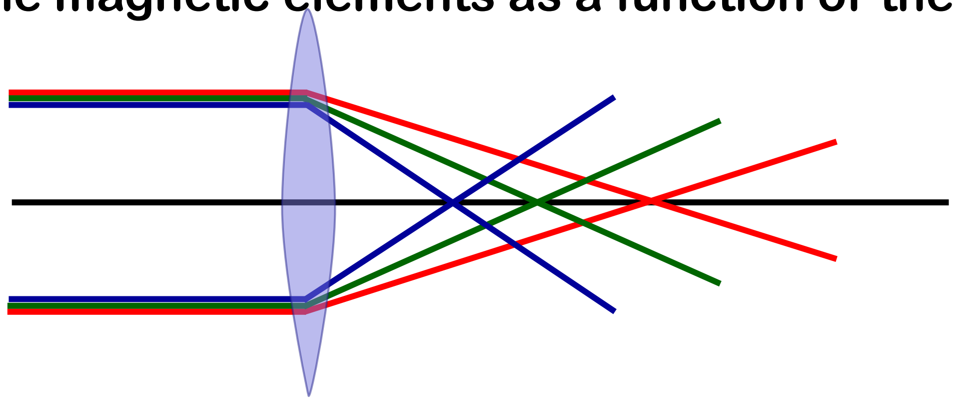
$$D'(s) = S'(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C'(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt$$

The length of the dispersive orbit can be computed as a function of the dispersion function. We also can define the so called momentum compaction factor as

$$\alpha_c = \frac{dC/C}{dp/p} = \frac{1}{C}\oint\frac{D(s)}{\rho(s)}ds$$

Chromaticity

Chromaticity describes the dependence of the betatron tune with the energy deviation of the particle. This is due to the different focussing strength of the magnetic elements as a function of the energy



Expanding the quadrupole strength as in

$$k_1 = \frac{e}{p}\frac{\partial B_y}{\partial x} = \frac{e}{p_0(1+\delta)}\frac{\partial B_y}{\partial x} = \frac{k_0}{1+\delta} \approx k_0(1-\delta+\delta^2-\ldots)$$

The betatron tunes change according to $\Delta Q = \frac{1}{4\pi}\oint \beta(s)\Delta K(s)ds$

The chromaticity is the derivative of the betatron tunes wrt to the relative energy change

$$Q' = \frac{dQ}{d\delta} = -\frac{1}{4\pi}\oint \beta_x(s)k_0(s)ds$$

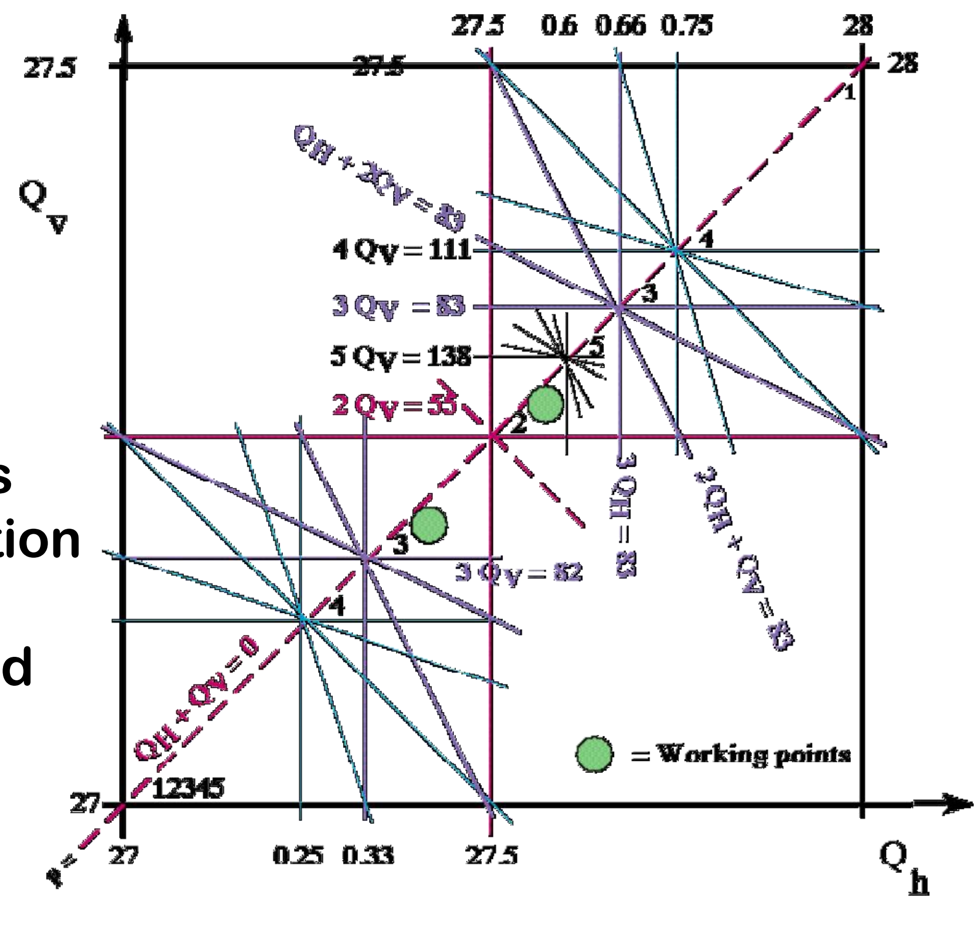
Resonances

Resonance conditions between the betatron tunes such as

$$m\,Q_x + n\,Q_z = p$$

$$|m| + |n|$$

is called the order of the resonance



Must be carefully avoided

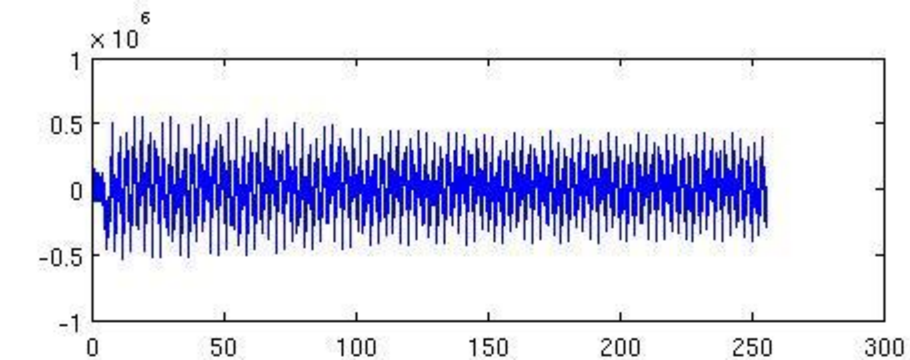
They can be potentially dangerous for the stability of the particle motion

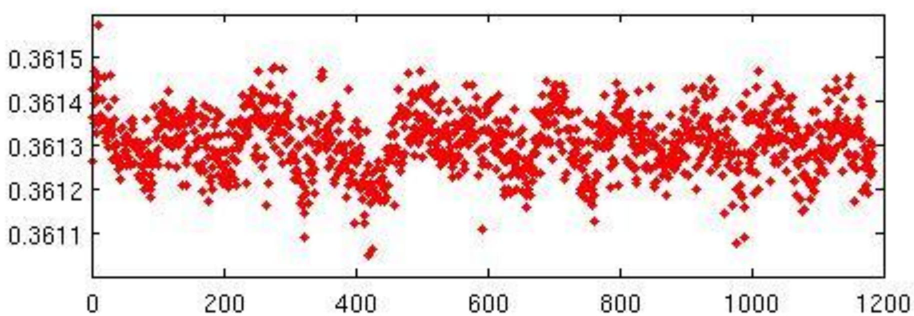
The analysis of the stability beyond the linear motion is a field where accelerator physics meets astronomy and galactic dynamics

Betatron tune measurements

A example of betatron oscillations recorded after a kick in the vertical plane at diamond.

256 turns are recorded: the time signals of many kicks is superimposed to check the reproducibility of the kick and of the oscillations, small variation in the betatron tunes are detected (2e-4).





The frequency corresponding to the peak of the amplitude of the FFT is the betatron tune

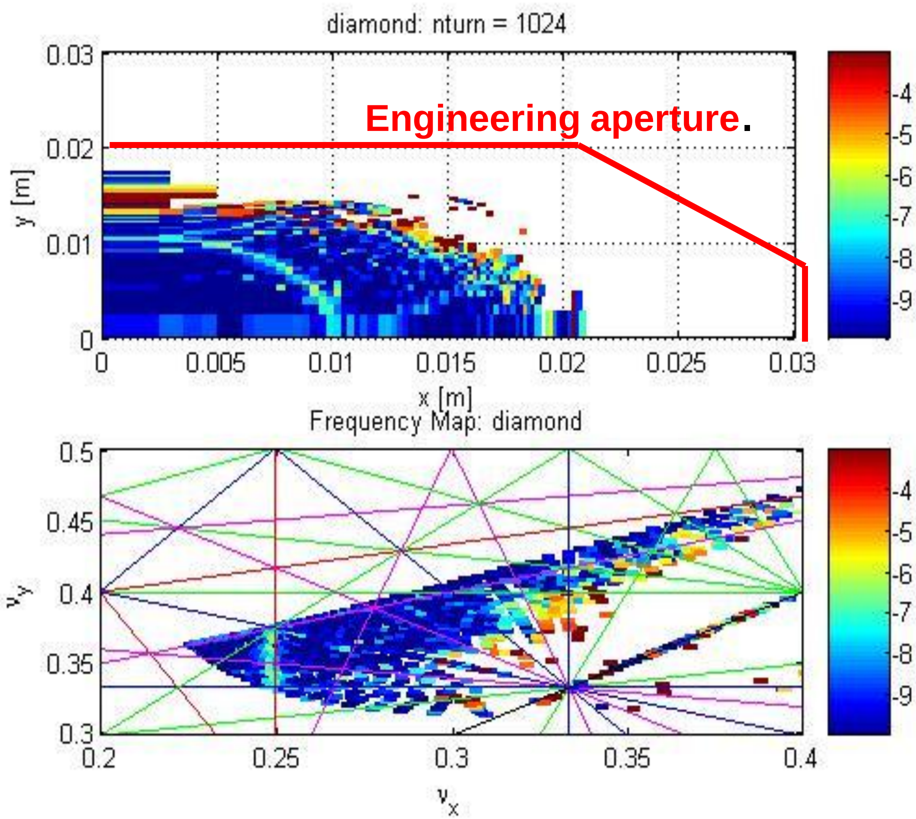
It can be measured form Beam Position Monitors (BPMs)

Example from Diamond: notice the ripple in the tune measurement due change in quadrupole terms in Hills' equations (vibrations – power supplies- ...)

Frequency Map Analysis

The Frequency Map Analysis is a technique introduced in Accelerator Physics form Celestial Mechanics (Laskar).

It allows the identification of dangerous non linear resonances during design and operation. Strongly excited resonances can destroy the Dynamic Aperture.



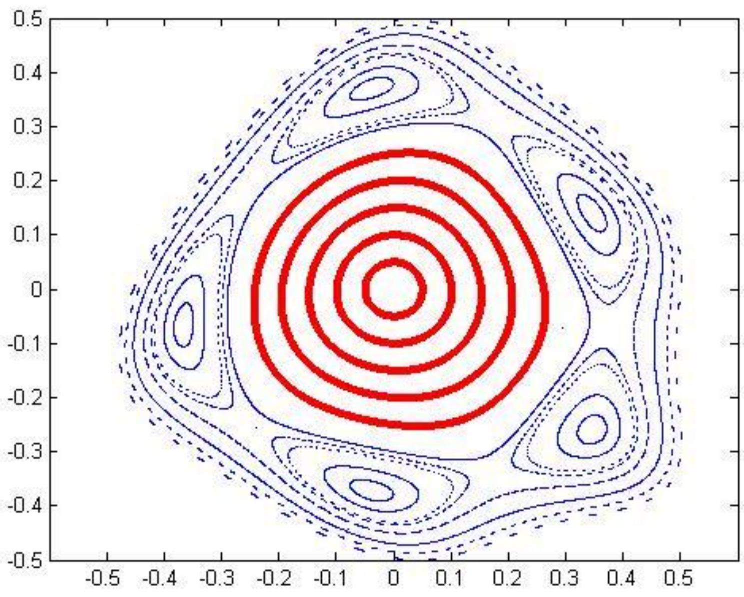
To each point in the (x, y) aperture there corresponds a point in the (Q_x, Q_y) plane

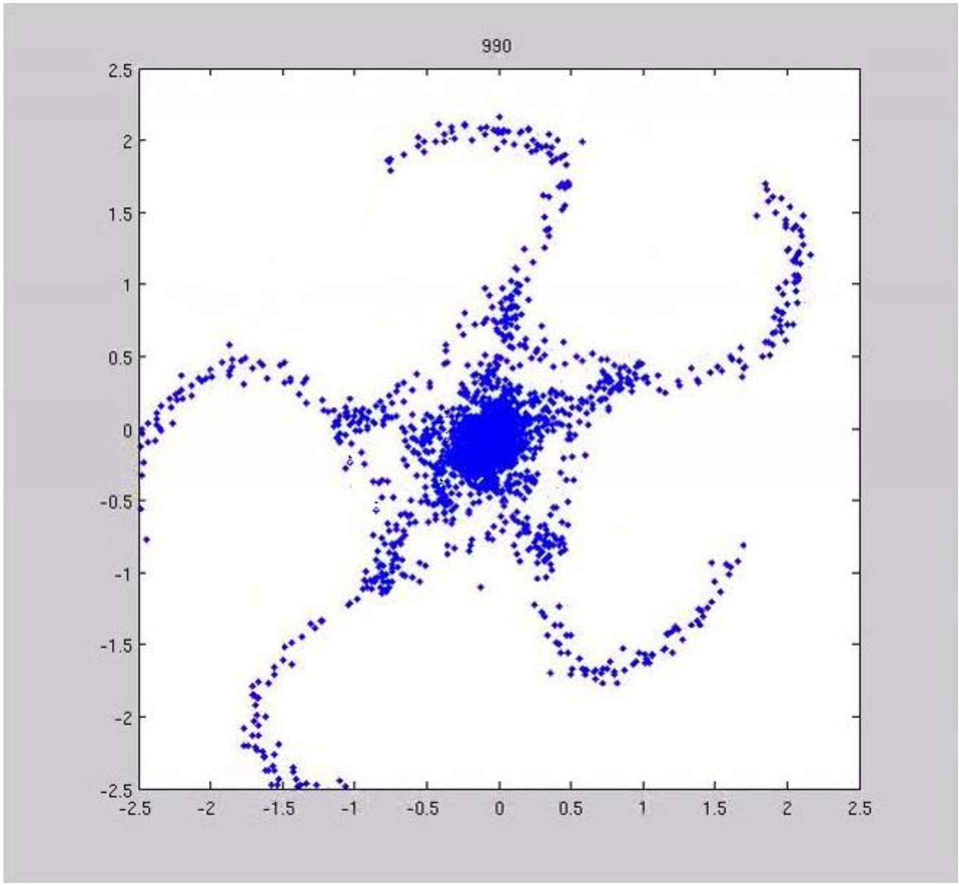
The colour code gives a measure of the stability of the particle (blue = stable; red = unstable)

The indicator for the stability is given by the variation of the betatron tune during the evolution: i.e. tracking N turns we compute the tune from the first N/2 and the second N/2

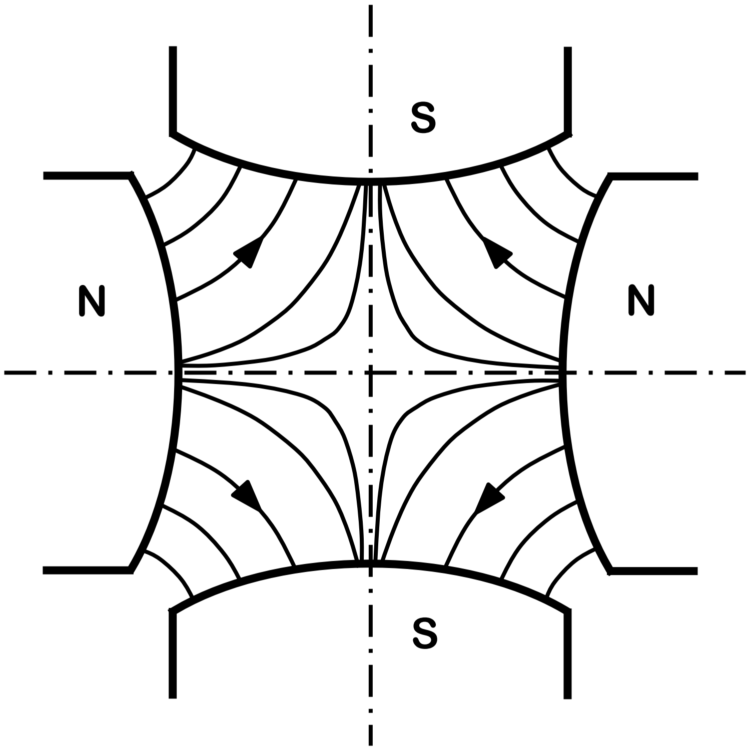
Chaotic motion beyond the linear approximation

Two examples of irregular and chaotic motion from simulations and from real measurement at the Diamond accelerator, showing a highly nonlinear motion with resonant islands and chaotic layers.

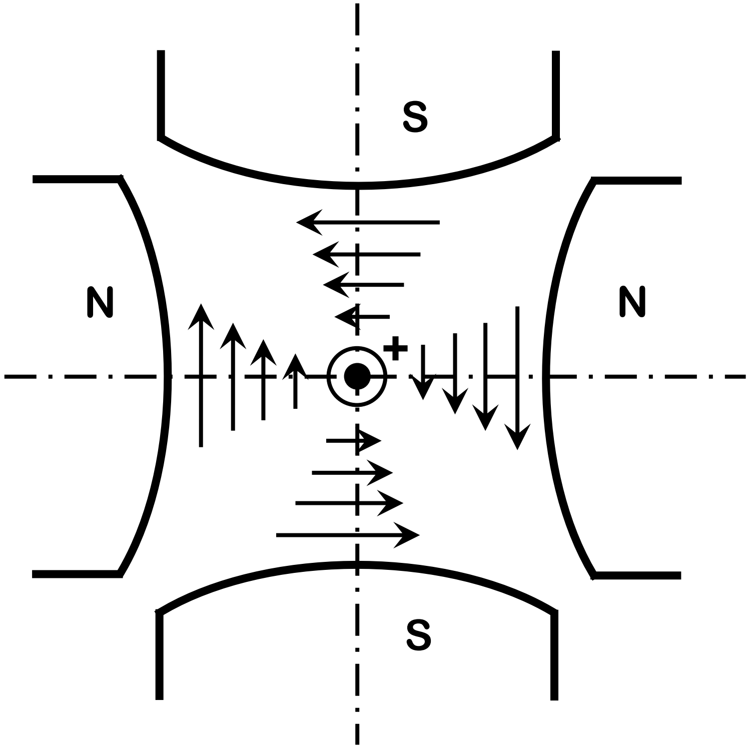




Coupling



Skew quadrupole field



Skew quadrupole forces

A standard way to create (or correct) coupling between x and y planes is to use skew quadrupoles

Other sources of coupling – solenoid (especially when solenoid overlaps with quadrupole field), offset of sextupoles, etc

Summary of the lecture

- In this lecture we discussed
  - Basics of beam dynamics (transverse)