Lecture 10:

Coherent Synchrotron Radiation

Yunhai Cai
SLAC National Accelerator Laboratory

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1D CSR Wakefield in Free Space

Wakefield due to CSR is given by

\[ W(z) = \frac{2}{3^{1/3}} \frac{\partial}{\partial z} \rho^{2/3} z^{-1/3} \]

For \( z > 0 \). It vanishes when \( z < 0 \) (force is on the particle ahead).

where \( \rho \) is the bending radius.

- Simplicity
- Universal
- In form of derivative
CSR Microbunching in Bunch Compressors

- CSR limits further improvement of longitudinal emittance and limits peak beam current below 3 kA
- 1D model results are in good agreement with data, as shown in the following BC1 examples
- 3D model may be necessary at much higher peak current

![Graph of CSR energy loss after BC1 measured with BPM](image1)

![Graph of horizontal emittance after BC1 vs. RF phase](image2)

Courtesy of Yuantao Ding
**CSR Instability in Electron Storage Rings**


Measured bursting threshold at ANKA
See M. Klein *et al*. PAC09, p4761 (2009)

Figure 1: MLS THz signals at the bursting threshold. Vertical axis: applied rf-voltage amplitude, horizontal axis: frequency of the detected THz signals. The colour indicates the THz signal intensity.

Scaling law for bunched beam:

\[
\sigma_z^{7/3} = \frac{c^2 Z_0}{8\pi^2} \xi^{th} (\chi) I_b^{th} \rho^{1/3} \left/ (V_{rf} \cos \phi_s f_{rf} f_{rev}) \right., \quad \xi^{th} (\chi) = 0.5 + 0.34 \chi, \quad \text{and} \quad \chi = \sigma_z \rho^{1/2}/h^{3/2}
\]

K. Bane, Y. Cai, and G. Stupakov, PRSTAB 13, 104402 (2010)
Transverse Force in Curved Geometry

Equation of motion:

\[ x'' + \frac{x}{\rho^2} = \frac{\delta}{\rho} + \frac{e}{c p_0 \beta_s} [E_x + \beta_y B_s - \beta_s (1 + \frac{x}{\rho}) B_y], \]
\[ y'' = \frac{e}{c p_0 \beta_s} [E_y + \beta_s (1 + \frac{x}{\rho}) B_x - \beta_x B_y] \]

- Curvature terms are conceptually important
- \( E_x, E_y, B_x, B_y, \) and \( B_s \) are the self-fields
- No explicit dependence on the potentials
- Equations are derived from the Hamiltonian by Courant-Synder
Lienard-Wiechert Formula

Space Charge

Radiated Field

\[ \vec{E} = e\left[\frac{\vec{n} - \vec{\beta}}{\gamma^2(1 - \vec{n} \cdot \vec{\beta})^3 R^2}\right]_{\text{ret}} + \left(\frac{e}{c}\right)\left[\frac{\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{\beta}}{(1 - \vec{n} \cdot \vec{\beta})^3 R}\right]_{\text{ret}}, \]

\[ \vec{B} = \vec{n} \times \vec{E} \]

- Space charge is suppressed by \(1/\gamma^2\)
- Identify radiated field with CSR
- Subject to retarded condition:

\[ t' = t - \frac{R}{c} \]
Electrical and Magnetic Fields

\[ E_s = \frac{e\beta^2 [\cos 2\alpha - (1 + \chi)][(1 + \chi)\sin 2\alpha - \beta\kappa]}{\rho^2 [\kappa - \beta(1 + \chi)\sin 2\alpha]^3} \]

\[ E_x = \frac{e\beta^2 \sin 2\alpha[(1 + \chi)\sin 2\alpha - \beta\kappa]}{\rho^2 [\kappa - \beta(1 + \chi)\sin 2\alpha]^3} \]

\[ B_y = \frac{e\beta^2 \kappa[(1 + \chi)\sin 2\alpha - \beta\kappa]}{\rho^2 [\kappa - \beta(1 + \chi)\sin 2\alpha]^3} \]

where

\[ \kappa = \frac{R}{\rho} = \sqrt{\chi^2 + 4(1 + \chi)\sin^2 \alpha}, \]

\[ \alpha = \theta / 2, \]

\[ \chi = x / \rho \]

- They are simplest expressions, especially in the denominator and chosen to suppress the numerical noise near the singularity.
Retarded Time and Longitudinal Position

Retarded Time:

\[ t' = t - \frac{R}{c} \]

Time of flight at position \( s \):

\[ \ell = v(t - t') - (s - s') \]

It is the variable for the wake. The arc distance to the source particle at the time \( t \). We derive its relation to \( \alpha \).

\[ \xi = \alpha - \frac{\beta}{2} \sqrt{x^2 + 4(1 + x)\sin^2 \alpha} \]

where \( \xi = -\ell/2\rho \) and \( \ell = z' - z \).
Solutions of the Retarded Condition

Expanding up to the fourth-order of $\alpha$ of the retarded condition, we have

$$\alpha^4 + \frac{3(1 - \beta^2 - \beta^2 \chi)}{\beta^2 (1 + \chi)} \alpha^2 - \frac{6\xi}{\beta^2 (1 + \chi)} \alpha + \frac{3(4\xi^2 - \beta^2 \chi^2)}{4\beta^2 (1 + \chi)} = 0$$

Numerical solution is on mesh: 512x512 using Mathematica taking several hours. The differences between the numeric and analytic solutions are at an order of $10^{-6}$. Here we have used $\gamma=500$. 

Yunhai Cai, SLAC 5/31/17
Analytical Solution of the Retarded Condition

In general, we want to find the roots of the depressed quartic equation:

\[ \alpha^4 + \nu \alpha^2 + \eta \alpha + \zeta = 0 \]

It has analytical solution discovered by Ferrari (1522-1565) by adding and subtracting a term to make a difference of two perfect squares. To find the term, we need to first find the roots of a third-order equation. A root \( m \) is given by,

\[
m = -\frac{\nu}{3} + \left( \frac{\xi}{3} + \frac{\nu^2}{36} \right) \Omega^{-1/3} + \Omega^{1/3}
\]

where

\[
\Omega = \frac{\eta^2}{16} - \frac{\xi \nu}{6} + \frac{\nu^3}{216} + \sqrt{\left( \frac{\eta^2}{16} - \frac{\xi \nu}{6} + \frac{\nu^3}{216} \right)^2 - \left( \frac{\xi}{3} + \frac{\nu^2}{36} \right)^3}
\]

The solution of \( \alpha \):

\[
\alpha = \begin{cases} 
\frac{1}{2} \left( \sqrt{2m + \sqrt{-2(m + \nu) - \frac{2\eta}{\sqrt{2m}}} \right) & \xi \geq 0 \\
\frac{1}{2} \left( -\sqrt{2m + \sqrt{-2(m + \nu) + \frac{2\eta}{\sqrt{2m}}} \right) & \xi < 0
\end{cases}
\]
The scaling with respect to $\gamma$ is different. Here we have used $\gamma=500$.

The centrifugal force is much hard to compute numerically because of the cancellation between the electric and magnetic forces.
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A Longitudinal Potential $\Psi_s$

Differentiate the retarded condition,

$$\xi = \alpha - \frac{\beta}{2} \sqrt{\chi^2 + 4(1 + \chi)\sin^2 \alpha}$$

We have,

$$d\xi = (1 - \frac{\beta(1 + \chi)\sin 2\alpha}{\sqrt{\chi^2 + 4(1 + \chi)\sin^2 \alpha}})d\alpha$$

Combining it with the longitudinal electric field $E_s$, we find

$$E_s d\xi = \frac{e\beta^2[\cos 2\alpha - (1 + \chi)][(1 + \chi)\sin 2\alpha - \beta \kappa]}{\rho^2 \kappa [\kappa - \beta(1 + \chi)\sin 2\alpha]^2} d\alpha$$

or

$$E_s = \frac{\partial \psi_s}{\partial \xi}$$

where

$$\psi_s(\xi, \chi) = \frac{e\beta^2(\cos 2\alpha - \frac{1}{1 + \chi})}{2\rho^2 [\kappa - \beta(1 + \chi)\sin 2\alpha]}$$
Transverse Force and Potential $\Psi_x$

Similarly,

$$\psi_x(\xi, \chi) = \frac{e^2 \beta^2}{2 \rho^2} \left\{ \frac{1}{|\chi| (1 + \chi)} \left[ (2 + 2 \chi + \chi^2) F(\alpha, \frac{-4(1 + \chi)}{\chi^2}) - \chi^2 E(\alpha, \frac{-4(1 + \chi)}{\chi^2}) \right] \right. $$

$$ + \left. \frac{\kappa^2 - 2 \beta^2 (1 + \chi)^2 + \beta^2 (1 + \chi)(2 + 2 \chi + \chi^2) \cos 2\alpha - \kappa \beta (1 + \chi) \sin 2\alpha [1 - \beta^2 (1 + \chi) \cos 2\alpha]}{[\kappa^2 - \beta^2 (1 + \chi)^2 \sin^2 2\alpha]} \right\},$$

where,

$$F_x = \frac{\partial \psi_x}{\partial \xi}$$

Curvature term

and,

$$F_x = \frac{e \beta^2 [\sin 2\alpha - (1 + \chi) \beta \kappa] [(1 + \chi) \sin 2\alpha - \beta \kappa]}{\rho^2 [\kappa - \beta (1 + \chi) \sin 2\alpha]^3}$$

• The Transverse force is the Lorentz force and plus the curvature term
• The curvature term is necessary for the analytical expression
• $F(\alpha, k)$ and $E(\alpha, k)$ are the incomplete elliptic integrals of the first and second kind
• $F_y = 0$, so the particles stay in the plane if they are initially in the horizontal plane
The scaling with respect to $\gamma$ is different. Here we have used $\gamma=500$.

The “logarithmic” singularity is clearly seen in the transverse potential along the line of $\chi=0$. 
Wakefields

From the equations of the motion, the changes of the momentum deviation and kick are given by,

\[ \delta' = \frac{r_e N_b}{\gamma} W_s(z, \chi), \]
\[ \chi'' = \frac{r_e N_b}{\gamma} W_x(z, \chi) \]

where \( r_e \) is the classical electron radius, \( N_b \) the bunch population, and the wakes,

\[ W_s(z, \chi) = \int \int Y_s \left( \frac{z - z'}{2\rho}, \chi - \chi' \right) \frac{\partial \lambda_b(z', \chi')}{\partial z'} dz' d\chi', \]
\[ W_x(z, \chi) = \int \int Y_x \left( \frac{z - z'}{2\rho}, \chi - \chi' \right) \frac{\partial \lambda_b(z', \chi')}{\partial z'} dz' d\chi', \]

with \( Y_s = 2\rho \Psi_s/(e\beta^2) \), \( Y_x = 2\rho \Psi_x/(e\beta)^2 \) and \( \lambda_b \) is the normalized distribution.

• These are additional changes when integrating through the bend.
\[ \rho = 1 \text{ m}, \; \gamma = 500, \; \sigma_x = \sigma_z = 10 \mu \text{m}, \; \Lambda = \frac{11}{24} \gamma E - 4 + \ln\left(\frac{2 \rho^2}{\sigma_x^2}\right) + \frac{13}{24} \ln\left(\frac{\sigma_z^2}{2 \rho^2}\right) \]
### Estimate of Emittance Growth

Increase of the projected emittance:

\[
\Delta \varepsilon_N = \frac{1}{2} \gamma \beta_x < (\Delta x' - <\Delta x'>)^2 > ,
\]

From the longitudinal contribution a bending magnet:

\[
\Delta \varepsilon_N = 7.5 \times 10^{-3} \frac{\beta_x}{\gamma} \left( \frac{N_b r_e L_B^2}{\rho^2} \right)^2 ,
\]

It leads to 38\% increase of the emittance for the last dipole. From the centrifugal force, we have

\[
\Delta \varepsilon_N = \frac{(-3 + 2\sqrt{3})}{24\pi} \frac{\beta_x}{\gamma} \left( \frac{\Lambda N_b r_e L_B}{\rho \sigma_z} \right)^2 ,
\]

This gives 29\% increase of the emittance.

The parameters for the last bend of BC2 in LCLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>$\gamma$</th>
<th>$\varepsilon_N$</th>
<th>$\sigma_z$</th>
<th>$N_b$</th>
<th>$\beta_x$</th>
<th>$\rho$</th>
<th>$L_B$</th>
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<td>10 $\mu$m</td>
<td>$10^9$</td>
<td>5 m</td>
<td>5 m</td>
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Summary

• The transverse force in the curvated coordinate is essentially the Lorentz force but with a substitution of the transverse magnetic field, $B_{x,y} \rightarrow (1+x/\rho)B_{x,y}$

• The curvature term play a key role for deriving the point-charge wakefield explicitly in terms of the incomplete elliptic integrals of the first and second kind

• Emittance growth due to the centrifugal force is at the same level of the contribution through the energy changes

• A steady-state theory of the coherent synchrotron radiation in two-dimensional free space is developed
References

1D theory:

2D and beyond:
3) G.V. Stupakov, “Effect of centrifugal transverse wakefield for microbunch in bend,” SLAC-PUB-8028, Revised March 2006
6) Ohmi’s talk in theory club, SLAC 2016
Acknowledgements

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