Lecture 3:

Chromatic Optics

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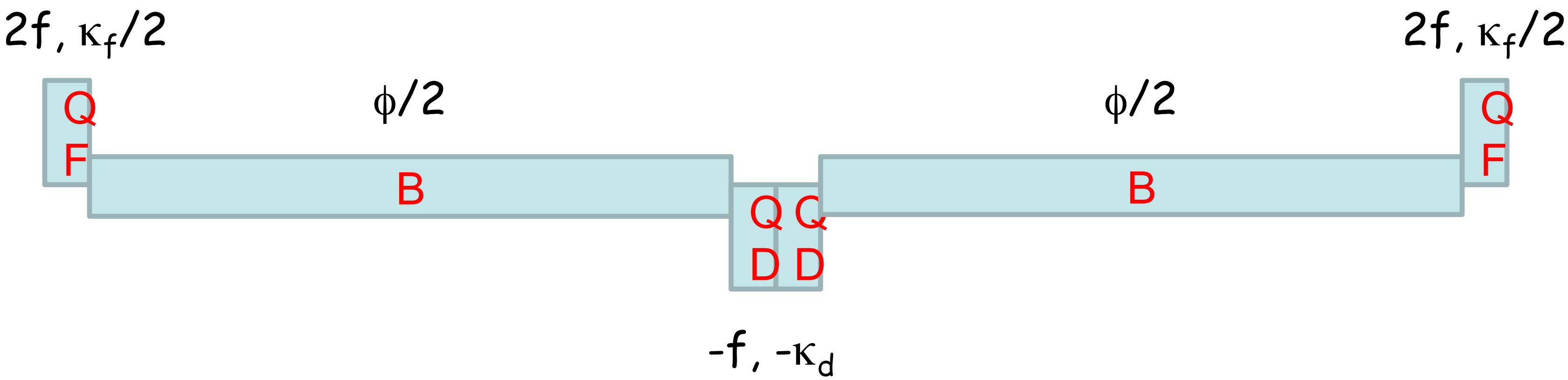
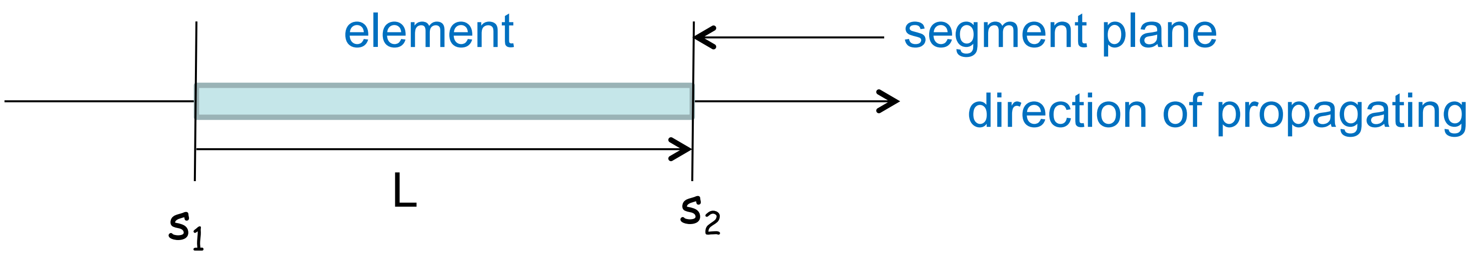
Periodic Cell: FODO



How to compute the Courant-Synder parameters and dispersions? For simplicity, we can use thin-lens approximation for quadrupoles, and small angle approximation for dipoles, and no gaps between any magnets.

What's the problem if we use these FODO cells to build entire ring? Why do we need to introduce sextupole magnets? How they work? Can we do better?

Concept of Transfer Map



$$z(s_1) = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ \ell \end{pmatrix}_{|s_1} \quad \mathcal{M}_{1\to 2} \quad \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ \ell \end{pmatrix}_{|s_2} = z(s_2)$$

$$z(s_2) = \mathcal{M}_{1\to 2}(z(s_1)).$$

abbreviated map notation

A set (six) of functions of canonical coordinates. It's called symplectic if its Jacob is symplectic.

Property of Symplectic Maps

Jacobian of a map:

$$J(\mathcal{M}) = \begin{pmatrix} \frac{\partial \mathcal{M}_1}{\partial x} & \frac{\partial \mathcal{M}_1}{\partial p_x} & \frac{\partial \mathcal{M}_1}{\partial y} & \frac{\partial \mathcal{M}_1}{\partial p_y} & \frac{\partial \mathcal{M}_1}{\partial \delta} & \frac{\partial \mathcal{M}_1}{\partial \ell} \\ \frac{\partial \mathcal{M}_2}{\partial x} & \frac{\partial \mathcal{M}_2}{\partial p_x} & \frac{\partial \mathcal{M}_2}{\partial y} & \frac{\partial \mathcal{M}_2}{\partial p_y} & \frac{\partial \mathcal{M}_2}{\partial \delta} & \frac{\partial \mathcal{M}_2}{\partial \ell} \\ \frac{\partial \mathcal{M}_3}{\partial x} & \frac{\partial \mathcal{M}_3}{\partial p_x} & \frac{\partial \mathcal{M}_3}{\partial y} & \frac{\partial \mathcal{M}_3}{\partial p_y} & \frac{\partial \mathcal{M}_3}{\partial \delta} & \frac{\partial \mathcal{M}_3}{\partial \ell} \\ \frac{\partial \mathcal{M}_4}{\partial x} & \frac{\partial \mathcal{M}_4}{\partial p_x} & \frac{\partial \mathcal{M}_4}{\partial y} & \frac{\partial \mathcal{M}_4}{\partial p_y} & \frac{\partial \mathcal{M}_4}{\partial \delta} & \frac{\partial \mathcal{M}_4}{\partial \ell} \\ \frac{\partial \mathcal{M}_5}{\partial x} & \frac{\partial \mathcal{M}_5}{\partial p_x} & \frac{\partial \mathcal{M}_5}{\partial y} & \frac{\partial \mathcal{M}_5}{\partial p_y} & \frac{\partial \mathcal{M}_5}{\partial \delta} & \frac{\partial \mathcal{M}_5}{\partial \ell} \\ \frac{\partial \mathcal{M}_6}{\partial x} & \frac{\partial \mathcal{M}_6}{\partial p_x} & \frac{\partial \mathcal{M}_6}{\partial y} & \frac{\partial \mathcal{M}_6}{\partial p_y} & \frac{\partial \mathcal{M}_6}{\partial \delta} & \frac{\partial \mathcal{M}_6}{\partial \ell} \end{pmatrix}$$

constant J matrix:

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Symplectic condition:

$$\tilde{J}(\mathcal{M}) \cdot J \cdot J(\mathcal{M}) = J$$

Specifically, R-matrix is given by J(*M*)$|_{x=px=y=py=d=l=0.}$ So it is sympletic as well.

Transfer Map
for Sector Bend with Small Bending Angle

The transfer map is given by:

$$\mathcal{M}_1 = x + \frac{L}{1+\delta}(p_x + \frac{\theta\delta}{2}),$$
$$\mathcal{M}_2 = p_x + \theta\delta,$$
$$\mathcal{M}_3 = y + \frac{L}{1+\delta} p_y,$$
$$\mathcal{M}_4 = p_y,$$
$$\mathcal{M}_5 = \delta$$
$$\mathcal{M}_6 = \ell + \theta x + \frac{L}{2(1+\delta)^2}[p_x^2 + p_y^2 + \theta(1+2\delta)(p_x + \frac{\theta\delta}{3})],$$

where L is the length of the magnet and θ bending angle. Note that it becomes map of drift if θ=0.

Transfer Map
for Thin Quadrupole, Sextupole

Transfer map is given by a kick:

$$\mathcal{M}_1 = x,$$

$$\mathcal{M}_2 = p_x - \frac{x}{f} - \frac{\kappa}{2}(x^2 - y^2),$$

$$\mathcal{M}_3 = y,$$

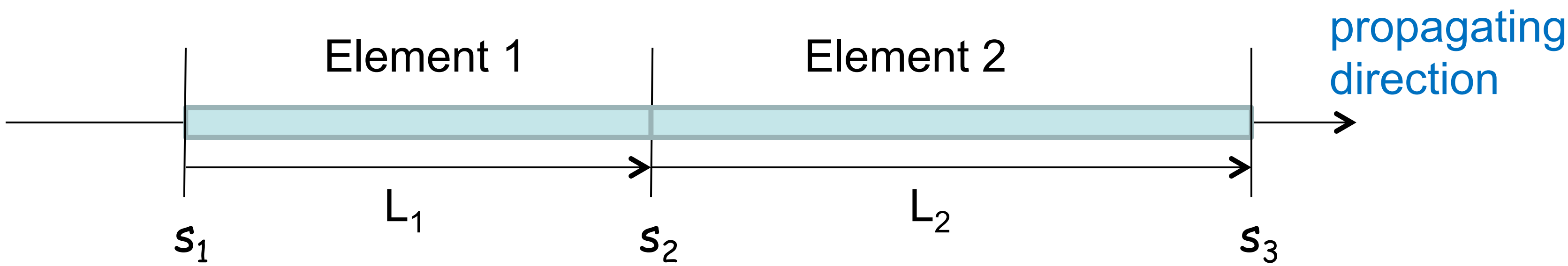
$$\mathcal{M}_4 = p_y + \frac{y}{f} + \kappa xy,$$

$$\mathcal{M}_5 = \delta$$

$$\mathcal{M}_6 = \ell,$$

where f is the focusing (in horizontal plane) length of quadrupole and κ, is the integrated strengths of sextupole.

Concatenation of Maps



If we have the transfer map for each individual elements:

$$z(s_2) = \mathcal{M}_{1\to 2}(z(s_1)),$$
$$z(s_3) = \mathcal{M}_{2\to 3}(z(s_2)).$$

Then the transfer map for the combined elements is given by

$$z(s_3) \underbrace{\boxed{= \mathcal{M}_{1\to 2} \circ \mathcal{M}_{2\to 3}}}_{\mathcal{M}_{1\to 3}} (z(s_1)) \equiv \mathcal{M}_{2\to 3}(\mathcal{M}_{1\to 2} z(s_1)),$$

nested functions

Courant-Snyder Parameters

Matrix of periodic system:

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Rotation matrix:

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

We have:

$$M = ARA^{-1}$$

where A^{-1} is a transformations from physical to normalized coordinates:

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}, A = \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

A is an "ascript" and is not unique. Since two-dimensional rotational group is commutative $AR(\theta)$ is also an "ascript". Courant and Synder choose to have $A_{12}=0$.

Linear Optics

Using the transfer map of the cell and the R-matrix, we find that the betatron phase advances in both planes are the same $\mu_x=\mu_y=\mu$ and given by,

$$\sin\frac{\mu}{2} = \frac{L}{4f},$$

where L is the cell length. The beta functions at the beginning:

$$\beta_x = \frac{L(1+\sin\frac{\mu}{2})}{\sin\mu}, \beta_y = \frac{L(1-\sin\frac{\mu}{2})}{\sin\mu},$$

and the periodical dispersion:

$$\eta_0 = \frac{L\varphi(1+\frac{1}{2}\sin\frac{\mu}{2})}{4\sin^2\frac{\mu}{2}}.$$

No surprises. They agree with the well-known results.

To the first-order of δ

Make a similar transformation to obtain the feed-down effects from the dispersive orbit,

$$\mathcal{M}_{\eta 0} = \mathcal{A}_{\eta 0} \circ \mathcal{M}_{cell} \circ \mathcal{A}^{-1}_{\eta 0},$$

where the dispersive map is given by,

$$\begin{aligned}
\mathcal{A}_1 &= x + \eta_0 \delta, \\
\mathcal{A}_2 &= p_x, \\
\mathcal{A}_3 &= y, \\
\mathcal{A}_4 &= p_y, \\
\mathcal{A}_5 &= \delta, \\
\mathcal{A}_6 &= \ell - \eta_0 p_x,
\end{aligned}$$

Introducing a Jacobian operator, we find the matrix with dependence of δ:

$$R_{\eta 0}(\delta) = J[\mathcal{M}_{\eta 0}] \equiv J(\mathcal{M}_{\eta 0})\,|_{x=px=y=py=l=0}$$

Like the R-matrix, it is symplectic.

Linear Chromaticity

Betatron phase advances up to the first-order of δ:

$$\mu_x(\delta) = \mu - \tan\frac{\mu}{2}[2 - \frac{1}{4\sin\frac{\mu}{2}}(\frac{1}{2} + \frac{1}{\sin^2\frac{\mu}{2}})(\kappa_f - \kappa_d)fL\varphi - \frac{3}{8\sin^2\frac{\mu}{2}}(\kappa_f + \kappa_d)fL\varphi]\delta,$$

$$\mu_y(\delta) = \mu - \tan\frac{\mu}{2}[2 - \frac{1}{4\sin\frac{\mu}{2}}(\frac{1}{2} - \frac{1}{\sin^2\frac{\mu}{2}})(\kappa_f - \kappa_d)fL\varphi - \frac{1}{8\sin^2\frac{\mu}{2}}(\kappa_f + \kappa_d)fL\varphi]\delta,$$

where κ_f, κ_d are the integrated strengths of the sextupoles. We can set their values:

$$\kappa_f = \frac{4\sin^2\frac{\mu}{2}}{fL\varphi(1 + \frac{1}{2}\sin\frac{\mu}{2})}, \kappa_d = \frac{4\sin^2\frac{\mu}{2}}{fL\varphi(1 - \frac{1}{2}\sin\frac{\mu}{2})},$$

to cancel the linear chromaticities in both planes. The settings are expected for the local compensation to the chromatic errors by quadrupoles.

To the second-order of δ

Make a similar transformation to obtain the feed-down effects from the dispersive orbit,

$$\mathcal{M}_{\eta 1} = \mathcal{A}_{\eta 1} \circ \mathcal{A}_{\eta 0} \circ \mathcal{M}_{cell} \circ \mathcal{A}^{-1}_{\eta 0} \circ \mathcal{A}^{-1}_{\eta 1},$$

where the new dispersive map is given by,

$$\mathcal{A}_1 = x + \eta_1 \frac{\delta^2}{2},$$

$$\mathcal{A}_2 = p_x,$$

$$\mathcal{A}_3 = y,$$

$$\mathcal{A}_4 = p_y,$$

$$\mathcal{A}_5 =,$$

$$\mathcal{A}_6 = \ell - \eta_1 p_x \delta,$$

where the first-order dispersion is found in the same way as the zeroth-order one, we have

$$\eta_1 = -\frac{f\phi}{2}.$$

Using the Jacobian operator, we find the matrix with dependence of δ:

$$R_{\eta 1}(\delta) = J[\mathcal{M}_{\eta 1}]$$

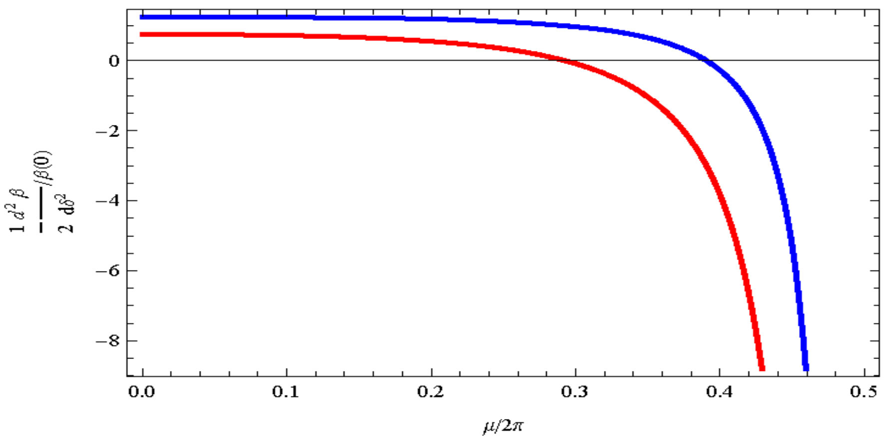
Like the R-matrix, it is symplectic.

Second-Order Beta Beating

The beta functions at the beginning of the cell, up to the second-order of δ:

$$\beta_x(\delta) = \beta_x[1-\delta+\frac{10-13\sin^2\frac{\mu}{2}-\sin^3\frac{\mu}{2}+3\sin^4\frac{\mu}{2}}{2(4-5\sin^2\frac{\mu}{2}+\sin^4\frac{\mu}{2})}\delta^2],$$

$$\beta_y(\delta) = \beta_y[1-\delta+\frac{3(2-3\sin^2\frac{\mu}{2}-\sin^3\frac{\mu}{2}+\sin^4\frac{\mu}{2})}{2(4-5\sin^2\frac{\mu}{2}+\sin^4\frac{\mu}{2})}\delta^2].$$

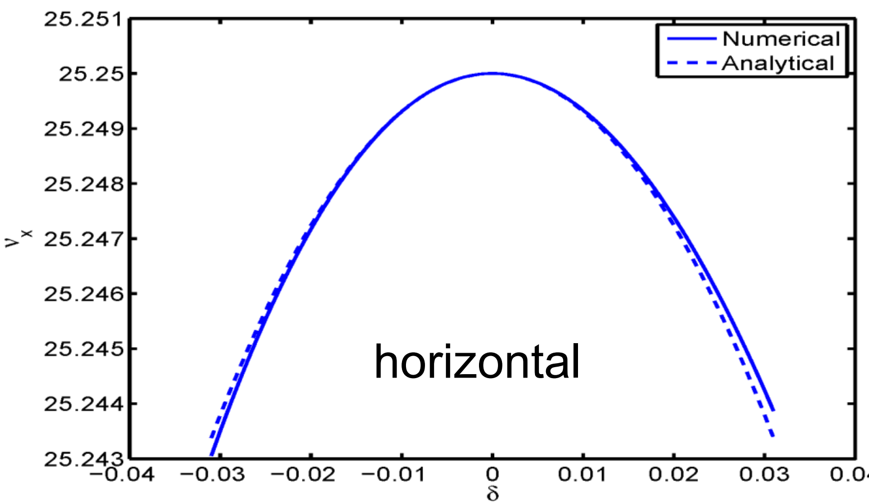


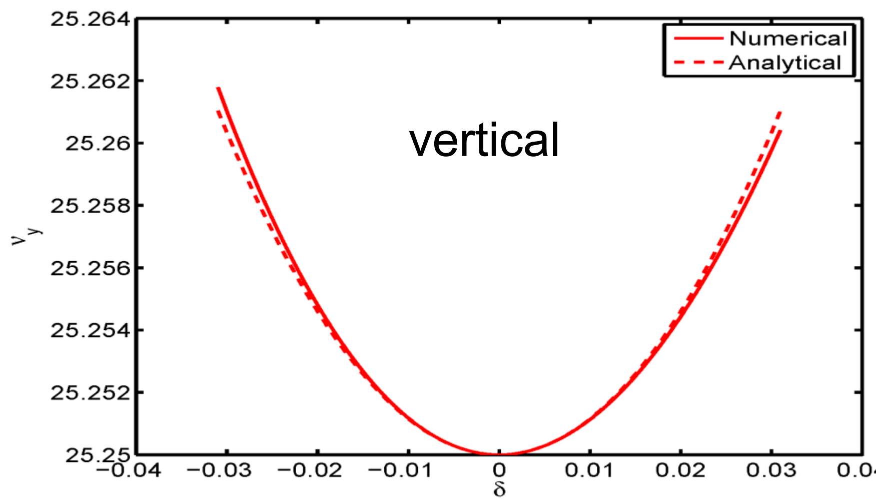
- half integer resonance seen
- not good if $\mu > 135^0$

Second-Order Chromatic Effects

The betatron phase advances up to the second-order of δ:

$$\mu_x(\delta) = \mu - \frac{\tan\frac{\mu}{2}\left(1-\frac{1}{2}\sin^2\frac{\mu}{2}\right)}{2\left(1-\frac{1}{4}\sin^2\frac{\mu}{2}\right)}\delta^2, \mu_y(\delta) = \mu + \frac{\tan\frac{\mu}{2}\left(1+\frac{1}{2}\sin^2\frac{\mu}{2}\right)}{2\left(1-\frac{1}{4}\sin^2\frac{\mu}{2}\right)}\delta^2.$$





Comparison to a numerical simulation in LEGO in a ring that consists of 101 90^0 cells.

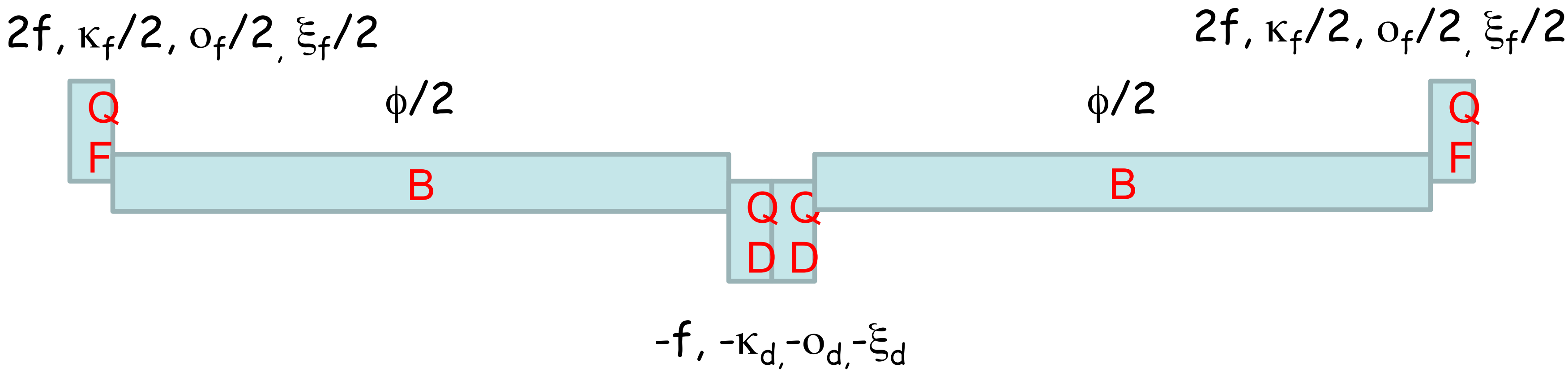
Transfer Map
for Thin Quadrupole, Sextupole, Octupole, and Decapole

Transfer map is given by a kick:

$$
\begin{aligned}
\mathcal{M}_1 &= x, \\
\mathcal{M}_2 &= p_x - \frac{x}{f} - \frac{\kappa}{2}(x^2 - y^2) - \frac{o}{6}x(x^2 - 3y^2) - \frac{\xi}{24}(x^4 - 6x^2y^2 + y^4), \\
\mathcal{M}_3 &= y, \\
\mathcal{M}_4 &= p_y + \frac{y}{f} + \kappa xy + \frac{o}{6}y(3x^2 - y^2) + \frac{\xi}{6}xy(x^2 - y^2), \\
\mathcal{M}_5 &= \delta \\
\mathcal{M}_6 &= \ell,
\end{aligned}
$$

where f is the focusing (in horizontal plane) length of quadrupole and κ, o, ξ are the integrated strengths of sextupole, octupole, and decapole respectively.

Cell with Nonlinear Elements



$$\kappa_f = \frac{4\sin^2\frac{\mu}{2}}{fL\phi(1+\frac{1}{2}\sin\frac{\mu}{2})}, \kappa_d = \frac{4\sin^2\frac{\mu}{2}}{fL\phi(1-\frac{1}{2}\sin\frac{\mu}{2})},$$

← Sextupoles

$$o_f = \frac{8\sin^5\frac{\mu}{2}}{fL^2\phi^2(1+\frac{1}{2}\sin\frac{\mu}{2})^3}, o_d = -\frac{8\sin^5\frac{\mu}{2}}{fL^2\phi^2(1-\frac{1}{2}\sin\frac{\mu}{2})^3},$$

← Octupoles

$$\xi_f = \frac{16\sin^6\frac{\mu}{2}(4-2\sin\frac{\mu}{2}+\sin^2\frac{\mu}{2})}{fL^3\phi^3(1+\frac{1}{2}\sin\frac{\mu}{2})^5}, \xi_d = \frac{16\sin^6\frac{\mu}{2}(4+2\sin\frac{\mu}{2}+\sin^2\frac{\mu}{2})}{fL^3\phi^3(1-\frac{1}{2}\sin\frac{\mu}{2})^5},$$

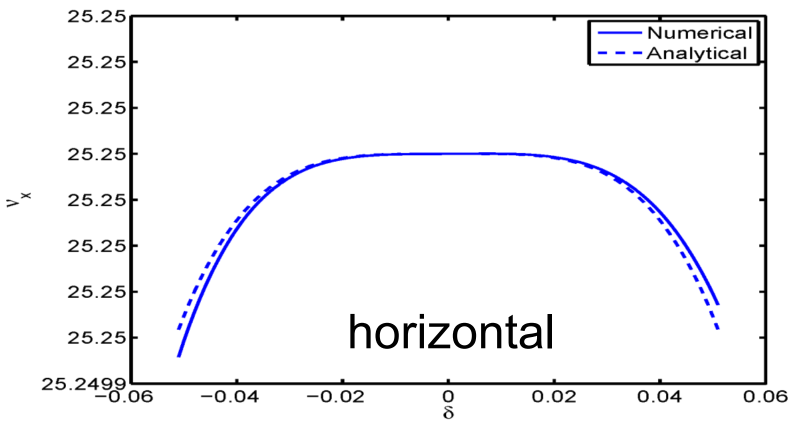
Decapoles

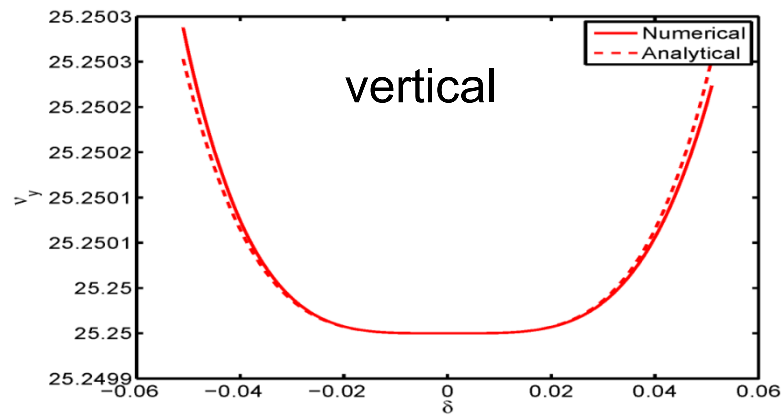
Four-Order Chromatic Effects

The betatron phase advances in the cell up to the fourth-order of δ:

$$\mu_x(\delta) = \mu + \frac{\tan\frac{\mu}{2}(-352 + 312\sin^2\frac{\mu}{2} + 60\sin^4\frac{\mu}{2} + \sin^6\frac{\mu}{2})}{12(4-\sin^2\frac{\mu}{2})^3}\delta^4,$$

$$\mu_y(\delta) = \mu + \frac{\tan\frac{\mu}{2}(992 + 840\sin^2\frac{\mu}{2} + 84\sin^4\frac{\mu}{2} + \sin^6\frac{\mu}{2})}{12(4-\sin^2\frac{\mu}{2})^3}\delta^4.$$





Comparison to a numerical simulation in LEGO in a ring that consists of 101 90^0 cells.

Beta Functions and Phase Advances within cell

The beta functions in region 0<s<L/2, up to the third-order of δ:

$$\beta_x(\delta,s) = \beta_x[1-\frac{4s}{L}\sin\frac{\mu}{2}(1-\frac{2s\sin\frac{\mu}{2}}{L(1+\sin\frac{\mu}{2})})](1-\delta+\delta^2+\delta^3),$$

$$\beta_y(\delta,s) = \beta_y[1+\frac{4s}{L}\sin\frac{\mu}{2}(1+\frac{2s\sin\frac{\mu}{2}}{L(1-\sin\frac{\mu}{2})})](1-\delta+\delta^2+\delta^3).$$

The betatron phase advances up to the third-order of δ:

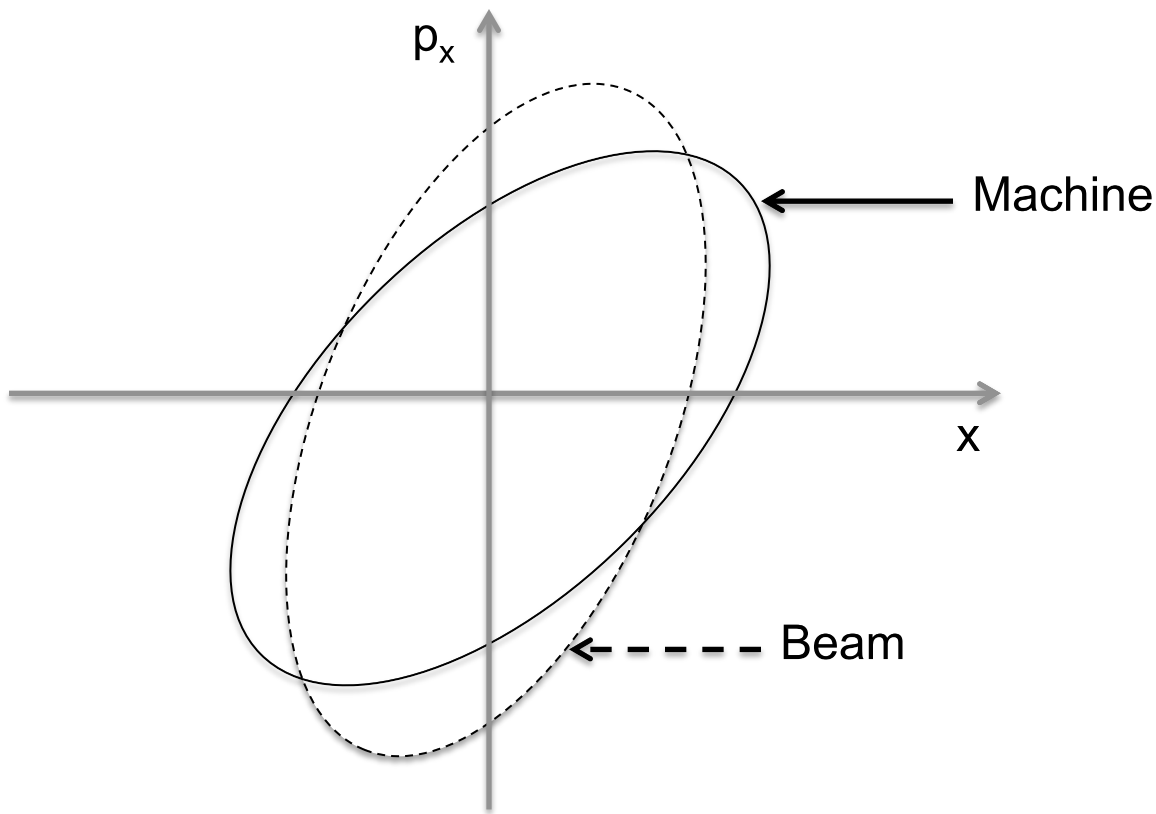
$$\psi_x(\delta,s) = \tan^{-1}[\frac{\tan\frac{\mu}{2}(1-\sin\frac{\mu}{2})\frac{2s}{L}}{1-\frac{2s}{L}\sin\frac{\mu}{2}}],$$

$$\psi_y(\delta,s) = \tan^{-1}[\frac{\tan\frac{\mu}{2}(1+\sin\frac{\mu}{2})\frac{2s}{L}}{1+\frac{2s}{L}\sin\frac{\mu}{2}}].$$

← no δ dependence !

Mismatch Parameter

$$\xi = \frac{1}{2}(\gamma_0\beta - 2\alpha_0\alpha + \beta_0\gamma)$$



Meaning of Mismatch Parameters

Transformations to normalized coordinates:

$$A_o^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

Beam sigma matrix:

$$\Sigma = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Transform it to the normalized space:

$$A_o^{-1} \Sigma A_o^{\mp 1} = \varepsilon \begin{pmatrix} \tilde{\beta} & -\tilde{\alpha} \\ -\tilde{\alpha} & \tilde{\gamma} \end{pmatrix} \xrightarrow{\text{Matched}} \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Mismatch parameters:

$$\xi = \frac{1}{2}(\tilde{\beta} + \tilde{\gamma})$$ ($M = \xi + \sqrt{\xi^2 - 1}$, Sands)

Mismatch Parameter and Montague Functions

Applying to a chromatic mismatch, we introduce:

$$\xi(\delta) = \frac{1}{2}[\gamma_0\beta(\delta) - 2\alpha_0\alpha(\delta) + \beta_0\gamma(\delta)]$$

as a function of momentum deviation δ. Then we have its Taylor expansion:

$$\xi(\delta) = 1 + \frac{1}{2}[(\frac{\beta'}{\beta_0})^2 + (\alpha' - \alpha_0\frac{\beta'}{\beta_0})^2]\delta^2 + O(\delta^3)$$

$$M(\delta) = 1 + \sqrt{(\frac{\beta'}{\beta_0})^2 + (\alpha' - \alpha_0\frac{\beta'}{\beta_0})^2}\delta + O(\delta^2)$$

Here we see how it related to Montague functions

$$A = \frac{\beta'}{\beta_0}, B = \alpha' - \alpha_0\frac{\beta'}{\beta_0}, W = \sqrt{A^2 + B^2}$$

Higher Order Chromatic Mismatch

Higher order terms are much more complicated. But if we assume that The first order deviations β' and α' have been zeroed out then we have

$$\xi(\delta) = 1 + \frac{1}{28}[(\frac{\beta''}{\beta_0})^2 + (\alpha'' - \alpha_0\frac{\beta''}{\beta_0})^2]\delta^2 + O(\delta^3)$$

$$A_n = \frac{\beta^{(n)}}{\beta_0}, B_n = \alpha^{(n)} - \alpha_0\frac{\beta^{(n)}}{\beta_0}, W_n = \sqrt{A_n^2 + B_n^2}$$

We found that these functions are very useful to describe the high order chromatic mismatch.

Summary

- Hamiltonian and symplectic maps are fundamental for the beam dynamics in storage rings, including the linear and chromatic optics.
- A perfect chromatic optics can be achieved order-by-order using high-order harmonics, such as sextupoles, octupoles, and decapoles in FODO cell.

References

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