Lecture 4:

Synchrotron Radiation

Yunhai Cai
SLAC National Accelerator Laboratory

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Lienard-Wiechert Formula

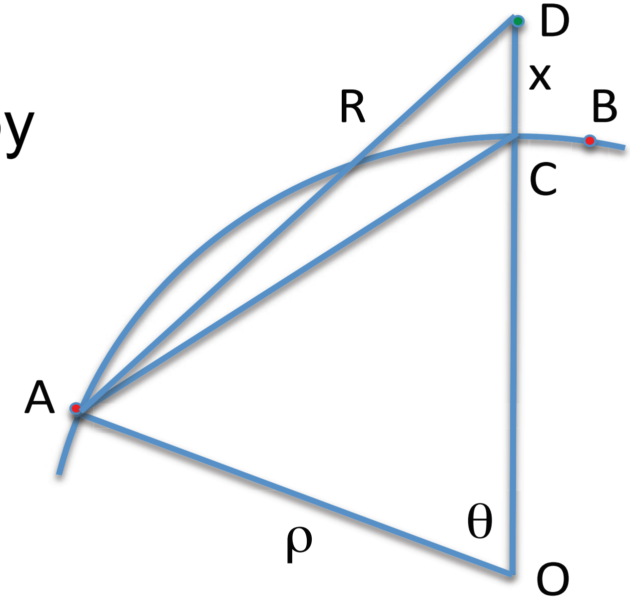
Space Charge ↓ Radiated Field ↓

$$\vec{E} = e[\frac{\vec{n}-\vec{\beta}}{\gamma^2(1-\vec{n}\cdot\vec{\beta})^3 R^2}]_{ret} + (\frac{e}{c})[\frac{\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}}{(1-\vec{n}\cdot\vec{\beta})^3 R}]_{ret},$$

$$\vec{B} = \vec{n}\times\vec{E}$$

- Space charge is suppressed by $1/\gamma^2$
- Identify radiated field with synchrotron radiation
- Subject to retarded condition:

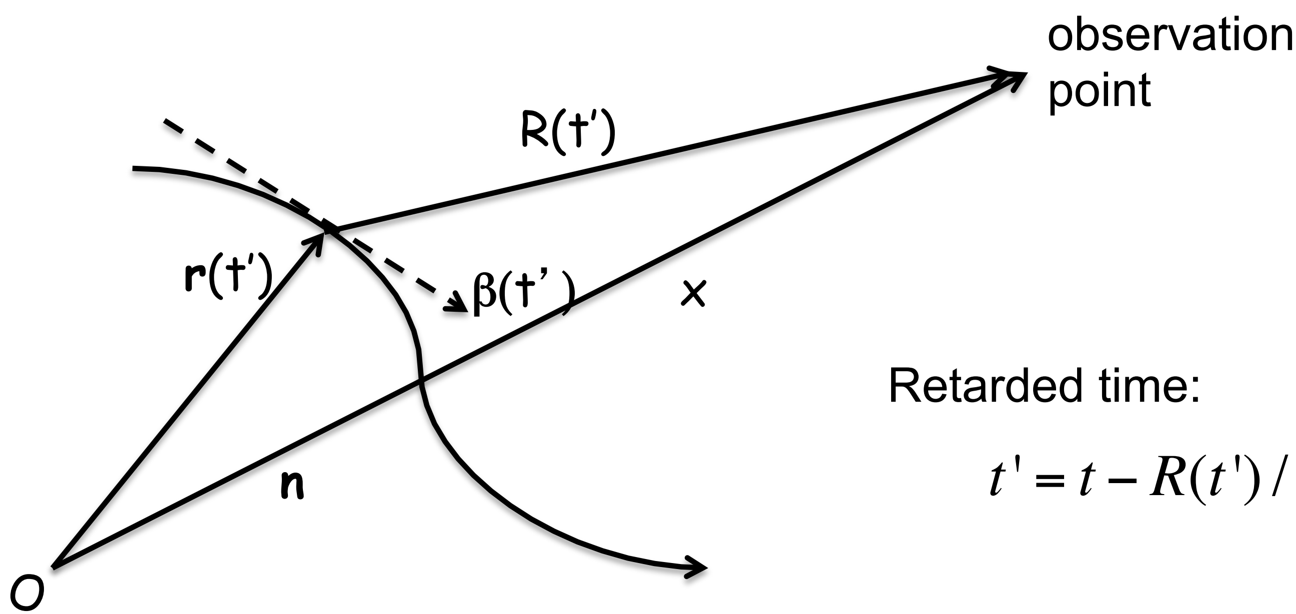
$$t' = t - \frac{R(t')}{c}$$



Spectral Distribution

In the far-field approximation, the intensity distribution is given by,

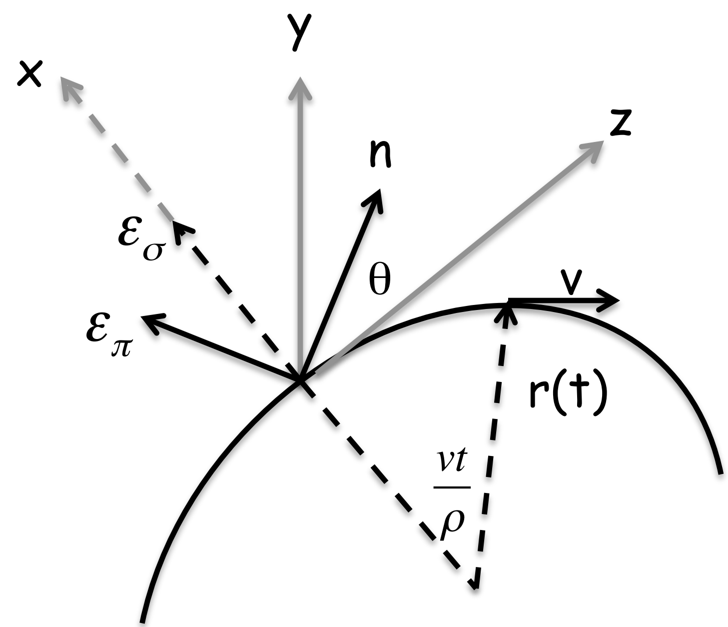
$$\frac{d^2 I}{d\omega d\Omega} = \frac{r_e m c \omega^2}{4\pi^2} \left| \int_{-\infty}^{\infty} \hat{n} \times [\hat{n} \times \vec{\beta}(t')] \exp[i\omega(t' - \hat{n} \cdot \vec{r}(t')/c)] dt' \right|^2$$



Retarded time:

$$t' = t - R(t')/c$$

Computing Radiation Spectrum



Radiation direction:

$$\hat{n} = (0, \sin\theta, \cos\theta)$$

Electron position:

$$\vec{r}(t) = (-\rho(1-\cos(\frac{vt}{\rho})), 0, \rho\sin(\frac{vt}{\rho}))$$

Its velocity:

$$\vec{\beta}(t) = (-\beta\sin(\frac{vt}{\rho}), 0, \beta\cos(\frac{vt}{\rho}))$$

Phase approximation:

$$\omega(t - \frac{\hat{n}\cdot\vec{r}(t)}{c}) = \omega[t - \frac{\rho}{c}\sin(\frac{vt}{\rho})\cos\theta] \approx \frac{\omega}{2}[(\frac{1}{\gamma^2} + \theta^2)t + \frac{c^2}{3\rho^2}t^3]$$

Vector integrand:

$$\hat{n}\times(\hat{n}\times\vec{\beta}) = \beta[\sin(\frac{vt}{\rho})\hat{\varepsilon}_\sigma + \cos(\frac{vt}{\rho})\sin\theta\hat{\varepsilon}_\pi] \approx \frac{ct}{\rho}\hat{\varepsilon}_\sigma + \theta\hat{\varepsilon}_\pi$$

Radiation Spectrum by Bending Magnet

Intensity distribution is given by,

σ mode ↓ π mode ↓

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3r_e mc}{4\pi^2}\gamma^2 (\frac{\omega}{\omega_c})^2 (1+\gamma^2\theta^2)^2 [K_{2/3}^2(\xi) + \frac{\gamma^2\theta^2}{1+\gamma^2\theta^2} K_{1/3}^2(\xi)]$$

where $K_{1/3}$ and $K_{2/3}$ are modified Bessel functions and their argument

$$\xi = \frac{1}{2}\frac{\omega}{\omega_c}(1+\gamma^2\theta^2)^{3/2}$$

Angle integrated intensity distribution is

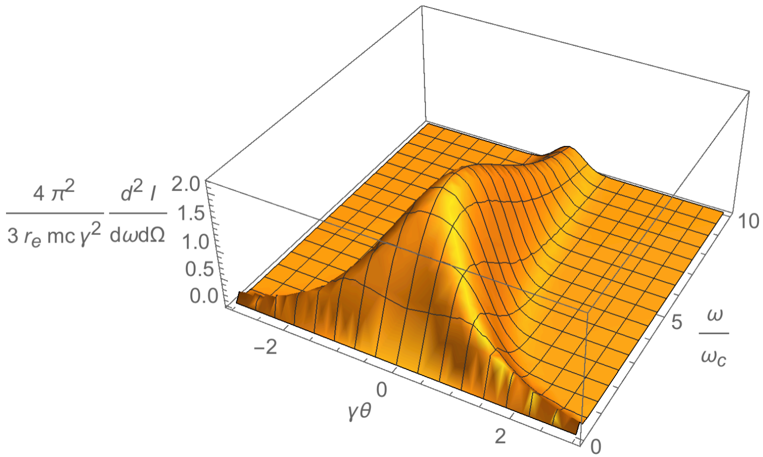
$$\frac{dI}{d\omega} = \sqrt{3} r_e mc\gamma \frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{\infty} K_{5/3}(x)dx$$

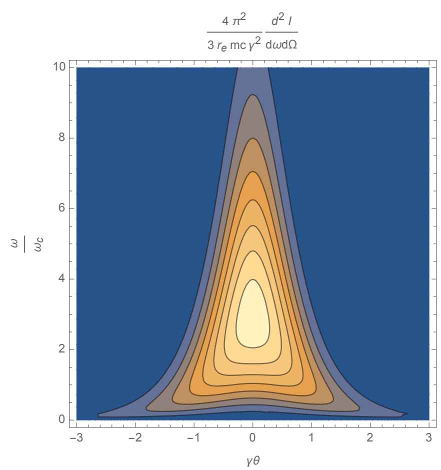
where the critical frequency is

$$\omega_c = \frac{3}{2}\gamma^3 (\frac{c}{\rho})$$

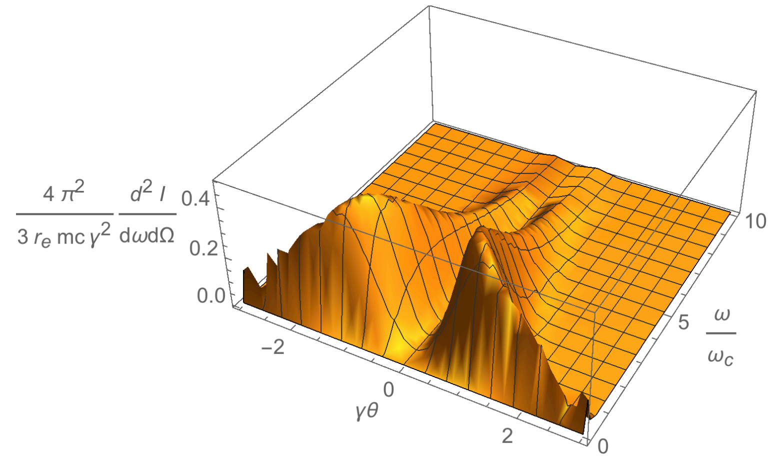
Intensity Distribution

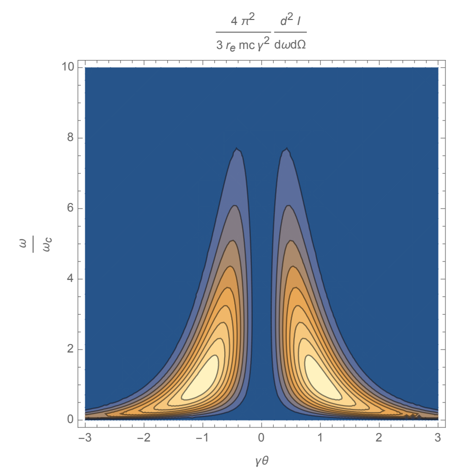
σ mode



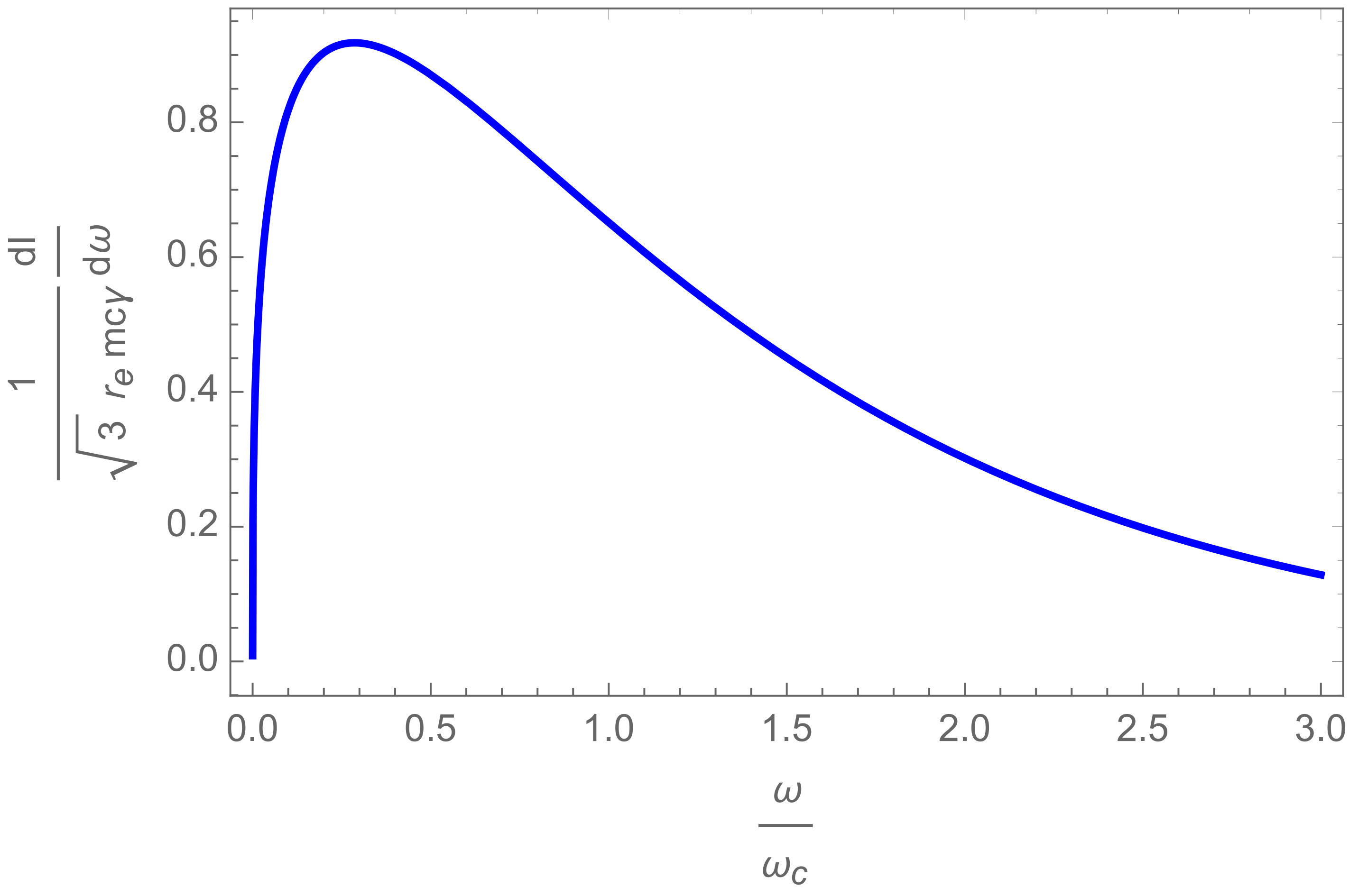


π mode





Radiation Spectrum



Beam Dynamics in Undulator

Electron velocity:

$$\frac{dx}{dt} = -\beta c \frac{K}{\gamma} \sin(k_p z)$$

$$\frac{dz}{dt} = \beta c [1 - \frac{K^2}{2\gamma^2} \sin^2(k_p z)]$$

Its position:

$$x(t) = \frac{K}{\gamma k_p} \cos(k_p \bar{\beta} c t)$$

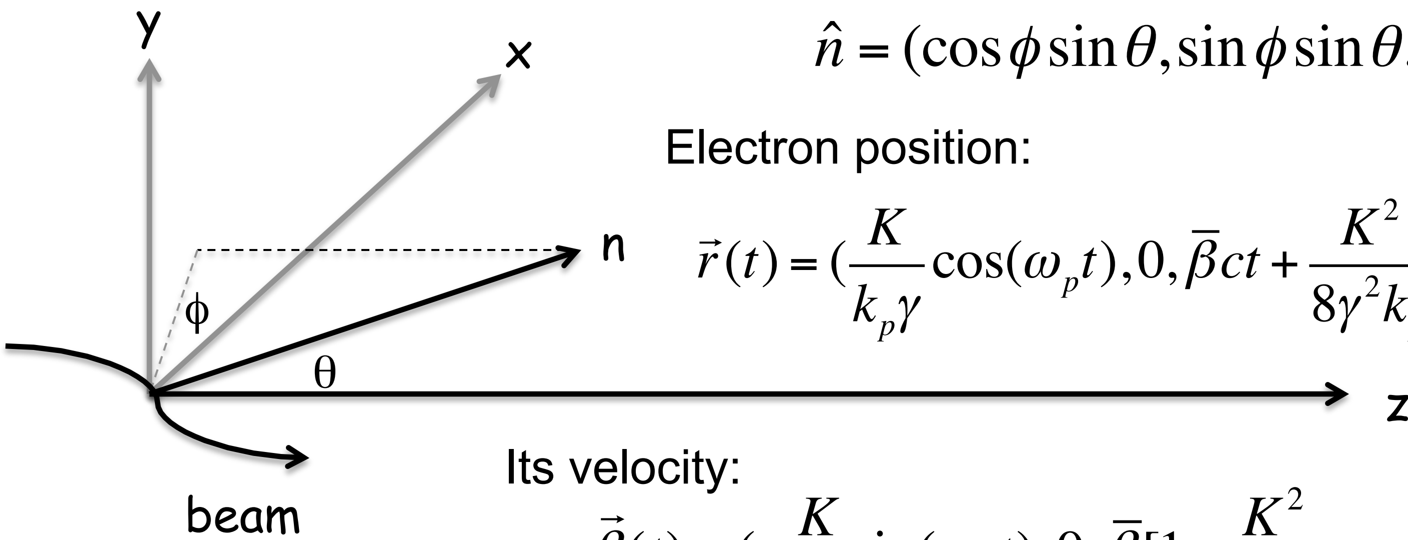
$$z(t) = \bar{\beta} c t + \frac{K^2}{8\gamma^2 k_p} \sin(2 k_p \bar{\beta} c t)$$

where the undulator parameter K and averaged velocity:

$$K = \frac{e B_0 \lambda_p}{2\pi m c^2}$$

$$\bar{\beta} = \beta (1 - \frac{K^2}{4\gamma^2})$$

Computing Spectrum of Undulator Radiation



Radiation direction:

$$\hat{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$$

Electron position:

$$\vec{r}(t) = (\frac{K}{k_p\gamma}\cos(\omega_p t), 0, \bar{\beta}ct + \frac{K^2}{8\gamma^2 k_p}\sin(2\omega_p t))$$

Its velocity:

$$\vec{\beta}(t) = (-\frac{K}{\gamma}\sin(\omega_p t), 0, \bar{\beta}[1 + \frac{K^2}{4\gamma^2}\cos(2\omega_p t)])$$

Phase approximation:

$$\omega(t - \frac{\hat{n}\cdot\vec{r}(t)}{c}) \approx \frac{\omega}{\omega_1}[\omega_p t - \frac{K\bar{\beta}\theta}{\gamma}\frac{\omega_1}{\omega_p}\cos\phi\cos(\omega_p t) - \frac{K^2\bar{\beta}}{8\gamma^2}\frac{\omega_1}{\omega_p}\sin(2\omega_p t)]$$

Vector integrand:

$$\hat{n}\times(\hat{n}\times\vec{\beta}) \approx \bar{\beta}\{[\theta\cos\phi + \frac{K}{\gamma}\sin(\omega_p t)]\hat{x} + \theta\sin\phi\hat{y}\}$$

Number of Photons within $\Delta\omega/\omega$

Total emitted photons after an electron passing through undulator is given by,

$$\frac{dN_{ph}(\omega)}{d\Omega} = \alpha\gamma^2\bar{\beta}^2 N_p^2 \frac{\Delta\omega}{\omega}\sum_{k=1}^{\infty} k^2 [\frac{\sin(\pi N_p \Delta\omega_k/\omega_1)}{\pi N_p \Delta\omega_k/\omega_1}]^2 [I_{\sigma,k} + I_{\pi,k}]$$

where,

$$I_{\sigma,k} = \frac{(2\gamma\theta\Sigma_1\cos\phi - K\Sigma_2)^2}{(1+\frac{K^2}{2}+\gamma^2\theta^2)^2}$$

$$I_{\pi,k} = \frac{(2\gamma\theta\Sigma_1\sin\phi)^2}{(1+\frac{K^2}{2}+\gamma^2\theta^2)^2}$$

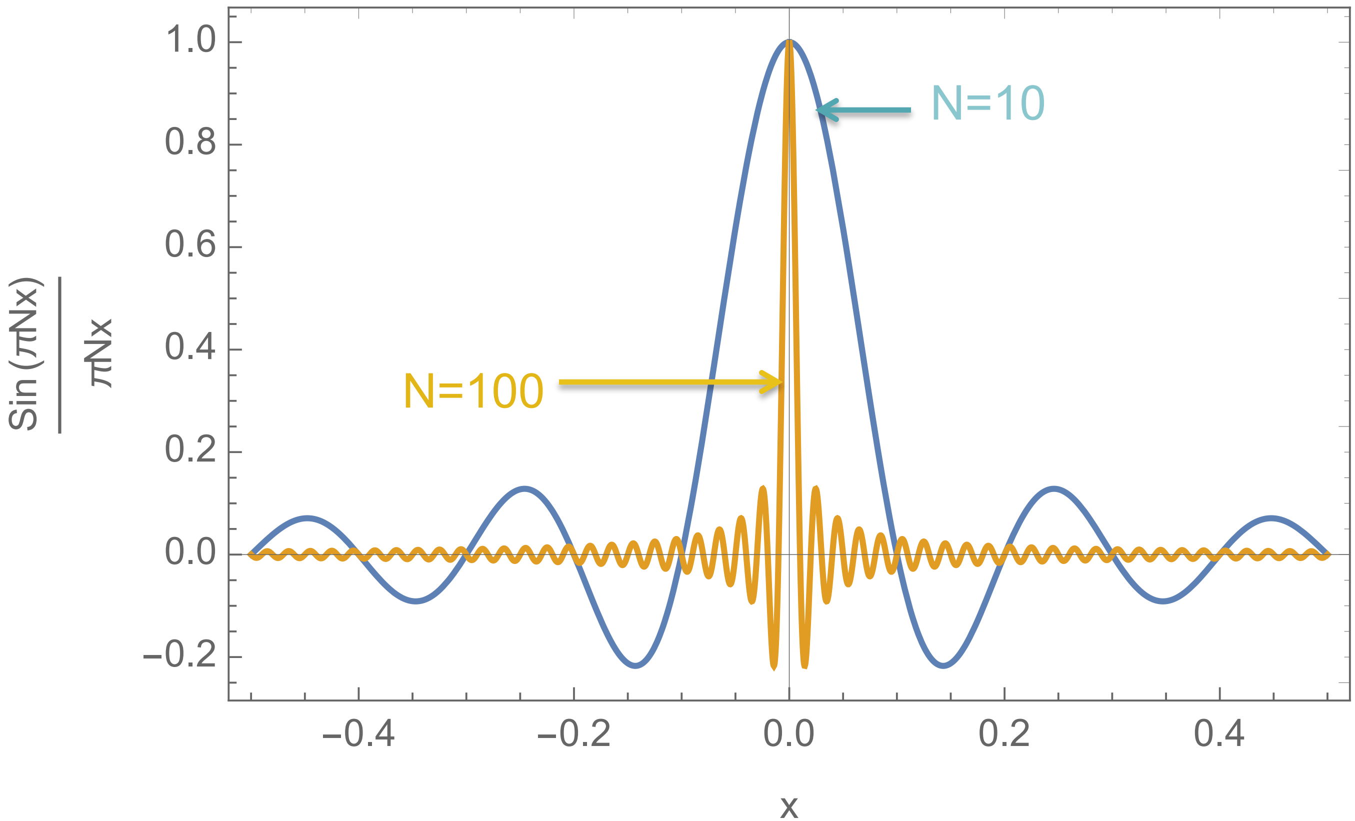
and

$$\Sigma_1 = \sum_{m=-\infty}^{\infty} J_{-m}(\mu) J_{k-2m}(\nu)$$

$$\Sigma_2 = \sum_{m=-\infty}^{\infty} J_{-m}(\mu)[J_{k-2m-1}(\nu) + J_{k-2m+1}(\nu)]$$

and J_n are Bessel functions.

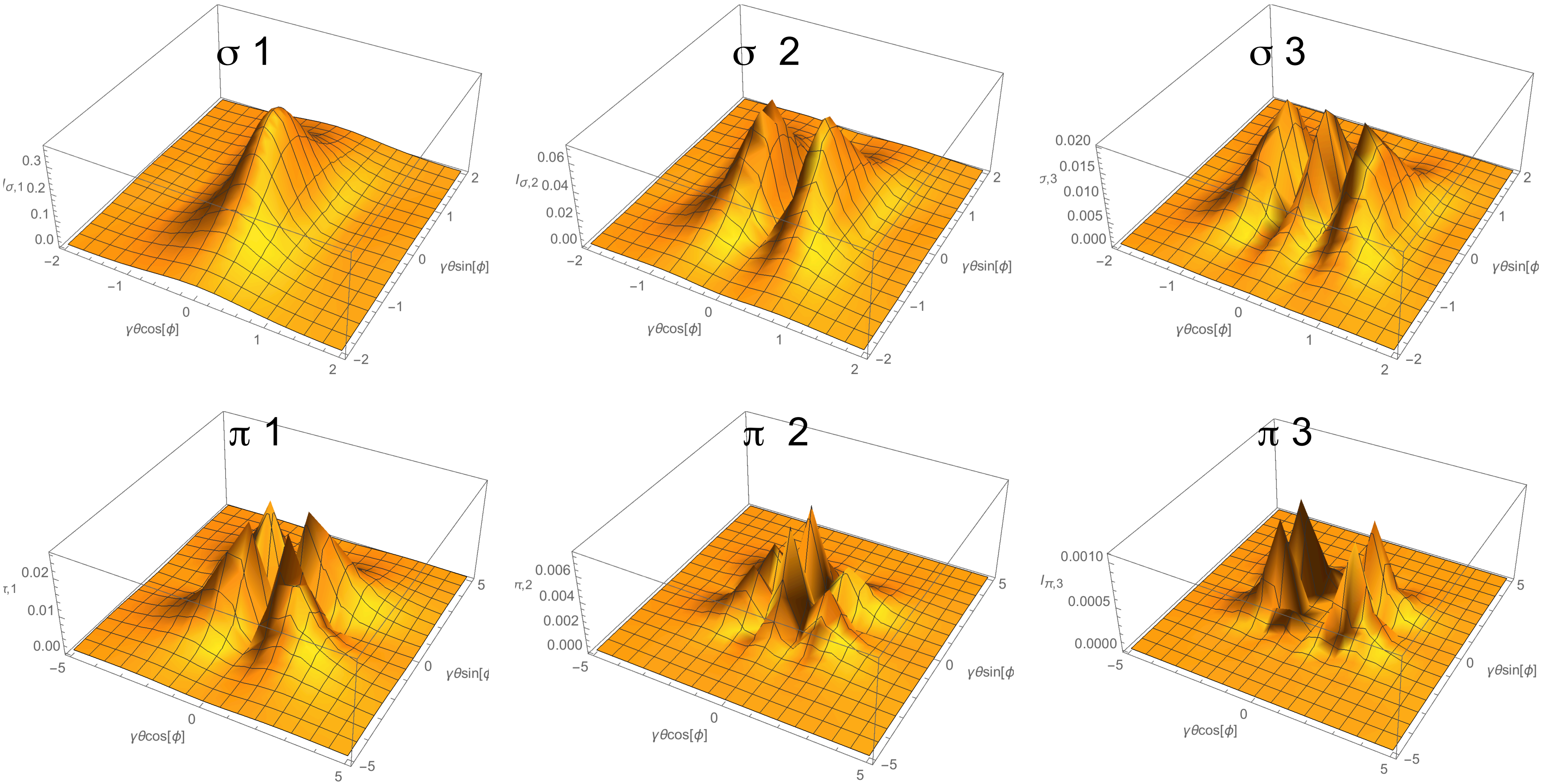
Interference Spectrum



$$\frac{\Delta\omega_k}{\omega_k} = \pm\frac{1}{kN_p}$$

first zeros near the origin define the width of the peak.

Radiation Distribution of σ and π Modes (K=1)



Forward Radiation

Total emitted photons after an electron passing through undulator is given by,

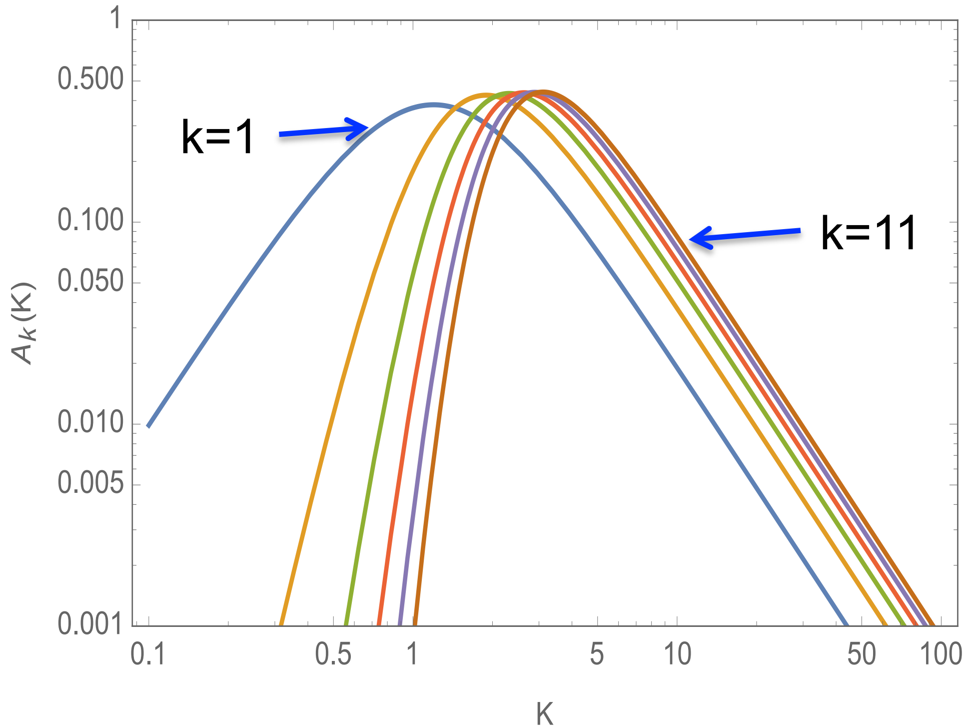
$$\frac{dN_{ph}(\omega)}{d\Omega} = \alpha\gamma^2 N_p^2 \frac{\Delta\omega}{\omega} \sum_{k=1}^{\infty} A_k(K)\left[\frac{\sin(\pi N_p \Delta\omega_k / \omega_1)}{\pi N_p \Delta\omega_k / \omega_1}\right]^2$$

where

$$A_k(K) = \frac{k^2 K^2}{(1+\frac{K^2}{2})^2} JJ^2$$

$$JJ = [J_{(k+1)/2}(\frac{kK^2}{4+2K^2}) - J_{(k-1)/2}(\frac{kK^2}{4+2K^2})]$$

Only odd σ modes contribute



Undulator parameter K should be between 1 to 4

Photon Flux

Flux at k[th] harmonics:

$$\frac{dN_{ph}(\omega_k)}{dt}\Big|_{\theta=0} = \frac{\pi}{2}\alpha N_p \frac{I}{e}\frac{\Delta\omega}{\omega_k} Q_k(K)$$

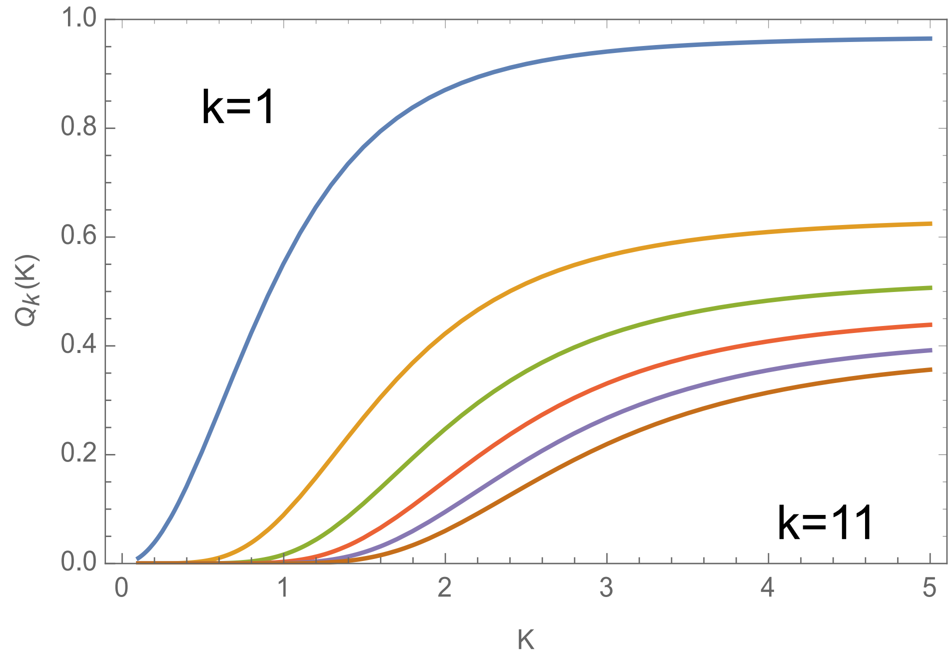
where

$$Q_k(K) = \frac{1+\frac{K^2}{2}}{k} A_k(K)$$

The rms opening angle:

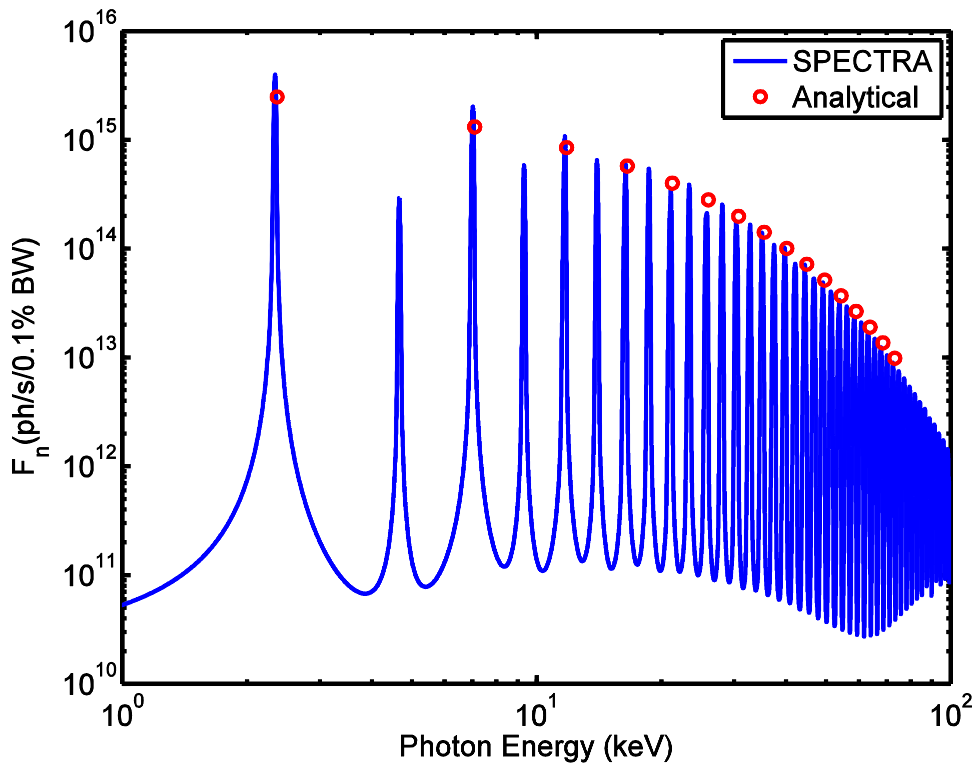
$$\sigma_{r'} \approx \frac{1}{2\gamma}\sqrt{\frac{1+\frac{K^2}{2}}{kN_p}} = \sqrt{\frac{\lambda_k}{2L}}$$

The forward cone: $d\Omega = 2\pi\sigma_{r'}^2$



Undulator parameter: K has to be large enough

Photon Flux of PEP-X



n[th] harmonic wavelength:

$$\lambda_n = \frac{\lambda_u}{2n\gamma^2}(1+\frac{K^2}{2})$$

$$F_n = \frac{\pi}{2}\alpha N_u Q_n(\frac{nK^2}{4+2K^2})\frac{\Delta\omega}{\omega}\frac{I}{e}$$

Gaussian Mode

The fundamental Gaussian mode can be written as

$$E(x,y,z) = E_0 \frac{w_0}{w(z)} \exp[-\frac{r^2}{w(z)}] \exp[-i(kz + k\frac{r^2}{2R(z)} - \phi(z))]$$

where

spot size: $w(z) = w_0 \sqrt{1+(z/z_R)^2}$

radius of curvature: $R(z) = z[1+(z/z_R)^2]$

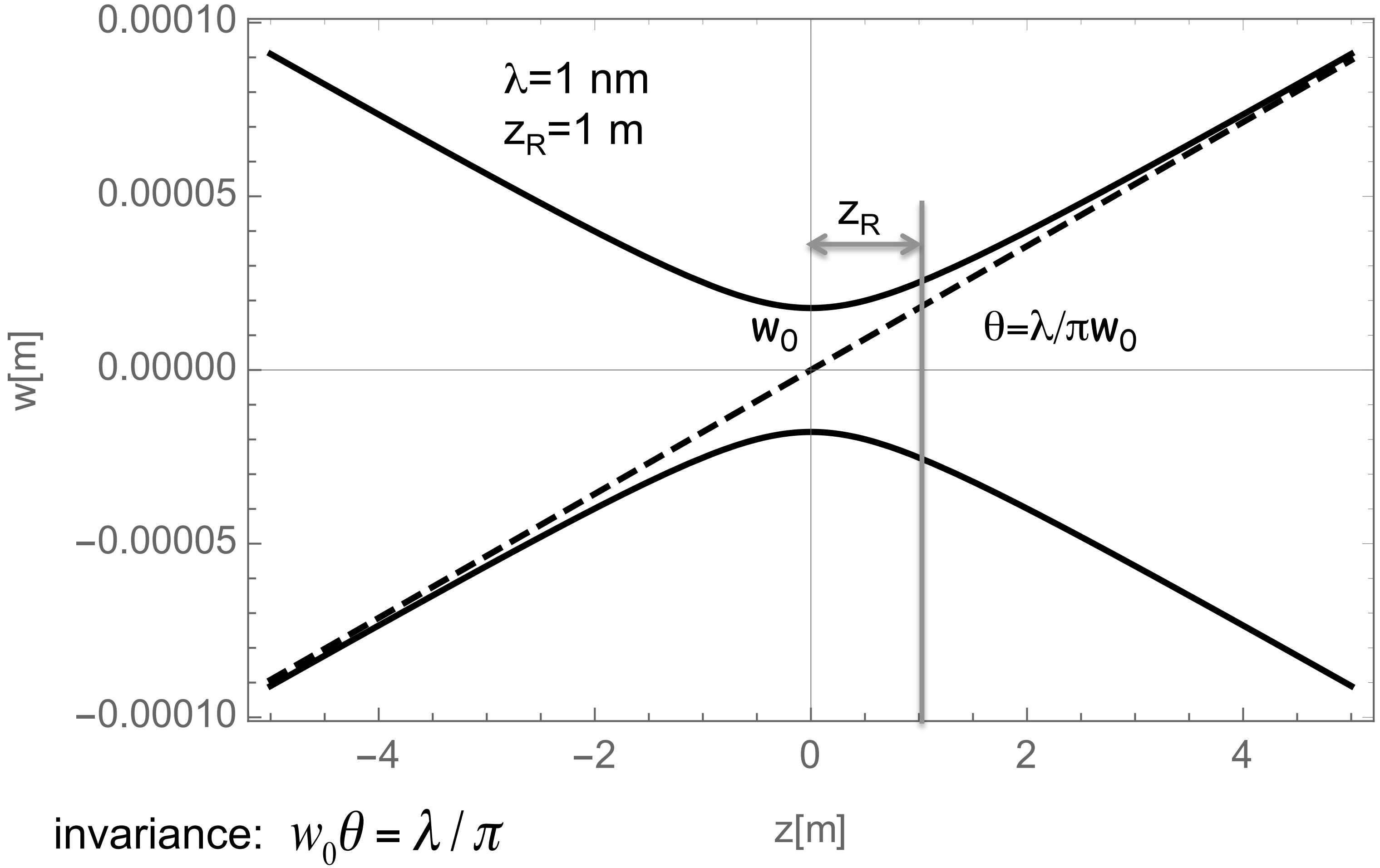
Guoy phase: $\phi(z) = \tan^{-1}(z/z_R)$

Rayleigh length: $z_R = \frac{\pi w_0^2}{\lambda}$

It is a solution of the paraxial wave equation:

$$(\nabla_\perp^2 - 2ik\frac{\partial}{\partial z})\psi(x,y,z) = 0$$

Visualization of a Gaussian Mode



invariance: $w_0\theta = \lambda / \pi$

Brightness of Gaussian Mode

For a Gaussian mode, its brightness distribution function is given by,

$$B(\vec{r},\vec{\varphi};0) = B_0 \exp[-\frac{\vec{r}^{\,2}}{2\sigma_r^2} - \frac{\vec{\varphi}^{\,2}}{2\sigma_{r'}^2}]$$

$$\sigma_r = w_0 / 2$$

$$\sigma_{r'} = \sigma_r / z_R$$

Then, we have

$$\sigma_r \sigma_{r'} = \lambda / 4\pi$$ emittance

$$\sigma_r / \sigma_{r'} = z_R$$ beta function

$$B_0 = \frac{F}{(2\pi\sigma_r\sigma_{r'})^2} = \frac{F}{(\lambda/2)^2}$$ ← coherence volume

Single Electron Brightness

Using the Gaussian mode as an approximation for the undulator source, we choice z_R=L/2π, so that,

$$\sigma_{r'} = \sqrt{\frac{\lambda_k}{2L}}$$

$$\sigma_r = \frac{\sqrt{2\lambda_k L}}{4\pi}$$

Its brightness function is given by,

$$B(\vec{r},\vec{\varphi};0) = B_0 \exp[-\frac{\vec{r}^2}{2\sigma_r^2} - \frac{\vec{\varphi}^2}{2\sigma_{r'}^2}]$$

and the photon flux is

$$F = \frac{\pi}{2}\alpha N_p \frac{I}{e}\frac{\Delta\omega}{\omega_k} Q_k(K)$$

Spectral Brightness of Electron Beam

Brightness of electron beam radiating at n^{th} (odd) harmonics in a undulator is given by

$$B_k = F_k / (4\pi^2 \Sigma_x \Sigma'_x \Sigma_y \Sigma'_y)$$

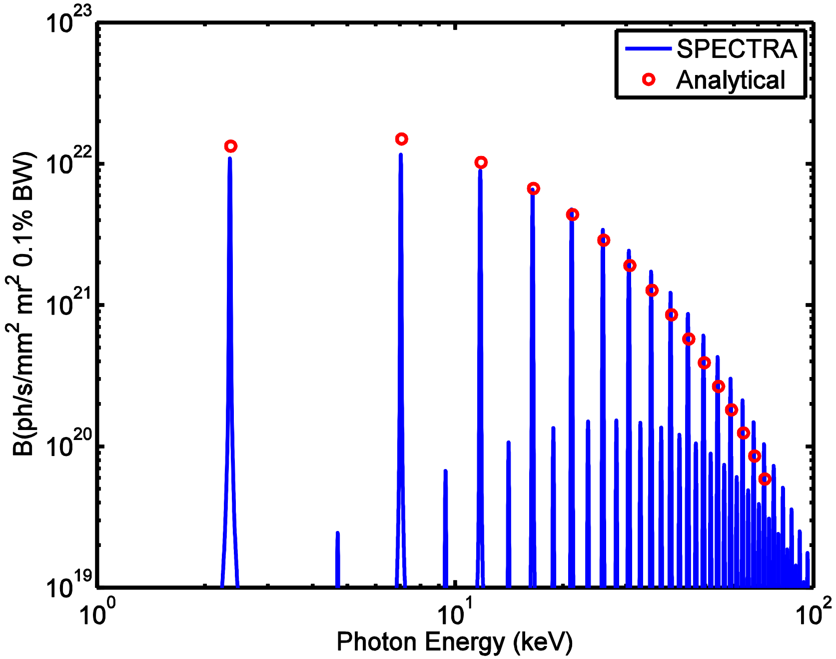
If the electron beam phase space is matched to those of photon's, the brightness becomes optimized

$$B_k = \frac{F_k}{4\pi^2(\varepsilon_x + \lambda_k / 4\pi)(\varepsilon_y + \lambda_k / 4\pi)}$$

Finally, even for zero emittances, there is an ultimate limit for the brightness

$$B_k = \frac{4F_k}{\lambda_k^2}$$

Spectral brightness of PEP-X



A diffraction limited ring at 1 angstrom or 8 pm-rad emittance

Coherent X-Ray Diffraction Imaging
with nanofocused Illumination
C.G. Schroer et al. PRL 101, 090801 (2008)

- Phone energy: 15.25 keV
- Coherent flux: 10^8 ph/s
- Exposure time: 60x10 s
- Resolution: 5 nm
- $\Delta E/E$: $1.4\text{x}10^{-4}$

The total number of photons D_c in the coherence volume available at a given source, however, is bounded from above by

$$D_c = F_c T = \mathrm{Br}\lambda^2 \frac{\Delta E}{E} T,$$

where F_c is the coherent flux, Br is the brilliance of the x-ray source, λ is the wavelength of the x rays, $\Delta E/E$ the degree of monochromaticity, and T the exposure time. For

Improvement of resolution scaled as $D_c^{1/4}$.

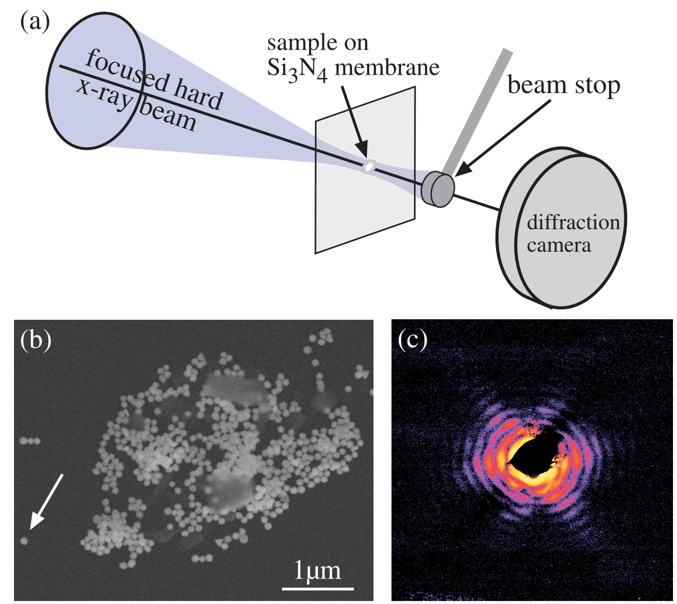
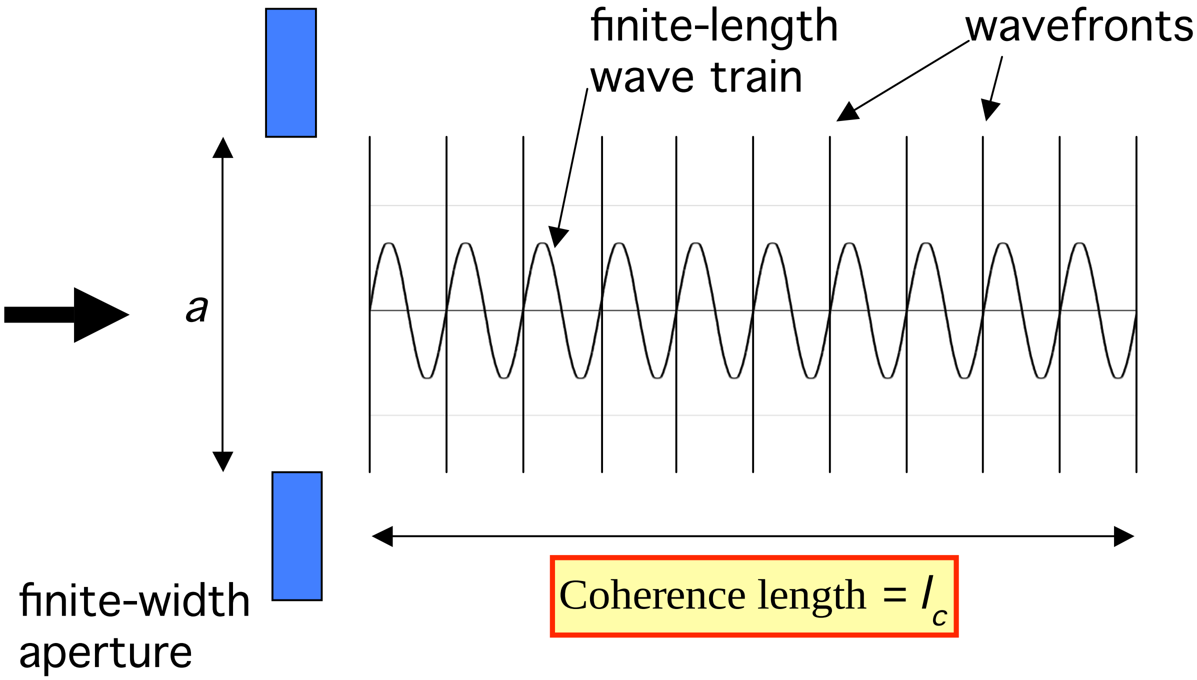


FIG. 1 (color online). (a) Schematic sketch of the coherent diffraction imaging setup with nanofocused illumination. (b) Scanning electron micrograph of gold particles (diameter ≈100 nm) deposited on a Si_3N_4 membrane. (c) Diffraction pattern (logarithmic scale) recorded of the single gold particle pointed to by the arrow in (b) and illuminated by a hard x-ray beam with lateral dimensions of about 100×100 nm^2. The maximal momentum transfer, both in horizontal and vertical direction, is $q = 1.65$ nm^{-1}.

THE DEGREE OF TEMPORAL COHERENCE IS DETERMINED BY THE LENGTH OF THE WAVE TRAIN (MONOCHROMATICITY)



For a wave train of N periods

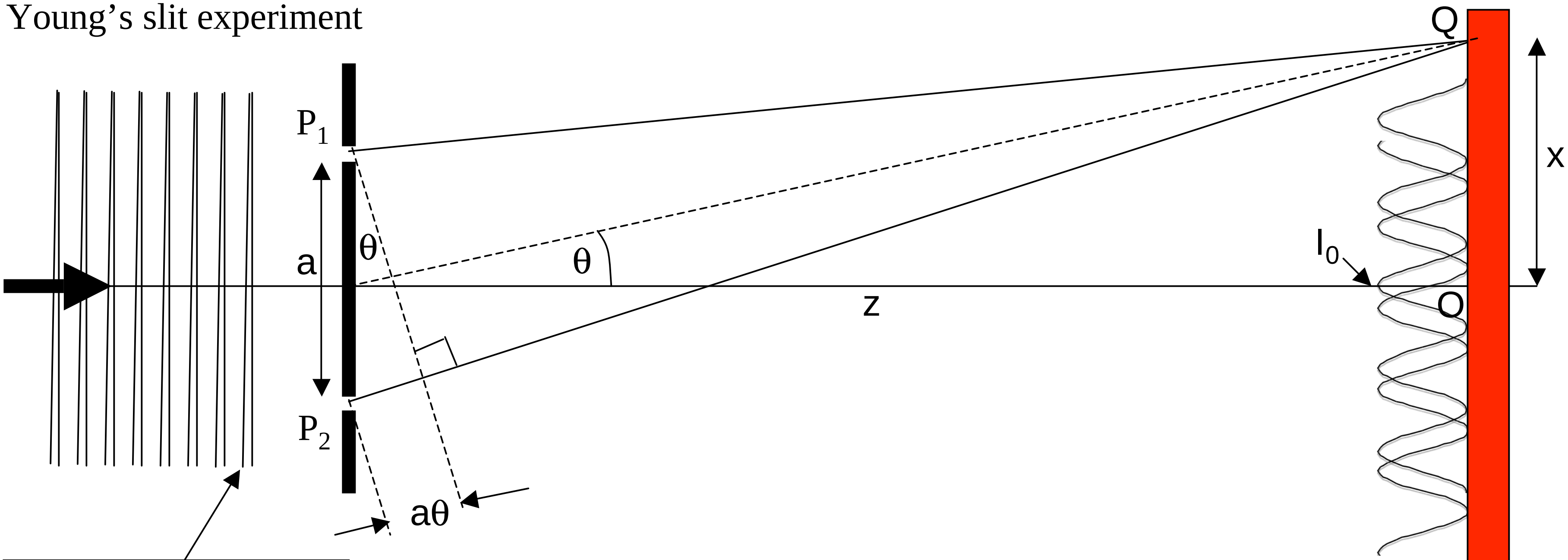
$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta\nu}{\nu} = \frac{\Delta E}{E} \approx \frac{1}{N}$$

$$l_c = N\lambda = \frac{\lambda^2}{\Delta\lambda}$$

- The main point is to make sure that the coherence length is long compared to all path differences between interfering rays in the experiment
- If this is done then the illumination is called ***quasimonochromatic*** and temporal coherence effects are removed from consideration

THE DEGREE OF SPATIAL COHERENCE IS DETERMINED BY THE DEGREE OF COLLIMATION

Young's slit experiment

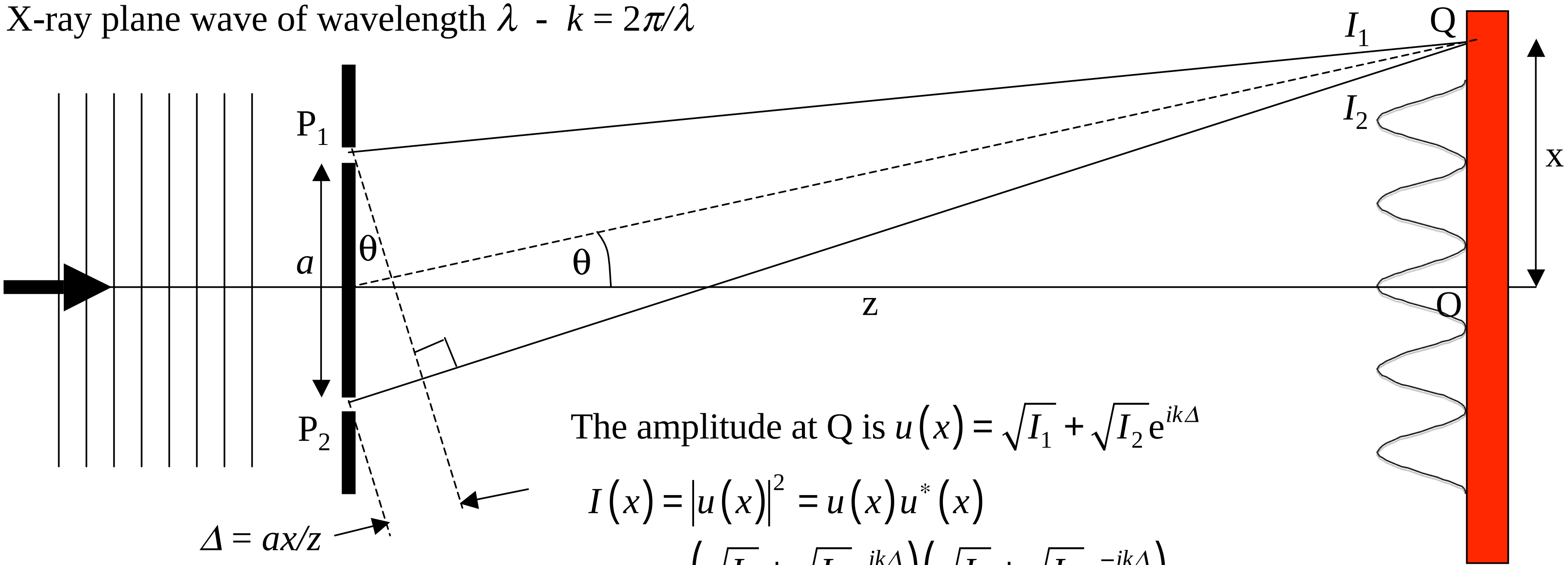


Second wave tilted by $\varepsilon=\lambda/(4a)$ giving an additional path lag of $\lambda/4$ of the signal from P_2 relative to that from P_1

- The fringe blurring caused by $\pm\lambda/4$ path change is considered tolerable so we say that P_1 and P_2 are "coherently" illuminated
- If the beam spread FULL angle is A (equals $\pm\varepsilon$) then the coherence width a is given by the $aA\approx\lambda/2$
- The equation $aA\approx\lambda/2$ is important and *defines* a spatially coherent beam

YOUNG'S SLITS EXPERIMENT IN COHERENT ILLUMINATION

X-ray plane wave of wavelength λ - $k = 2\pi/\lambda$



The amplitude at Q is $u(x) = \sqrt{I_1} + \sqrt{I_2}\,e^{ik\Delta}$

$$I(x) = |u(x)|^2 = u(x)u^*(x)$$

$$= \left(\sqrt{I_1} + \sqrt{I_2}\,e^{ik\Delta}\right)\left(\sqrt{I_1} + \sqrt{I_2}\,e^{-ik\Delta}\right)$$

$$I(x) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(k\Delta)$$

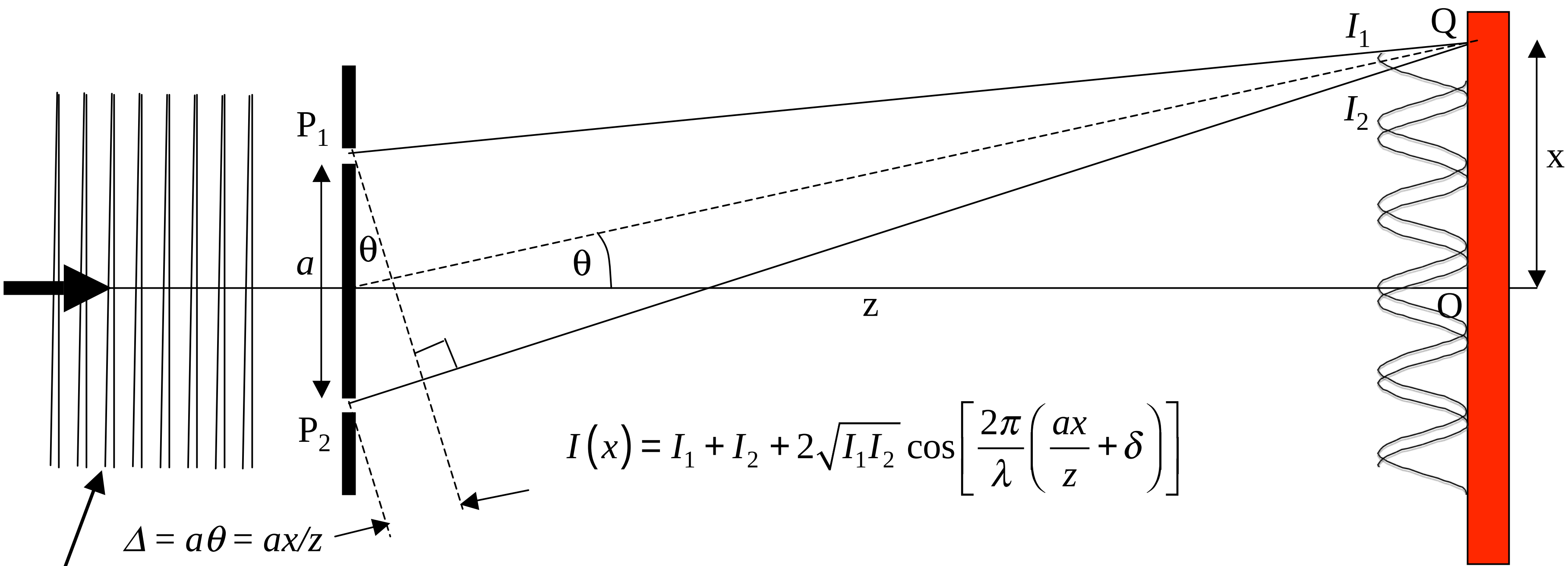
$$= I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left(\frac{2\pi a x}{\lambda z}\right)$$

In this case P_1 and P_2 are coherently iluminated

The fringe visibility **V** is given by

$$\mathbf{V} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{4\sqrt{I_1 I_2}}{2(I_1 + I_2)} \boxed{= 1} \text{ when } I_1 = I_2$$

INTRODUCE A TILTED WAVE TO REPRESENT IMPERFECT COLLIMATION

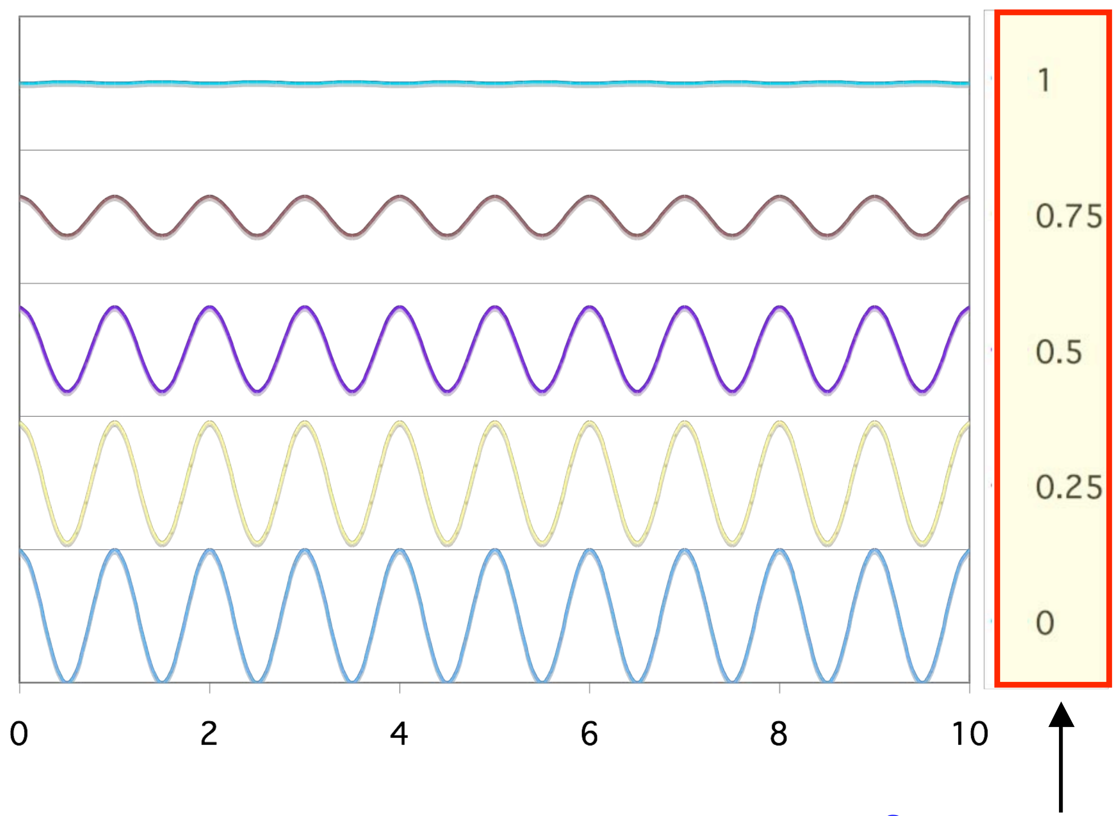


$$I(x) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\left[\frac{2\pi}{\lambda}\left(\frac{ax}{z} + \delta\right)\right]$$

Second wave tilted so that the path difference Δ at P_2 becomes $\Delta + \delta$

- The visibility remains = 1
- The fringes shift down
- A point illumination by an extended source receives a finite angular spread and a **range** of values of δ
- The fringe systems due to all the values of δ are then averaged together
- Resulting in a blurring of the fringes and reduced visibility (contrast)

FRINGE CONTRAST WHEN THE ILLUMINATING BEAM HAS ANGULAR SPREAD



δ range in wavelengths

- The graphs show the loss of fringe contrast when the fringe patterns with all δ values in the given δ range were averaged from $-\delta/2$ to $+\delta/2$
- δ range equals zero is the coherent case
- Note that there is no change in the phase of the fringes because the angular spread was symmetrical
- The zero and maximum of the intensity for each plotted fringe pattern are the axes immediately above and below the plot
- When δ equals one wavelength for example the total beam angular spread is $1\lambda/P_1P_2$

THE UNDULATOR ONE-ELECTRON PATTERN

- The on-axis monochromatic one-electron pattern emitted by an undulator is a spatially-coherent beam - also known as a diffraction-limited beam or a wave mode

- We will model it as a Gaussian laser mode with RMS intensity width and angular width equal to σ_r and $\sigma_{r'}$ - so that the ***width-angle product or emittance*** is given by

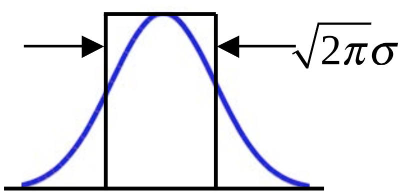
$$\sigma_r \sigma_{r'} = \frac{\lambda}{4\pi}$$

- We will rearrange this using the fact that a rectangle of width $\sqrt{2\pi}\sigma$ and height 1 has equal area to a Gaussian of RMS width σ and height 1 - thus we get

$$\left(\sqrt{2\pi}\sigma_r\right)\left(\sqrt{2\pi}\sigma_{r'}\right) = \frac{\lambda}{2}$$

$$\Delta_c \Delta_c' = \frac{\lambda}{2}$$

Worth remembering this



- Where $\Delta_c = \sqrt{2\pi}\sigma_r$ and $\Delta_c' = \sqrt{2\pi}\sigma_{r'}$ - this is the relation you use to choose beam-line slit widths to get a coherent beam

- This is now the same as our earlier representation of a spatially coherent beam

$$aA = \frac{\lambda}{2}$$

References

1) J.D. Jackson, *Classical Electrodynamics*, Third Edition, John Wiley & Son, Inc. 1999
2) H. Wiedemann, *Synchrotron Radiation*, Springer-Verlag Berlin Heidelberg 2003
3) Kwang-Je Kim, “Characteristics of Synchrotron Radiation,” AIP Proc. No. 184 (AIP, New York, 1989), pp. 565–632
4) Malcolm Howells, ESRF lecture series of coherent X-ray and their applications