Lecture 7:

Lie Algebra

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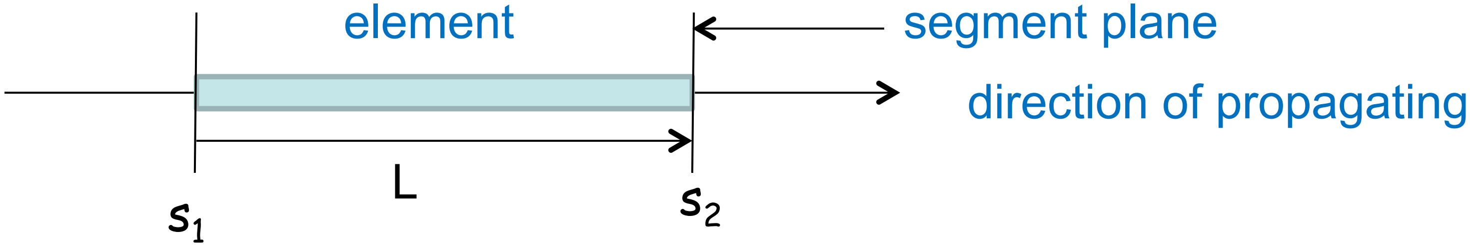
Concept of Transfer Map



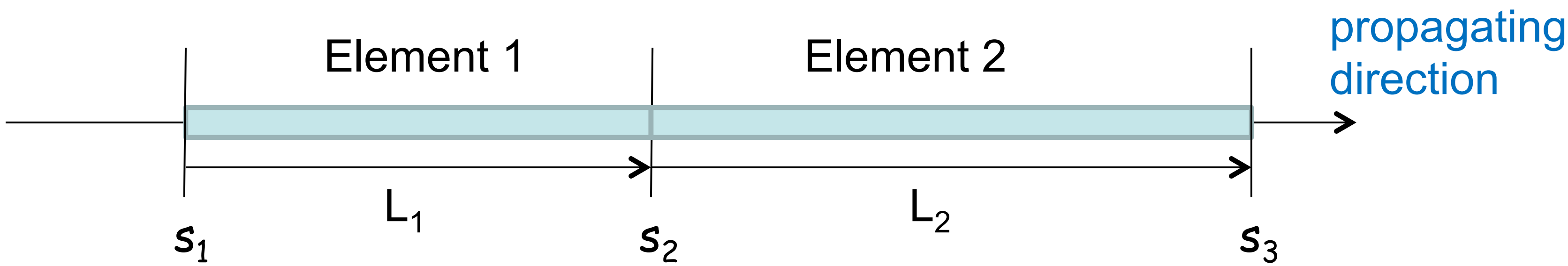
$$z(s_1) = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ \ell \end{pmatrix}_{|s_1} \quad \mathcal{M}_{1\to 2} \quad \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \delta \\ \ell \end{pmatrix}_{|s_2} = z(s_2)$$

$$z(s_2) = \mathcal{M}_{1\to 2}(z(s_1)).$$

abbreviated map notation

A set (six) of functions of canonical coordinates. It's called symplectic if its Jacob is symplectic.

Concatenation of Maps



If we have the transfer map for each individual elements:

$$z(s_2) = \mathcal{M}_{1\to2}(z(s_1)),$$
$$z(s_3) = \mathcal{M}_{2\to3}(z(s_2)).$$

Then the transfer map for the combined elements is given by

$$z(s_3) \underbrace{\boxed{= \mathcal{M}_{1\to2} \circ \mathcal{M}_{2\to3}}}_{\mathcal{M}_{1\to3}} (z(s_1)) \equiv \mathcal{M}_{2\to3}(\mathcal{M}_{1\to2} z(s_1)),$$

nested functions

Poisson Bracket

Given coordinate q_i, and its conjugate momentum p_i, the Poisson bracket is defined as,

$$[f,g] = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

Fundamental brackets:

$$[q_i, q_j] = 0$$

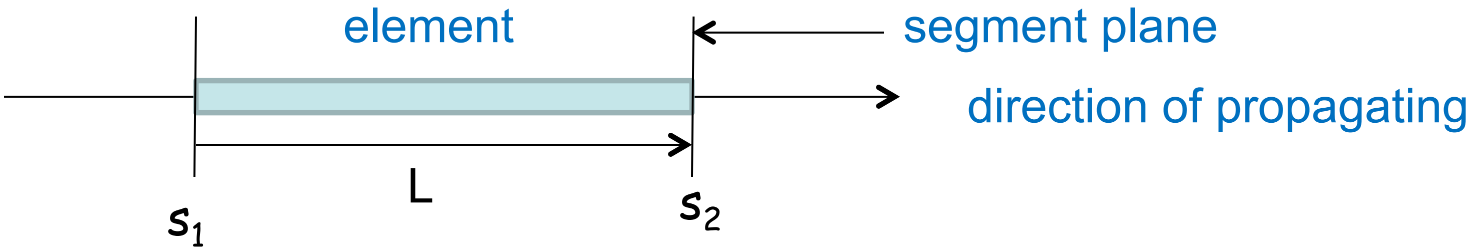
$$[p_i, p_j] = 0$$

$$[q_i, p_j] = \delta_{ij}$$

It is closely resemble the commutator in Quantum mechanics. It acts like a derivative with respect to its conjugate, for example,

$$[q_1, f] = \frac{\partial f}{\partial q_1}$$

Taylor Series and Exponential Lie Operator



For any function f(s), we have the Taylor expansion

$$f(s_2) = \sum_{n=0}^{\infty} \frac{L^n}{n!} \frac{d^n f}{ds^n}\Big|_{s_1} \equiv e^{L\frac{d}{dx}} f(s)\Big|_{s_1}$$

a symbolic notation

In particular, if there is no explicit dependent of s in the function f(s), namely f(s) = f(x(s),p_x(s),…), we have

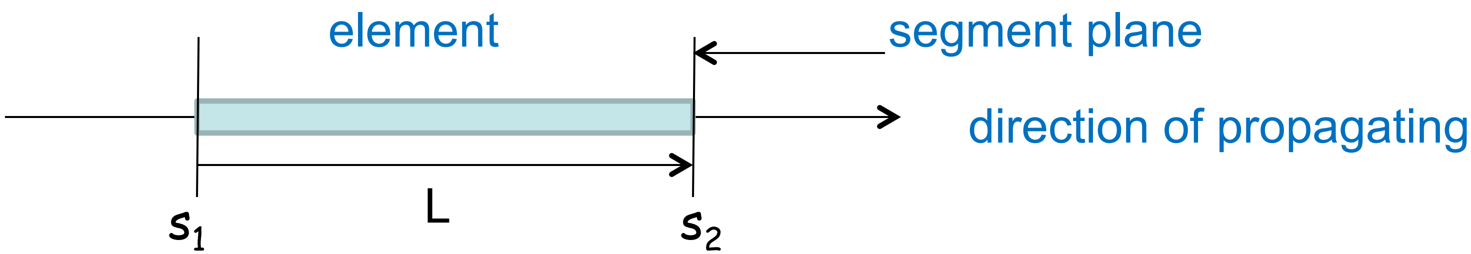
$$\frac{df}{ds} = -[H, f] \equiv -:H:f,$$

another symbolic notation

Used Hamiltonian equation and the definition of the Poisson bracket. Combining these symbolic notations, we have the exponential Lie operator

$$f(s_2) = e^{-L:H:} f(s)\Big|_{s_1}$$

Lie Operator as a Transfer Map



In the previous slide, we have shown that

$$f(s_2) = e^{-L:H:} f(s)\big|_{s_1}.$$

If we apply this formula to a particular function: z=x, or p_x, or y, or p_y, or δ or l, and then we have

$$z(s_2) = e^{-L:H:} z(s_1).$$

Therefore, this exponential Lie operator is a transfer map. We have

$$\mathcal{M}_{1\to 2} = e^{-L:H:}$$

An Example of a Drift

Hamiltonian in the paraxial approximation is given by

$$H_D = \frac{1}{2(1+\delta)}(p_x^2 + p_y^2).$$

It is easy to show that the exponential Lie operator indeed generates the transfer map we have found by solving the Hamiltonian equation. Namely, we have

$$x_f = e^{-L:H_D:}x_{|i} = x_i + \frac{p_{xi}}{1+\delta_i}L,$$

$$p_{xf} = e^{-L:H_D:}p_{x|i} = p_{xi},$$

$$y_f = e^{-L:H_D:}y_{|i} = y_i + \frac{p_{yi}}{1+\delta_i}L,$$

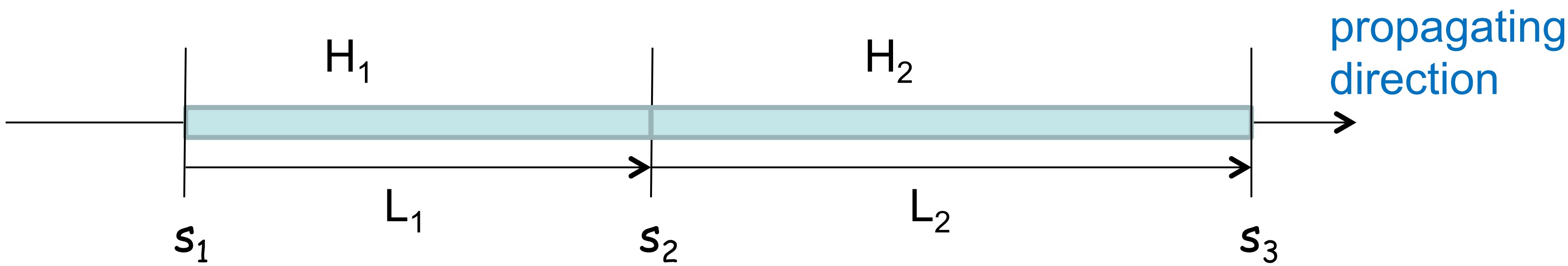
$$p_{yf} = e^{-L:H_D:}p_{y|i} = p_{yi},$$

$$\delta_f = e^{-L:H_D:}\delta_{|i} = \delta_{i,}$$

$$\ell_f = e^{-L:H_D:}\ell_{|i} = \ell_i + \frac{L}{2(1+\delta_i)^2}(p_{xi}^2 + p_{yi}^2).$$

However, most time, it is easier to obtain the transfer map by solving the Hamiltonian equation.

Lie Operators and Map Concatenation



just shown

It is obvious that

$$f(s+L) = e^{-L:H:} f(x, p_x, \ldots) = f(e^{-L:H:} x, e^{-L:H:} p_x, \ldots) = f(x(s+L), p_x(s+L), \ldots)$$

just shown

obviously true

The Lie operator acts only on the arguments of function . This precisely the definition of the map concatenation we introduced early. So we have

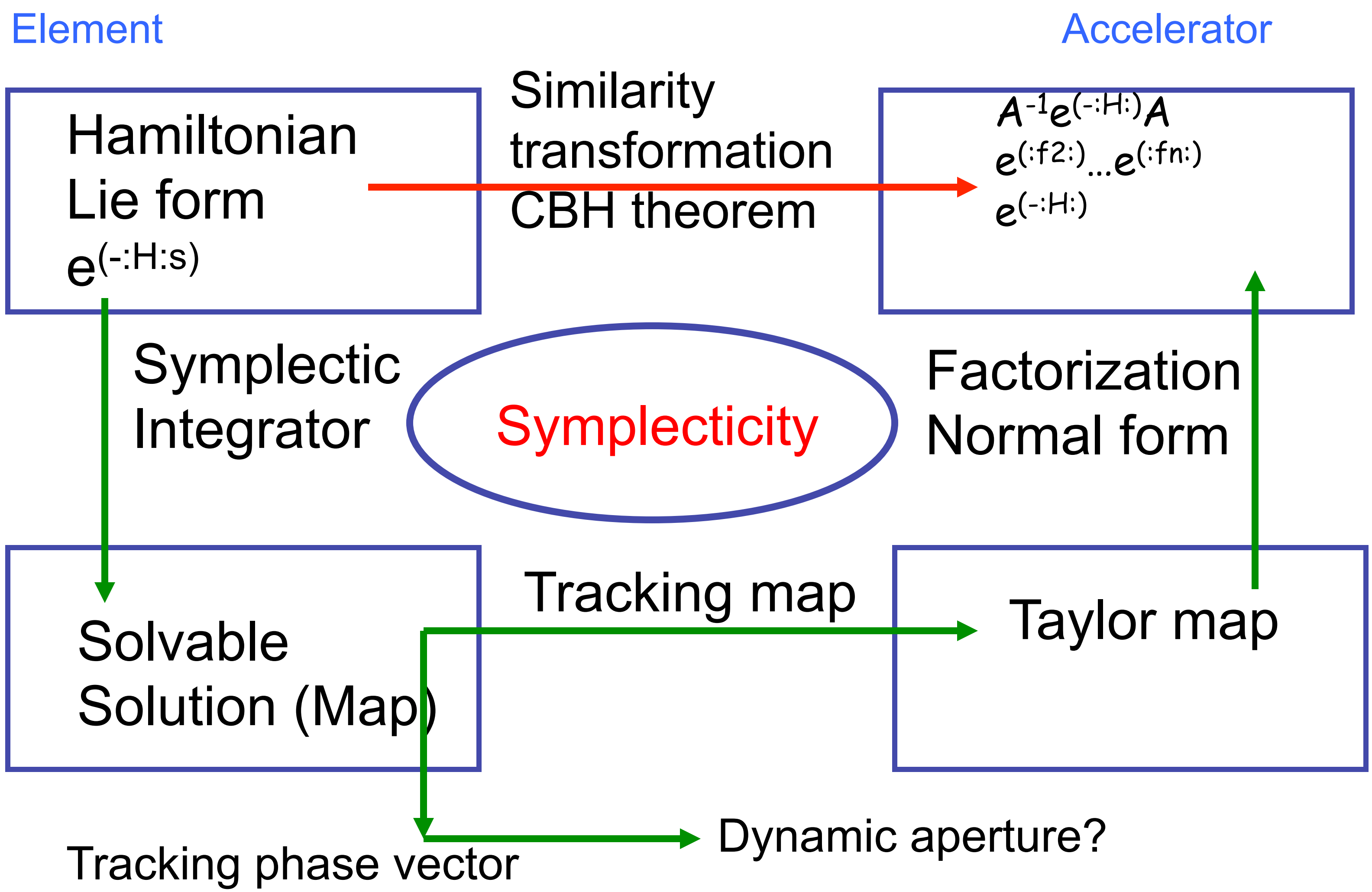
$$\mathcal{M}_{1\to 3} = \mathcal{M}_{1\to 2} \circ \mathcal{M}_{2\to 3} = e^{-L_1:H_1:} e^{-L_2:H_2:}.$$

The dot is removed because Lie operator automatically has the property.

Map and Matrix

- Concatenation of linear maps leads to multiplication of matrices
- Be careful about the order
  - Map: first is first $\mathcal{M}_{1\to 2} \circ \mathcal{M}_{2\to 3}$
  - Matrix: last is first $M_{2\to 3} \cdot M_{1\to 2}$
- Maps are not limited to the linear ones. In Hamiltonian system, maps are symplectic.
- If one uses Taylor map as an approximation. Its zeroth order can be considered as a reference orbit. Therefore their concatenation includes the feed-down effects of magnets
  - For example, a horizontal offset of sextupole generates a quadrupole field

Lie Method Bases Analysis and Tracking Code



Similarity Transformation

$$e^{:A:}e^{:B:}e^{-:A:} = e^{:e^{:A:}B:}$$

Here is a proof. Set $f=e^{-:A:}g$, so we have

$$\begin{aligned}
e^{:A:}e^{:B:}f &= e^{:A:}\sum \frac{1}{n!}[B,[B,\ldots[B,f]\ldots]]] \\
&= \sum \frac{1}{n!}e^{:A:}[B,[B,\ldots[B,f]\ldots]] \\
&= \sum \frac{1}{n!}[e^{:A:}B,e^{:A:}[B,\ldots[B,f]\ldots]] \\
&= \sum \frac{1}{n!}[e^{:A:}B,[e^{:A:}B,\ldots[e^{:A:}B,e^{:A:}f]\ldots]] \\
&= \sum \frac{1}{n!}[e^{:A:}B,[e^{:A:}B,\ldots[e^{:A:}B,g]\ldots]] \\
&= e^{:e^{:A:}B:}g
\end{aligned}$$

We used $e^{:A:}[f_1,f_2]=[e^{:A:}f_1,e^{:A:}f_2]$ ($e^{:A:}[x,p_x]=[e^{:A:}x,e^{:A:}p_x]$)

The Cambell-Baker-Hausdorf (CBH) Theorem

To combine two exponential Lie operators, we have

$$e^{:A:}e^{:B:} = e^{:A+B+\frac{1}{2}[A,B]+\ldots:}$$

The bracket notes the Poisson bracket. This theorem can be shown easily using the definition of the exponential Lie operator and the Jacob identity for the Poisson brackets:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

In general, it should be considered as a part of perturbation theory. It is good when A and B are small.

$$e^{:A:}e^{:B:} = e^{:A+B:}$$ (actually, this is an exact result)

This a necessary condition for the exponential Lie operator being a transfer map of the element that can be described by a Hamiltonian.

Linear Similarity Transformation

A specially useful transformation is given by

$$\mathcal{M}e^{:B(z):}\mathcal{M}^{-1} = e^{:B(\mathcal{M}z):}.$$

Here, *M* is a symplectic linear map. As an example, let us to consider a pair of identical thin lens sextupoles with integrated strength S_2,separated by –I transformation. For simplicity, we limit to the transverse dimensions. The transfer map is given by

$$e^{-:\frac{s_2}{6}(x^3-3xy^3):}(-I)e^{-:\frac{s_2}{6}(x^3-3xy^3):}$$

$$=(-I)(-I)e^{-:\frac{s_2}{6}(x^3-3xy^3):}(-I)e^{-:\frac{s_2}{6}(x^3-3xy^3):}$$

$$=(-I)e^{-:\frac{s_2}{6}((-x)^3-3(-x)(-y)^3):}e^{-:\frac{s_2}{6}(x^3-3xy^3):}$$

$$=(-I)$$

← Similarity transformation

← CBH theorem

We obtain the well-known result by Karl Brown.

Courant-Synder Invariance

It can be shown that the Lie operator:

$$\exp[-\mu_x : \frac{(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2)}{2}]:$$

is the transfer map of the Courant-Synder matrix:

$$M_x = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma_x \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix}$$

Gaussian beam distribution:

$$\rho_G(x, p_x) \propto \exp[-\frac{(\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2)}{2\lambda_x}]$$

Calculate Nonlinear Hamiltonian

Consider a set of multipoles separated by linear maps. We can represent the nonlinear transfer map by

$$\mathcal{M}_{0,1} e^{-:V_1(z)} \mathcal{M}_{1,2} e^{-:V_2(z)} \ldots \mathcal{M}_{n-1,n} e^{-:V_n(z)} \mathcal{M}_{n,n+1}$$

$$= \mathcal{M}_{0,1} e^{-:V_1(z):} \mathcal{M}_{1,2} e^{-:V_2(z):} \ldots \mathcal{M}_{n-1,n} \mathcal{M}_{n,n+1} \mathcal{M}_{n,n+1}^{-1} e^{-:V_n(z):} \mathcal{M}_{n,n+1}$$

$$= \mathcal{M}_{0,1} e^{-:V_1(z):} \mathcal{M}_{1,2} e^{-:V_2(z):} \ldots \mathcal{M}_{n-1,n} \mathcal{M}_{n,n+1} e^{-:V_n(\mathcal{M}_{n,n+1}^{-1} z):}$$

similarity transformation

$$= \mathcal{M}_{0,n+1} e^{-:V_1(\mathcal{M}_{1,n+1}^{-1} z):} e^{-:V_2(\mathcal{M}_{2,n+1}^{-1} z):} \ldots e^{-:V_n(\mathcal{M}_{n,n+1}^{-1} z):}$$

$$= \mathcal{M}_{0,n+1} e^{-:H_{NL}:}$$

CBH theorem

z is the coordinates at the end.

We can use the similarity transformation and CBH theorem to obtain an effective Hamiltonian so that the transfer map consists of a linear map followed by an exponential Lie operator for the nonlinearity. It is very useful for understanding of the nonlinear effects and their compensation. Clearly, it is an approximation to a real accelerator.

Perturbation of Sextupoles

For a sextupole magnet, we have the potential

$$V_S(x,y) = \frac{S_2}{6}(x^3 - 3xy^2).$$

So

$$V_S(\mathcal{M}_{i,n+1}^{-1} z) = \frac{S_2}{6}(x_i^3 - 3x_i y_i^2),$$

with

$$x_i = \sqrt{\beta_x(s_i)}(\cos\Delta\psi_{i,n+1}x - \sin\Delta\psi_{i,n+1}p_x),$$
$$y_i = \sqrt{\beta_y(s_i)}(\cos\Delta\phi_{i,n+1}y - \sin\Delta\phi_{i,n+1}p_y),$$

where x,p_x,y,p_y are the normalized coordinates at n+1 position. The effective Hamiltonian based on the CBH theorem is

$$H_{NL} = \sum_{i=1}^{n} \frac{S_2^i}{6}(x_i^3 - 3x_i y_i^2) + \frac{1}{2}\sum_{i<j}^{n} \frac{S_2^i S_2^j}{36}[(x_i^3 - 3x_i y_i^2),(x_j^3 - 3x_j y_j^2)]$$

The first term is the same as the first-order canonical perturbation theory. Except the reference point is the end rather then the beginning.

Nonlinear Hamiltonian of Sextupoles

The Poisson bracket can be evaluated, we have

$$H_{NL} = \sum_{i=1}^{n} \frac{S_2^i}{6}(x_i^3 - 3x_i y_i^2)$$

$$+\frac{1}{2}\sum_{i<j}^{n} S_2^i S_2^j \sqrt{\beta_{x,i}\beta_{x,j}} \{\sin(\Delta\phi_{i,n+1} - \Delta\phi_{j,n+1})\beta_{y,i}\beta_{y,j} x_i x_j y_i y_j$$

$$+\sin(\Delta\psi_{i,n+1} - \Delta\psi_{j,n+1})(\beta_{x,i}x_i^2 - \beta_{y,i}y_i^2)(\beta_{x,j}x_j^2 - \beta_{y,j}y_j^2)/4\}$$

We see that two sextupoles generate octopole like terms.

Second-Order Symplectic Integrators

Separate Hamiltonian into two exactly solvable parts:

$$H = H_0 + H_1$$

Approximation with symplectic integrators:

propagator

$$e^{-:H:L} = \prod_{i=1}^{n} e^{-:H:\Delta s} \approx \prod_{i=1}^{n} [e^{-\frac{:H_0:}{2}\Delta s} e^{-:H_1:\Delta s} e^{-\frac{:H_0:}{2}\Delta s} + O(\Delta s)^3]$$

exact kick error

1. The residual term can be easily see from the CBH theorem
2. It becomes the exact solution at the limit of infinite number of segments
3. Preserves symplectic condition during the integration

$$e^{-\frac{1}{2}\Delta s:H_0:} e^{-\Delta s:H_1:} e^{-\frac{1}{2}\Delta s:H_0:} = e^{-\frac{1}{2}\Delta s:H_0:} e^{-\Delta s:(H_1-\frac{1}{2}H_0)+\frac{\Delta s^2}{4}[H_1,H_0]:+O(\Delta s)^3:}$$

$$= e^{-\Delta s:(H_1+H_0):+O(\Delta s)^3}$$

Summary

- Lie algebra is a powerful method for nonlinear analysis. It is equivalent to the Hamiltonian perturbation
- Exponential Lie operator is a representation of the transfer map
  – Used to derive symplectic integrator
  – Define transfer map of a beamline
- Similarity transformation and the CBH theorem are two important tools in the Lie method
  – Derive the nonlinear Hamiltonian
- Courant-Synder invariance can be used as a quadratic polynomial for a Lie operator

References

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