Magnetic dipole moment and BMT equation

D. Rubin
USPAS
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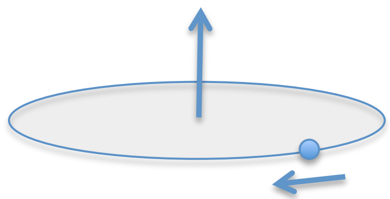
Magnetic moment

$$\mu = \frac{e\hbar}{2mc}$$

Fundament unit of magnet moment $\left[\frac{IA}{c}\right]$



Charge e
Mass m
Ang Mom $\hbar$

=> Magnetic moment of muon

$$\mu = g\frac{e\hbar/2}{2m_\mu c}$$

$$\frac{p^2}{2m}\psi = E\psi$$

Schrodinger Equation - ψ is a scalar function

With minimal coupling

$$\frac{(\mathbf{p} - \frac{e}{c}\mathbf{A})^2}{2m}\psi = (E - e\phi)\psi$$

Pauli suggested turning scalar wave function into two-component spinor with the substitution

$$\frac{1}{2m}[\sigma \cdot (\mathbf{p} - \frac{e}{c}\mathbf{A})]^2\psi = (E - e\phi)\psi$$

$$\frac{1}{2m}[\mathbf{p} - \frac{e}{c}\mathbf{A}]^2\psi - \frac{e\hbar}{2mc}\sigma \cdot \mathbf{B}\psi = (E - e\phi)\psi$$

using $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k \qquad (x_i, p_j) = i\hbar\delta_{ij}$

$$\mu = \frac{e\hbar}{2mc}2\mathbf{s}, \text{ where } \mathbf{s} = \frac{\sigma}{2} \to g = 2$$

Dirac found spin was essential for compatibility with relativity

Dirac equation(s) in momentum space

$$(E - \sigma \cdot \mathbf{p})\chi^+ = m\chi^-$$

$$(E + \sigma \cdot \mathbf{p})\chi^- = m\chi^+$$

and $\psi = \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}$

Substitute one into the other to get Klein Gordon equation

$$(E^2 - \mathbf{p}^2)\chi^\pm = m^2\chi^\pm$$

In the Ultra-relativistic limit $pc \gg m$

$$\frac{\sigma \cdot \mathbf{p}}{E}\chi^\pm = \pm\chi^\pm$$

$\chi^\pm$ Are eigenkets of helicity with eigenvalues $\frac{1}{2}$

In the nonrelativistic limit, $pc \ll m$ Consider an alternative basis for

$$\Psi = \frac{1}{2}(\chi^+ + \chi^-)$$
$$\Phi = \frac{1}{2}(\chi^+ - \chi^-)$$

$$\begin{aligned} E\Psi - \sigma \cdot \mathbf{p}\Phi &= m\Psi \\ E\Phi - \sigma \cdot \mathbf{p}\Psi &= -m\Phi \end{aligned}$$

$$\Phi \approx \frac{\sigma \cdot \mathbf{p}}{2m}\Psi$$

$$\rightarrow \frac{(\sigma \cdot \mathbf{p})}{2m}\Psi = (E - m)\Psi$$

Just like Pauli's hypothesis

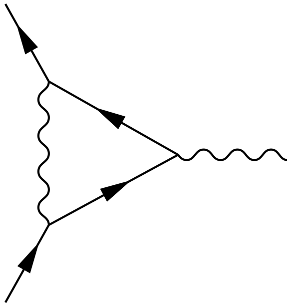
Dirac equation predicts g=2

But then what is there to measure?

Dirac field $\chi^{\pm}$ and E-M field $\mathbf{A}$ are classical fields − continuous throughout space

Interaction $\mu \cdot \mathbf{B}$ is likewise classical

Quantization of fields by introducing creation and annihilation operators for muons photons =>



$$\frac{g}{2} - 1 \sim \frac{\alpha}{2\pi}$$

Differential equation for spin in E&M fields

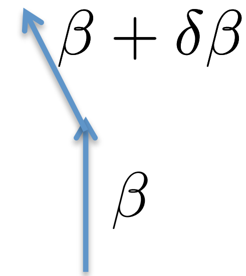
In the rest frame of the muon $$\frac{d\mathbf{s}}{dt'} = \frac{ge}{2mc}\mathbf{s} \times \mathbf{B}'$$

In the lab frame $$t' \to \gamma t, \quad \mathbf{B}' \to \gamma(\mathbf{B} - \beta \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1}\beta(\beta \cdot \mathbf{B})$$

And because the rest frame of the muon is accelerating, there is a Thomas precession of the rest frame with frequency ω_T

$$\left.\frac{d\mathbf{G}}{dt}\right)_{non-rot} = \left.\frac{d\mathbf{G}}{dt}\right)_{rot} + \omega_{\mathbf{T}} \times \mathbf{G}$$

Thomas precession



$$x' = A_{boost}(\vec{\beta})x$$

$$x'' = A_{boost}(\vec{\beta} + \delta\vec{\beta})x$$

$$x'' = A_b(\vec{\beta} + \delta\vec{\beta})A_b^{-1}(\vec{\beta})x'$$

$$x'' = A_b(\vec{\beta} + \delta\vec{\beta})A_b(-\vec{\beta})x'$$

To first order in $\delta\vec{\beta}$

$$A_b(\vec{\beta} + \delta\vec{\beta})A_b^{-1}(\vec{\beta}) = A_b(\vec{\Delta}\beta)R(\vec{\Delta}\Omega)$$

$$\boldsymbol{\Delta}\tilde{\beta} = \gamma^2\delta\vec{\beta}_{||} + \gamma\delta\vec{\beta}_{\perp}$$

$$\Delta\vec{\Omega} = \frac{\gamma^2}{\gamma + 1}\beta \times \delta\beta$$

To the observer in the rest frame, the frame is rotating by $\Delta\Omega$
In each time interval δt during which the $\beta \to \beta + \delta\beta$

This equation works if there is a boost but not a rotation. $\frac{d\mathbf{s}}{dt'} = \frac{ge}{2mc}\mathbf{s} \times \mathbf{B}'$

We find a non-rotating rest frame by boosting but not rotation

$$x''' = A_b(\Delta\vec{\beta})x'$$
$$\Rightarrow x''' = R(-\Delta\Omega)A_b(\vec{\beta} + \delta\vec{\beta})x$$

The nonrotating rest frame rotates in the lab frame by $-\Delta\Omega$

$$\Delta\vec{\Omega} = \frac{\gamma^2}{\gamma+1}\beta \times \delta\beta$$

Thomas precession $\omega_T = -\lim_{\delta t \to 0}\frac{\Delta\Omega}{\delta t} = \frac{\gamma^2}{\gamma+1}\dot{\tilde{\beta}} \times \tilde{\beta}$

Thomas precession for circular motion

$$\omega_T = -\lim_{\delta t \to 0} \frac{\Delta\Omega}{\delta t} = \frac{\gamma^2}{\gamma+1} \dot{\tilde{\beta}} \times \tilde{\beta}$$

$$a = \frac{v^2}{r}, \omega = \frac{v}{r}$$

$$\begin{aligned} \omega_T &= \frac{\gamma^2}{\gamma+1} \left(\frac{v^2}{rc} \right) \frac{v}{c} \\ &= \frac{\gamma^2}{\gamma+1} \beta^2 \frac{\omega}{c^2} \\ &= (\gamma - 1)\omega \end{aligned}$$

Thomas-BMT equation

In the rest frame of the muon $$\frac{d\mathbf{s}}{dt'} = \frac{ge}{2mc}\mathbf{s} \times \mathbf{B}'$$

In the lab frame $$\frac{d\mathbf{s}}{dt} = \frac{1}{\gamma}\frac{ge}{2mc}\mathbf{s} \times \mathbf{B}'$$

$$\mathbf{B}' \rightarrow \gamma(\mathbf{B} - \beta \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1}\beta(\beta \cdot \mathbf{B})$$

The rest frame of the muon is accelerating. There is a Thomas precession of the rest frame with frequency ω_T

$$\left.\frac{d\mathbf{G}}{dt}\right)_{non-rot} = \left.\frac{d\mathbf{G}}{dt}\right)_{rot} + \omega_{\mathbf{T}} \times \mathbf{G}$$

$$\frac{d\mathbf{s}}{dt} = \frac{1}{\gamma}\frac{ge}{2mc}\mathbf{s}\times\gamma(\mathbf{B} - \beta \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1}\beta(\beta\cdot\mathbf{B}) - \mathbf{s}\times\frac{\gamma^2}{\gamma+1}\dot{\beta}\times\beta$$

$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \beta \times \mathbf{B}\right)$$

$$\rightarrow \frac{d\vec{\beta}}{dt} = \frac{e}{\gamma mc}[\mathbf{E} + \beta\times\mathbf{B} - \beta(\beta\cdot\mathbf{E})]$$

$$\frac{d\mathbf{s}}{dt} = \frac{e}{mc}\mathbf{s}\times\left[\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - \left(\frac{g}{2} - 1\right)\frac{\gamma}{\gamma+1}(\beta\cdot\mathbf{B})\beta - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\beta\times\mathbf{E}\right]$$

$$\frac{d\mathbf{s}}{dt} = \mathbf{s}\times\mathbf{\Omega_p}$$

$$\mathbf{\Omega_p} = \frac{e}{mc}\left[\left(\frac{g}{2} - 1 + \frac{1}{\gamma}\right)\mathbf{B} - \left(\frac{g}{2} - 1\right)\frac{\gamma}{\gamma+1}(\beta\cdot\mathbf{B})\beta - \left(\frac{g}{2} - \frac{\gamma}{\gamma+1}\right)\beta\times\mathbf{E}\right]$$

$$\frac{d\mathbf{s}}{dt} = \mathbf{s}\times\mathbf{\Omega_p}$$

$$\rightarrow \frac{d\vec{\beta}}{dt} = \frac{e}{\gamma mc}[\mathbf{E} + \beta\times\mathbf{B} - \beta(\beta\cdot\mathbf{E})]$$

Suppose $\mathbf{E} \rightarrow \mathbf{E}_\perp$

$$\frac{d\vec{\beta}}{dt} = \frac{e}{\gamma mc}[\mathbf{E}_\perp + \beta\times\mathbf{B}] = \frac{e}{\gamma mc}\beta\times[(\mathbf{E}\times\beta)\left(\frac{\gamma^2}{\gamma^2-1}\right) + \mathbf{B}] = \beta\times\mathbf{\Omega_c}$$

$$\mathbf{\Omega_p} - \mathbf{\Omega_c} = \frac{e}{mc}\left[\left(\frac{g}{2} - 1\right)\mathbf{B} - \left(\frac{g}{2} - 1\right)\frac{\gamma}{\gamma+1}(\beta\cdot\mathbf{B})\beta - \left(\frac{g}{2} - 1 - \frac{1}{\gamma^2-1}\right)\beta\times\mathbf{E}\right]$$

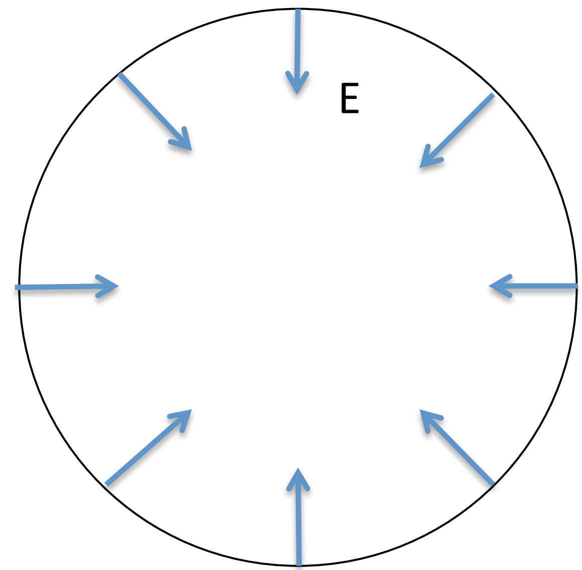
$$\frac{d}{dt}(\hat{\beta}\cdot\mathbf{s}) = \hat{\beta}\cdot\frac{d\mathbf{s}}{dt} + \frac{1}{\beta}[\mathbf{s} - (\hat{\beta}\cdot\mathbf{s})\hat{\beta}]\cdot\frac{d\beta}{dt}$$

$$\frac{d}{dt}(\hat{\beta}\cdot\mathbf{s}) = -\frac{e}{mc}\mathbf{s}_{\perp}\cdot\left[\left(\frac{g}{2}-1\right)\hat{\beta}\times\mathbf{B} + \left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\mathbf{E}\right]$$

Magic momentum

Imagine an all electric ring

A muon, momentum $\mathbf{p}$ moves in circular orbit



$$\frac{d}{dt}(\hat{\beta}\cdot\mathbf{s}) = -\frac{e}{mc}\mathbf{s}_{\perp}\cdot\left[\left(\frac{g}{2}-1\right)\hat{\beta}\times\mathbf{B}+\left(\frac{g\beta}{2}-\frac{1}{\beta}\right)\mathbf{E}\right]$$

As velocity increases …

No precession at low velocity
(no B-field in rest frame)

Suppose g=2
B-field in rest frame increases with velocity
As velocity -> c, precession frequency catches up with cyclotron freq.

g>2
precession frequency catches up more quickly and matches at magic momentum