Momentum distribution

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The amplitude dependence of the tune shift as determined by tracking is plotted in Figure 1 along with the analytic calculation. Analytic and numerical results are in reasonable agreement.

The turn by turn sum of trajectories with and without quad nonlinearities are shown in Figures 4 and 5 respectively. The amplitude of the oscillation is about 3.9 cm. The signal at amplitude is with all quad multipoles included. The amplitude dependence of the tune shift as determined by tracking is shown in Figure 3. The signal at amplitude is with purely linear quadrupole fields, (no multipoles). The red line is with all multipoles included. The amplitude of the oscillation is about 4 cm. For one of the particles, turn out quad multipoles. The oscillation amplitude is modulated at the tune difference. Note that if the tunes for two different muons are split, then the particles are 180° out of phase.

The decoherence time (in units of turns) is \( \frac{1}{340} \) turns.

Amplitude \([m]\)

\begin{tabular}{c|c|c|c|c|c}
Amplitude \([m]\) & -0.08 & -0.06 & -0.04 & 0.02 & 0.04 & 0.06 & 0.08 \\
\hline
Amplitude \([m]\) & -0.08 & -0.06 & -0.04 & 0.02 & 0.04 & 0.06 & 0.08 \\
\end{tabular}
4 Decoherence in a distribution

Consider a distribution of 1000 muons with 95% horizontal and vertical emittance of 40mm-mrad.
Suppose the residual coherent betatron oscillation amplitudes are 10mm horizontally and about 1mm vertically.
Due to the beta mismatch the width of the distribution varies from 6 to 13 mm and the height from 8 to 16mm.
As described above, the effect of the quadrupole nonlinearity is to introduce an amplitude dependent tuneshift.
Vertical and horizontal centroid and width over the first 2000 turns are shown in Figure 6.
Here the quad nonlinearity is turned off so there is no decoherence. The decoherence is evident in the evolution of the distribution with quad nonlinearity restored as shown in Figure 7.
The amplitude of the coherent horizontal betatron oscillation shrinks by a factor of two in 2000 turns.
The variation in the width and height of the distribution also "damps" on a time scale of 1000 turns.
Note that the particles remain bunched, as they all have the same energy.

Figure 4: Sum of displacements with and without quad multipoles (nonlinearities) when the oscillation amplitude is 3.9 cm as in Figure 2. $\Delta Q = 1/838$ turns.

Figure 5: Sum of displacements with and without quad multipoles when the oscillation amplitude is 3.4 cm. $\Delta Q = 1/2470$ turns.
Fast rotation signal on an imaginary detector plane for a single muon

\[ S(t) = \sum_{n=0}^{\infty} \delta(t - nT) \]

And for an off momentum muon

\[ S(t) = \sum_{n=0}^{\infty} \delta(t - nT(1 + \Delta)) \]

For a distribution of momenta

\[ S(t) = \sum_{n=0}^{\infty} \int \rho(\Delta) \delta(t - nT(1 + \Delta))d\Delta \]

Gaussian distribution of momenta

\[ S(t) = \sum_{n=0}^{\infty} \int \frac{e^{-\Delta^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0} \delta(t - nT(1 + \Delta))d\Delta \]

\[ = \sum_{n=0}^{\infty} \int \frac{e^{-\Delta^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0} \delta(\Delta - \left(\frac{t}{nT} - 1\right)) \frac{d\Delta}{nT} \]

\[ = \sum_{n=0}^{\infty} \frac{e^{-\left(\frac{t}{nT} - 1\right)^2/(2\Delta_0^2)}}{\sqrt{2\pi}\Delta_0 nT} \]
Figure 8: Fast rotation signal 0-20 µs. (See script for plotting with gnuplot in Appendix.)

Figure 9: Fast rotation signal from 5 µs to 10 µs

Figure 10: Fast rotation signal from 15 µs to 20 µs.
Our ‘perfect’ signal extends over all time and is periodic
The real measured signal
• Starts a few turns beyond $t_0$
• Muons decay
• Our only signal is when the muon decays (rather than on every turn)
To extract momentum distribution from the fast rotation signal
Consider Fourier transform

We use $\cos$ assuming symmetry about $t=0$

$$F(\omega, t_0) = \int_0^\infty S(t, \Delta_0) \cos \omega(t - t_0) dt$$

$$= \sum_{n=0}^{\infty} \int_0^\infty \frac{e^{-(\frac{t}{nT} - 1)^2/(2\Delta_0^2)}}{\sqrt{2\pi\Delta_0 nT}} \cos \omega(t - t_0) dt$$

$$F(\omega, t_0) = \sum_{n=0}^{\infty} e^{-\omega^2(nT)^2\Delta_0^2/2} \cos \omega(nT - t_0)$$
The fourier transform reproduces the Gaussian momentum distribution.
The measured fast rotation signal starts and ends

The fourier transform is not entirely characterized by harmonics of the rotation frequency but includes all frequencies
Including frequencies that are unphysical, and outside the aperture

\[
\hat{S}(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_0}^{\infty} S(t) \cos \omega(t - t_0) dt
\]

\[
S(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \hat{S}(\omega) \cos \omega(t - t_0) d\omega
\]

Start collecting data at \( t_s \)

\[
\hat{S}_1(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_s}^{\infty} S(t) \cos \omega(t - t_0) dt
\]

Missing bit

\[
\Delta(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_0}^{t_s} S(t) \cos \omega(t - t_0) dt
\]
\[ \Delta(\omega) = \sqrt{\frac{2}{\pi}} \int_{t_0}^{t_s} S(t) \cos \omega(t - t_0) dt = \frac{2}{\pi} \int_{t_0}^{t_s} \int_{0}^{\infty} \hat{S}(\omega') \cos \omega'(t - t_0) \cos \omega(t - t_0) d\omega' dt \]

\[ \Delta(\omega) = \frac{2}{\pi} \int_{t_0}^{t_s} \sum_{n=1}^{\infty} \int_{-\omega_n^+}^{\omega_n^+} S(\omega') \cos \omega'(t - t_0) \cos \omega(t - t_0) d\omega' dt \]

Include only frequencies (or harmonics) within the aperture

\[ \hat{S}(\omega) = S_1(\omega) + \Delta(\omega) \]
Fourier Analysis

1) Fit 9-parameter wiggle

2) Divide out background signals → ω_α, τ_μ and CBO

Making FR spectrum

Producing frequency distribution

Producing radial distribution and estimating #

Partial-time Fourier analysis approach possible because in a weak focusing ring:

Frequency ↔ Radius

From Antoine:

Parabola correction term to account for missing time between...