Lecture 1
Preliminaries

January 24, 2019

Recommended literature

1. A. Chao. *Physics of Collective Beam Instabilities in High Energy Rings*, 1993 (avalable from `http://www.slac.stanford.edu/~achao/wileybook.html`).
2. P. Wilson. *Introduction to Wakefields and Wake Potentials*. AIP Conference Proceedings 184, 1989 (`http://www.slac.stanford.edu/~stupakov/uspas19/1989_Wilson.pdf`).
3. A. Wolsky. "Beam Dynamics in High Energy Particle Accelerators", Imperial College Press, 2014 (see lectures 6,7,8 from USPAS 2013 at `http://pcwww.liv.ac.uk/~awolski/`)
4. K. Y. Ng. *Physics of Intensity Dependent Beam Instabilities*, World Scientific, 2005.
5. J. D. Jackson. Classical Electrodynamics. John Wiley, New York, 1962, 1974, and 1998.

Maxwell's equations

- We will use the MKS system of units.

Classical electrodynamics in vacuum is governed by the Maxwell equations. In the MKS system of units, the equations are

$$\nabla \cdot \boldsymbol{D} = \rho, \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{1.1}$$

where ρ is the charge density, $\boldsymbol{j}$ is the current density, with $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$, $\boldsymbol{H} = \boldsymbol{B}/\mu_0$. $\boldsymbol{D}$ is called the electric displacement, $\boldsymbol{B}$ is called the magnetic induction, and $\boldsymbol{H}$ is called the magnetic field. The term $\partial \boldsymbol{D}/\partial t$ is often called the displacement current.

Maxwell's equations are linear: the sum of two solutions, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$ and $\boldsymbol{E}_2$, $\boldsymbol{B}_2$, is also a solution corresponding to the sum of densities $\rho_1 + \rho_2$, $\boldsymbol{j}_1 + \boldsymbol{j}_2$.

Boundary conditions

Proper boundary conditions should be specified in each particular case. On a surface of a good conducting metal the boundary condition requires that the tangential component of the electric field is equal to zero, $\boldsymbol{E}_t|_S = 0$ (we will talk more about the boundary condition in L5).

In some cases one can neglect the boundaries. We then need to solve the Maxwell equations in *free space*. This usually means that the fields tend to zero at infinity.

We will talk more about the boundary conditions on the surface of a good conductor; they are often called the Leontovich boundary condition.

SI versus Gaussian system of units

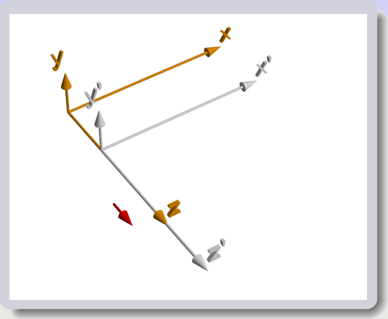
We will use the *vacuum impedance* Z_0

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377 \text{ Ohm} \tag{1.2}$$

In CGS units $Z_0 = 4\pi/c$.
We also have

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{1.3}$$

The Lorentz transformation



Consider two coordinate systems, K (laboratory) and K' (beam). The system K' is moving with velocity v in the z direction relative to the system K. The coordinates of an event in both systems are related by the Lorentz transformation

$$\begin{aligned} x &= x', \quad y = y' & x' &= x, \quad y' = y\,, \\ z &= \gamma(z' + \beta c t') & z' &= \gamma(z - \beta c t)\,, \\ t &= \gamma(t' + \beta z'/c) & t' &= \gamma(t - \beta z/c) \end{aligned} \tag{1.4}$$

where $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$.
The total energy of a relativistic particle is

$$E = \gamma m c^2 \tag{1.5}$$

The electron mass is $m_e = 0.511\ \mathrm{MeV}/c^2$, for the proton $m_p = 0.938\ \mathrm{GeV}/c^2$.

The Lorentz factor γ

Our interest is with *relativistic* beams. In the limit $\gamma \gg 1$, a useful approximation is

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2} \tag{1.6}$$

Example: 10 GeV electrons have $\gamma \approx 2 \times 10^4$, so $1 - \beta \approx 1.3 \times 10^{-9}$ ($c - v = 1.4$ km/h). 7 TeV protons in LHC have $\gamma = 7.4 \times 10^3$.

Lorentz contraction and time dilation

Two events occurring in the moving frame at the same point and separated by the time interval $\Delta t'$ will be measured by the lab observers as separated by Δt,

$$\Delta t = \gamma \Delta t'$$

This is the effect of relativistic *time dilation*. Example: the lifetime of muons is 2.4 μs $\rightarrow$ muon collider.

An object of length ℓ' aligned in the moving frame with the z' axis will have the length ℓ in the lab frame:

$$\ell = \frac{\ell'}{\gamma}$$

This is the effect of relativistic *contraction*. The length in the direction transverse to the motion is not changed. Example: the bunch length in the beam frame.

Lorentz transformation of fields

The electromagnetic field is transformed from K' to K according to the following equations

$$
\begin{aligned}
E_z &= E'_z, \qquad & \boldsymbol{E}_\perp &= \gamma\left(\boldsymbol{E}'_\perp - \boldsymbol{v}\times\boldsymbol{B}'\right)\,, \\
B_z &= B'_z, \qquad & \boldsymbol{B}_\perp &= \gamma\left(\boldsymbol{B}'_\perp + \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}'\right)\,,
\end{aligned}
\tag{1.7}
$$

where $\boldsymbol{E}'_\perp$ and $\boldsymbol{B}'_\perp$ are the components of the electric and magnetic fields perpendicular to the velocity $\boldsymbol{v}$: $\boldsymbol{E}'_\perp = (E_x, E_y)$, $\boldsymbol{B}'_\perp = (B_x, B_y)$.

A big drawback of the MKS system of units is that E and B have different dimensions ($[E] = [cB]$). In special theory of relativity whether you have E or B depends on the frame of reference. $B = 1$ T (a neodymium magnet) corresponds to the electric field of $E = 300$ MV/m.

Vector and scalar potentials

It is often convenient to express the fields in terms of the *vector potential* $\boldsymbol{A}$ and the *scalar potential* ϕ:

$$\begin{aligned} \boldsymbol{E} &= -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} &= \nabla \times \boldsymbol{A} \end{aligned} \tag{1.8}$$

Substituting these equations into Maxwell's equations, we find that the second and the third equations are satisfied identically. We only need to take care of the first and the fourth equations.

The electromagnetic potentials $(\phi/c, \boldsymbol{A})$ are transformed exactly as the 4-vector $(ct, \boldsymbol{r})$:

$$\begin{aligned} A_x &= A'_x \\ A_y &= A'_y \\ A_z &= \gamma\left(A'_z + \frac{v}{c^2}\phi'\right) \\ \phi &= \gamma(\phi' + vA'_z) \end{aligned} \tag{1.9}$$