Lecture 2
EM field of relativistic particles in free space.
Field of a relativistic beam moving in free space

January 24, 2019

Lecture outline

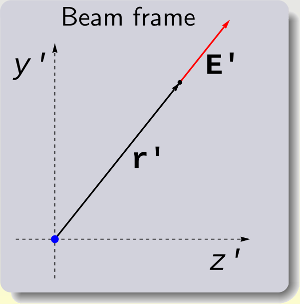
- Electromagnetic field of a relativistic particle moving along a straight line.
- Interaction of relativistic charges in free space.
- Transverse and longitudinal electromagnetic fields of a relativistic bunch.
- 3D Gaussian bunch

Relativistic field of a particle moving with constant velocity

Consider a point charge q moving with a constant velocity $\boldsymbol{v}$ along the z axis. We are interested in the case of a relativistic velocity, $v \approx c$, or $\gamma \gg 1$. In the *particle frame of reference* it has a static Coulomb field,

$$\boldsymbol{E}' = \frac{1}{4\pi\epsilon_0} \frac{q\boldsymbol{r}'}{r'^3}$$

(the prime indicates quantities in the reference frame where the particle is at rest).



Lorentz transformation of the field

To find the electric and magnetic fields in the lab frame we will use the Lorentz transformation (1.4) for coordinates and time, and the transformation for the fields (1.7). We have $\boldsymbol{B}' = 0$: $E_x = \gamma E_x'$, $E_y = \gamma E_y'$, and $E_z = E_z'$. We also need to transform vector $\boldsymbol{r}'$ into the lab frame using Eqs. (1.4). For the length of this vector we have

$$r' = \sqrt{x'^2 + y'^2 + z'^2} = \sqrt{x^2 + y^2 + \gamma^2 (z - vt)^2}$$

The Cartesian coordinates of $\boldsymbol{E}$ are

$$\begin{aligned} E_x &= \frac{1}{4\pi\epsilon_0} \frac{q\gamma x}{(x^2 + y^2 + \gamma^2 (z - vt)^2)^{3/2}} \\ E_y &= \frac{1}{4\pi\epsilon_0} \frac{q\gamma y}{(x^2 + y^2 + \gamma^2 (z - vt)^2)^{3/2}} \\ E_z &= \frac{1}{4\pi\epsilon_0} \frac{q\gamma (z - vt)}{(x^2 + y^2 + \gamma^2 (z - vt)^2)^{3/2}} \end{aligned} \tag{2.1}$$

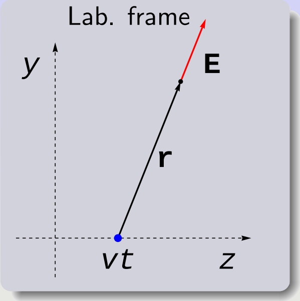
Electric and magnetic fields of a relativistic point charge

These three equations can be combined into a vectorial one

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q\boldsymbol{r}}{\gamma^2\mathcal{R}^3} \tag{2.2}$$

Here vector $\boldsymbol{r}$ is drawn from the current position of the particle to the observation point, $\boldsymbol{r} = (x, y, z - vt)$, and $\mathcal{R}$ is given by

$$\mathcal{R} = \sqrt{(z - vt)^2 + (x^2 + y^2)/\gamma^2}$$

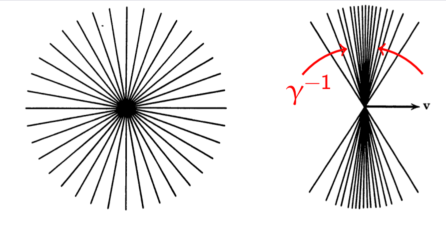


As follows from Eqs. (1.7), a moving charge carries magnetic field

$$\boldsymbol{B} = \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E} \tag{2.3}$$

The magnetic field is directed azimuthally around the direction of motion.

Field lines



particle's frame lab frame

Within a narrow cone with the angular width $\sim 1/\gamma$ with respect to the transverse plane the field is large, $E \sim q\gamma/r^2$. On the axis the field is weak, $E \sim 1/r^2\gamma^2$. The absolute value of the magnetic field is almost equal to that of the electric field$/c$.

Limit $v \to c$

In some problems we can neglect the small angular width of the electromagnetic field of a relativistic particle and consider it as an infinitely thin "pancake", $E \propto \delta(z - ct)$. This approximation formally corresponds to the limit $v \to c$. Because the field is directed along the vector drawn from the current position of the charge, more precisely, we can write $\boldsymbol{E} = A\boldsymbol{\rho}\delta(z - ct)$ where $\boldsymbol{\rho} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y$ and A is a constant which is determined by the requirements that the areas under the curves $E_x(z)$ and $E_y(z)$ agrees with the ones given by Eq. (2.2) in the limit $\gamma \to \infty$.

Taking the limit $v \to c$, we find for the fields ($\rho = \sqrt{x^2 + y^2}$)

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{2q\boldsymbol{\rho}}{\rho^2}\delta(z - ct)$$

$$\boldsymbol{B} = \frac{1}{c}\hat{\boldsymbol{z}} \times \boldsymbol{E} \qquad (2.4)$$

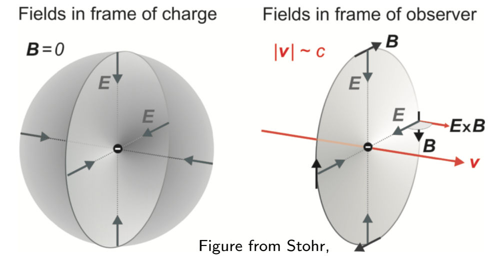
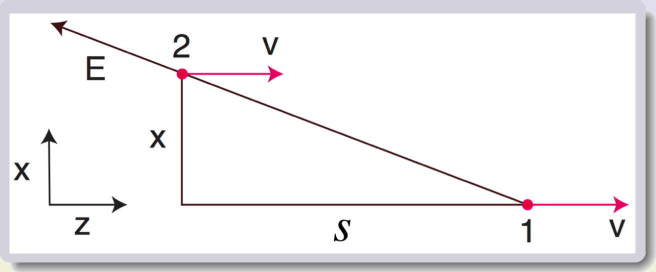


Figure from Stohr, Siegmann, "Magnetism"

Interaction of Moving Charges in Free Space

Let us now consider a *source* particle of charge q moving with velocity v, and a *test* particle of *unit* charge moving behind the leading one on a parallel path at a distance s with an offset x. We want to find the force which the source particle exerts on the test one.



A leading particle 1 and a trailing particle 2 traveling in free space with parallel velocities $\boldsymbol{v}$. Shown also is the coordinate system x, z.

Interaction of moving charges in free space

The longitudinal force is

$$F_\ell = E_z = -\frac{1}{4\pi\epsilon_0}\frac{qs}{\gamma^2(s^2 + x^2/\gamma^2)^{3/2}} \tag{2.5}$$

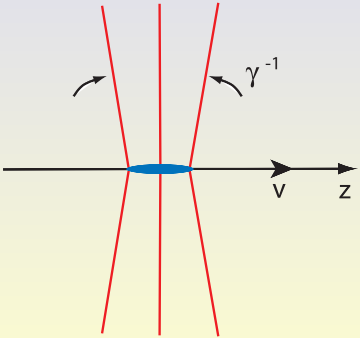
and the transverse force is

$$F_t = E_x - vB_y = \frac{1}{4\pi\epsilon_0}\frac{qx}{\gamma^4(s^2 + x^2/\gamma^2)^{3/2}} \tag{2.6}$$

In accelerator physics, the force $\boldsymbol{F}$ is often called *the space charge force*. The longitudinal force decreases as γ^{-2} when γ increases (for $s \gtrsim x/\gamma$). For the transverse force, if $s \gg x/\gamma$, $F_t \sim \gamma^{-4}$, and for $s = 0$, $F_t \sim \gamma^{-1}$. Hence, in the limit $\gamma \to \infty$, the electromagnetic interaction in free space between two particles on parallel paths vanishes.

Field of a long-thin relativistic bunch of particles

We now consider a relativistic bunch moving in free space. The bunch length σ_z is much larger[1] than the bunch transverse size $\sigma_z \gg \sigma_\perp$ (a *line charge*). The bunch is moving in the longitudinal direction along the z axis with a relativistic factor $\gamma \gg 1$. What is the electric field of this bunch?



[1] In what frame of reference?

Field outside of the bunch

We first calculate the radial electric field outside of the bunch at distance ρ from the z axis. Assuming that $\rho \gg \sigma_\perp$ we can neglect the transverse size of the beam and represent it as a collection of point charges located on the z axis. Each such charge generates the electric field given by Eq. (2.2). From this equation we find that the radial component dE_ρ created by an infinitesimally small charge dq' located at coordinate z' is

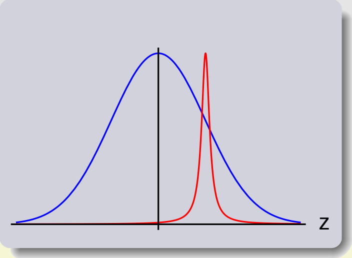
$$dE_\rho(z, z', \rho) = \frac{1}{4\pi\epsilon_0} \frac{\rho dq'}{\gamma^2((z - z')^2 + \rho^2/\gamma^2)^{3/2}} \tag{2.7}$$

where z and $\rho = \sqrt{x^2 + y^2}$ refer to the observation point. To find the field of the bunch we assume that the bunch 1D distribution function is given by $\lambda(z)$ $\left(\int \lambda(z) dz = 1\right)$, so that the charge dq' within dz' is equal to $Q\lambda(z')dz'$, with Q the total charge of the bunch.

Small distance from the bunch

For the field, we need to add contributions of all elementary charges in the bunch:

$$E_\rho(z,\rho) = \int dE_\rho(z,z',\rho) = \frac{Q\rho}{4\pi\epsilon_0\gamma^2}\int_{-\infty}^{\infty}\frac{\lambda(z')dz'}{((z-z')^2+\rho^2/\gamma^2)^{3/2}} \qquad (2.8)$$



The function $((z-z')^2+\rho^2/\gamma^2)^{-3/2}$ in this integral [red] has a sharp peak at $z' \approx z$ of width $\Delta z \sim \rho/\gamma$ at $z = z'$. At distances $\rho \ll \sigma_z\gamma$ from the bunch the width of the peak is smaller than the width of the distribution function σ_z [blue], and we can replace it by the delta function:

$$\frac{1}{((z-z')^2+\rho^2/\gamma^2)^{3/2}} \to \frac{2\gamma^2}{\rho^2}\delta(z-z'). \qquad (2.9)$$

Small distance from the bunch

The factor in front of the delta function on the right hand side follows from the requirements that the area under the functions on the left hand side and on the right hand side should be equal, and from the mathematical identity

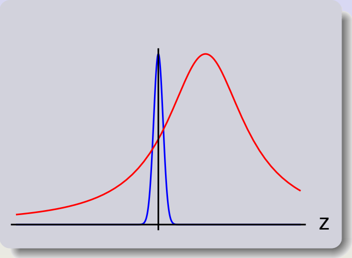
$$\int_{-\infty}^{\infty} \frac{dz'}{((z-z')^2+a^2)^{3/2}} = \frac{2}{a^2}$$

The approximation (2.9) is equivalent to using Eqs. (2.4) instead of (2.2). The result is

$$E_\rho(z,\rho) = \frac{1}{4\pi\epsilon_0}\frac{2Q\lambda(z)}{\rho} \tag{2.10}$$

We see that the factor γ does not enter this formula—this agrees with our expectation because Eqs. (2.4) are valid in the limit $\gamma \to \infty$.

Large distance from the bunch

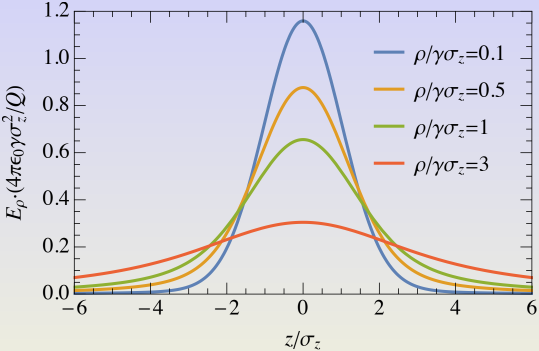


In the opposite limit, $\rho \gg \sigma_z\gamma$, we can replace $\lambda(z)$ in Eq. (2.8) by the delta function $\delta(z)$, which gives the field of a point charge

$$E_\rho(z,\rho) = \frac{1}{4\pi\epsilon_0}\frac{Q\rho\gamma}{(z^2\gamma^2+\rho^2)^{3/2}} \tag{2.11}$$

In the intermediate region, $\rho \sim \sigma_z\gamma$, the result is shown on the next slide for a Gaussian distribution function $\lambda(z) = (1/\sqrt{2\pi}\sigma_z)e^{-z^2/2\sigma_z^2}$.

Field distribution as a function of distance

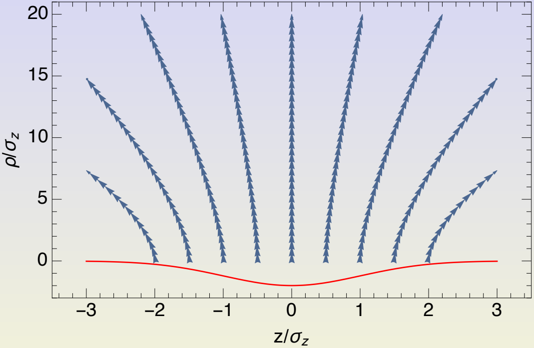


Transverse electric field of a relativistic bunch with Gaussian distribution for various values of the parameter $\rho/\sigma_z\gamma$. The field is normalized by $(4\pi\epsilon_0)^{-1}Q/\gamma\sigma_z^2$.
The magnetic field of the beam

$$\boldsymbol{B} = \frac{v}{c^2}\hat{\boldsymbol{z}} \times \boldsymbol{E}\,.$$

For an axisymmetric beam this means azimuthal magnetic field $B_\theta = \beta E_\rho / c$.

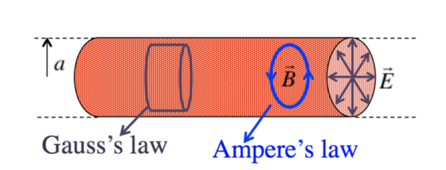
Electric field lines



Electric field lines of a thin relativistic bunch with $\gamma = 10$. The red line at the bottom shows the longitudinal Gaussian charge distribution in the bunch.

Transverse field inside the bunch

If we want to find the transverse field inside the bunch, we need to drop the assumption of an infinitely thin bunch. Let us assume a beam of radius a with a uniform charge density en_0 inside. The beam moves with velocity v along the z axis. We can find the radial electric field E_ρ and the magnetic field B_θ from Gauss's and Ampere's laws, respectively.



$$E_\rho = \frac{en_0}{2\epsilon_0}\rho, \quad B_\theta = \frac{en_0\beta}{2\epsilon_0 c}\rho \qquad (2.12)$$

The e.m. transverse force is

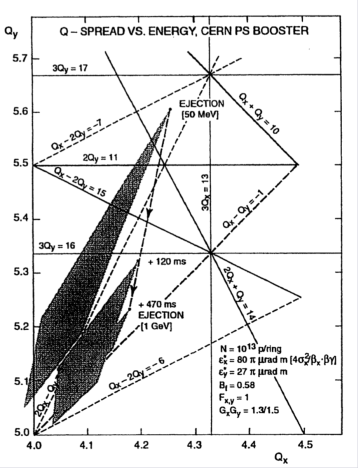
$$F_\rho = e(E_\rho - c\beta B_\theta) = \frac{e}{\gamma^2}\frac{en_0}{2\epsilon_0}\rho \qquad (2.13)$$

There is a cancellation of the electric and magnetic forces that results in the small factor γ^{-2}.

These formulas are also valid for a long-thin bunch ($\sigma_\perp \ll \gamma\sigma_z$) — to find the fields at z one needs to substitute the local value $n_0(z) = \frac{\lambda(z)}{\pi a^2}$.

Space charge tune shift in CERN PS

This force is called the *direct space charge force*. It results in the *incoherent tune shift* in a circular accelerators.



Example for a space-charge limited synchrotron: betatron tune diagram and areas covered by direct space-charge tune spreads at injection, intermediate, and extraction energies for the CERN Proton Synchrotron Booster. During acceleration, space charge gets weaker and the "necktie" area shrinks, enabling the external machine tunes to move the "necktie" to an area clear of betatron resonances.

Longitudinal field inside the bunch

What is the longitudinal electric field inside the bunch? If we neglect the transverse size of the beam and assume the same infinitely-thin-beam approximation we used above, we can try to integrate the longitudinal field of a unit point charge

$$dE_z(z, z') = \frac{dq'}{4\pi\epsilon_0\gamma^2} \frac{z - z'}{|z - z'|^3}$$

as we did above for the transverse field:

$$\begin{aligned} E_z(z) &= \int dE_z(z, z') \\ &= \frac{Q}{4\pi\epsilon_0\gamma^2} \int dz' \lambda(z') \frac{z - z'}{|z - z'|^3} \end{aligned} \tag{2.14}$$

but the integral diverges at $z' \to z$. This divergence indicates that one has to take into account the finite transverse size of the beam which we denote by $\sigma_\perp$.

Estimate of the longitudinal field

We can still extract an order of magnitude estimate from Eq. (2.14). Do not approach to the singularity at the distance smaller than some $\Delta z_{\min}$ and estimate $\lambda(z') \sim 1/\sigma_z$. This gives[2]

$$E_z \sim \frac{1}{4\pi\epsilon_0} \frac{Q}{\sigma_z^2 \gamma^2} \log \frac{\sigma_z}{\Delta z_{\min}}$$

One can show that $\Delta z_{\min} \sim \sigma_\perp/\gamma$ and a crude estimate for E_z is:

$$E_z \sim \frac{1}{4\pi\epsilon_0} \frac{Q}{\sigma_z^2 \gamma^2} \log \frac{\sigma_z \gamma}{\sigma_\perp} \tag{2.15}$$

This estimate is valid in the limit $\sigma_\perp/\gamma\sigma_z \ll 1$. Formally, it diverges in the limit of infinitely thin beam ($\sigma_\perp \to 0$), but in reality the effect of the longitudinal electric field for relativistic beams is often small because of the factor γ^{-2} (the so called *space charge effect*).

[2] To arrive at this result it helps to replace $\frac{z-z'}{|z-z'|^3} \to \frac{\partial}{\partial z'} \frac{1}{|z-z'|}$ in Eq. (2.14) and integrate by parts.

Electric field of a 3D Gaussian distribution

A bunch of charged particles in accelerator physics is often represented as having a Gaussian distribution function in all three directions so that the charge density ρ (do not confuse with the transverse radius) is

$$\rho(x, y, z) = \frac{Q}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} e^{-x^2/2\sigma_x^2 - y^2/2\sigma_y^2 - z^2/2\sigma_z^2} \tag{2.16}$$

where σ_x, σ_y, and σ_z are the rms bunch lengths in the corresponding directions. What is the electric field of such bunch? This problem can be solved exactly in the beam frame where the beam is at rest, and the field is purely electrostatic.

Due to the Lorentz transformations the bunch length in the beam frame is γ times longer than in the lab frame, $\sigma_{z,b} = \gamma\sigma_{z,\mathrm{lab}}$. We assume that this factor is already taken into account and $\sigma_z = \sigma_{z,b}$ in (2.16) is the bunch length in the beam frame.

Electric field of a 3D Gaussian distribution

The electrostatic potential ϕ_b in the beam frame of reference satisfies the Poisson equation

$$\nabla^2 \phi_b = -\frac{\rho_b}{\epsilon_0}$$

whose solution can be written as

$$\phi_b(x, y, z) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_b(x', y', z') dx' dy' dz'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}} \qquad (2.17)$$

It is not an easy problem to carry out the three-dimensional integration in this equation. A trick that reduces Eq. (2.17) to a one-dimensional integral is to use the following identity

$$\frac{1}{R} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\lambda^2 R^2/2} d\lambda \qquad (2.18)$$

Electric field of a 3D Gaussian distribution

Assuming that $R = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2}$ and replacing $1/R$ in Eq. (2.17) with Eq. (2.18) we first arrive at the four-dimensional integral

$$\phi_b = \frac{1}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\int_0^\infty d\lambda \int e^{-\lambda^2[(x-x')^2+(y-y')^2+(z-z')^2]/2} \\ \times \rho_b(x', y', z')dx'dy'dz' \qquad (2.19)$$

With the Gaussian distribution (2.16) the integration over x', y' and z' can now be easily carried out, e.g.,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda^2(x-x')^2} e^{-\frac{x'^2}{2\sigma_x^2}}\, dx' = \frac{\sqrt{2\pi}}{\sqrt{\lambda^2 + \sigma_x^{-2}}} e^{-\frac{x^2\lambda^2}{2(\lambda^2\sigma_x^2+1)}}$$

Electric field of a 3D Gaussian distribution

which gives for the potential

$$\phi_b = \frac{1}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\frac{Q}{\sigma_x\sigma_y\sigma_{z,b}}\int_0^\infty d\lambda \frac{e^{-\frac{x^2\lambda^2}{2(\lambda^2\sigma_x^2+1)}}\, e^{-\frac{y^2\lambda^2}{2(\lambda^2\sigma_y^2+1)}}\, e^{-\frac{z^2\lambda^2}{2(\lambda^2\sigma_{z,b}^2+1)}}}{\sqrt{\lambda^2+\sigma_x^{-2}}\sqrt{\lambda^2+\sigma_y^{-2}}\sqrt{\lambda^2+\sigma_{z,b}^{-2}}} \tag{2.20}$$

This integral is much easier to evaluate numerically, and it is often used in numerical simulations to calculate the field of charged bunches. There are various useful limiting cases of this expression, such as $\sigma_x = \sigma_y$ (axisymmetric beam) or $\sigma_x, \sigma_y \ll \sigma_{z,b}$ (a long, thin beam) that can be analyzed. Eq. (2.15) can be derived from this formula in the limit of thin beam.

Electric field of a 3D Gaussian distribution

Having found the potential in the beam frame, it is now easy to transform it to the laboratory frame using the Lorentz transformation. First we have to recall that $\sigma_{z,b}$ is the bunch length in the beam frame equal to $\gamma\sigma_{z,\mathrm{lab}}$. Second, from the Lorentz transformations (1.9) we see that the potential in the lab frame is γ times larger than in the beam frame (note that $A'_z = 0$). Third, we need to transform the coordinates x, y, z to the lab frame. The x and y coordinates are not transformed however z should be replaced by $\gamma(z_{\mathrm{lab}} - vt_{\mathrm{lab}})$. The resulting expression is (we drop all "lab" subscripts in what follows)

$$\phi = \frac{Q}{4\pi\epsilon_0}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda^2\sigma_x^2+1}\sqrt{\lambda^2\sigma_y^2+1}\sqrt{\lambda^2\sigma_z^2+\gamma^{-2}}} \times e^{-\frac{x^2\lambda^2}{2(\lambda^2\sigma_x^2+1)}} e^{-\frac{y^2\lambda^2}{2(\lambda^2\sigma_y^2+1)}} e^{-\frac{(z-vt)^2\lambda^2}{2(\lambda^2\sigma_z^2+\gamma^{-2})}} \qquad (2.21)$$

The vector potential in the lab frame has only z component and is equal to $A_z = v\phi/c^2$.

Bassetti-Erskine formula

In the limit $\sigma_z \to \infty$ the expression for the potential and the components of the field, E_x and E_y can be simplified and expressed in terms of the complex error function[3]

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{\zeta^2}\,d\zeta\right)$$

Here are the expressions for the fields

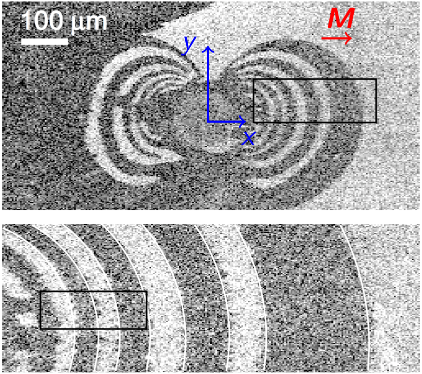
$$E_x = \frac{Q}{2\varepsilon_0\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Im}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]} w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

$$E_y = \frac{Q}{2\varepsilon_0\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Re}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]} w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

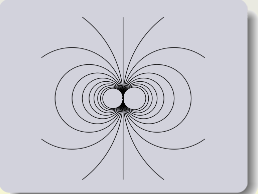
This is often a good approximation because in many cases the bunches are relatively long.

[3] M. Bassetti and G.A. Erskine, "Closed expression for the electrical field of a two-dimensional Gaussian charge", CERN-ISR-TH/80-06 (1980).

The beam field can be visualized in experiment[4]



The beam passes through a thin Fe film on GaAs surface. $\sigma_z = 630$ μm, beam cross section 6×9 μm, beam energy 28 GeV, $Q \approx 2$ nC. The film is initially magnetized in $-x$ direction. The final magnetization (and the darkness) is defined by $\int B_y dt$.



The pattern boundaries are lines of constant value of

$$\int B_y dt = C\frac{x}{\rho^2} = C\frac{x}{x^2+y^2}$$

Lines $B_y = \mathrm{const}$ are circles that go through the center of the beam.

[4] C. Stamm et al. PRL **94**, 197603 (2005)