Lecture 4
Wakefield in a bunch of particles. Loss and kick factors, impedance

January 24, 2019

Lecture outline

- Loss factor and kick factor
- Impedance, its properties
- Cavity impedance

Wake field in a bunch of particles

Consider a beam consists of N particles with the distribution function $\lambda(z)$ defined so that $\lambda(z)dz$ gives the probability to find a particle near the point z, $\int \lambda(z)dz = 1$. Here we define positive z in the head of the beam, and negative z in the tail. A particle located at z will interact with all other particles of the beam through the wake[11]:

$$\Delta p_z(z) = -\frac{Ne^2}{c}\int_z^\infty dz'\lambda(z')w_\ell(z'-z)$$

Note that we neglect the dependence of the wake on the transverse coordinates of the test and source particles, implicitly assuming that the beam is thin (the line charge model of the beam). In relativistic limit the energy change is $\Delta\mathcal{E}(z) = c\Delta p_z$,

$$\Delta\mathcal{E}(z) = -Ne^2\int_z^\infty dz'\lambda(z')w_\ell(z'-z) \tag{4.1}$$

The negative sign here means that, with our convention on the signs, a positive wake means energy loss.

[11] Here we explicitly assume that the wake is behind the source particle; in a more general case use $\int_{-\infty}^{\infty} dz' \ldots$.

Wake field in a bunch of particles

Later we will use the *wake field of the bunch* W,

$$W_\ell(z) = \int_z^\infty dz' \lambda(z') w_\ell(z' - z) \tag{4.2}$$

Note the relation $\Delta\mathcal{E}(z)/Ne^2 = -W_\ell(z)$. We can similarly define a transverse wake field of the bunch, W_t.

Two important integral characteristics of the strength of the wake are the average value of the energy loss $\Delta\mathcal{E}_{\rm av}$ (per particle) and the rms energy spread, $\Delta\mathcal{E}_{\rm rms}$, generated by the wake,

$$\Delta\mathcal{E}_{\rm av} = \int_{-\infty}^\infty dz \Delta\mathcal{E}(z)\lambda(z)$$

and

$$\Delta\mathcal{E}_{\rm rms} = \left[\int_{-\infty}^\infty dz (\Delta\mathcal{E}(z) - \Delta\mathcal{E}_{\rm av})^2 \lambda(z)\right]^{1/2}$$

The energy loss for the whole bunch is $N\Delta\mathcal{E}_{\rm av}$.

Loss factor

The *loss factor* is defined as

$$\varkappa_{\text{loss}} = -\frac{1}{Ne^2}\Delta\mathcal{E}_{\text{av}} \tag{4.3}$$

(the minus sign is chosen to make the loss factor positive).

Let us calculate $\varkappa_{\text{loss}}$ for a constant wake, $w_\ell = w_0$. From (4.2) we have

$$W_\ell(z) = w_0 \int_z^\infty dz' \lambda(z')$$

and[12]

$$\varkappa_{\text{loss}} = w_0 \int_{-\infty}^\infty dz \lambda(z) \int_z^\infty dz' \lambda(z') = \frac{1}{2} w_0$$

This explains why the factor $\varkappa$ in Eq. (3.11) is also called the loss factor — this is the loss factor in the sense of (4.3) *for very short bunches*.

[12] To calculate the integral use $\lambda(z) = d\psi(z)/dz$ with $\psi(-\infty) = 0$ and $\psi(\infty) = 1$.

Transverse kick in a bunch of particles

Consider a beam passing through an element with an offset y which has transverse wake $\bar{w}_t(s)$. What is the deflection angle θ at the exit?

$$\theta(z) = \frac{\Delta p_\perp(z)}{p} = \frac{1}{cp}\int_z^\infty dz' Ne\lambda(z') \cdot ey\bar{w}_t(z'-z)$$
$$= y\frac{Ne^2}{\gamma mc^2}\int_z^\infty dz'\lambda(z')\bar{w}_t(z'-z)$$

The averaged over the distribution function deflection angle is

$$\theta_{\mathrm{av}} = \langle\theta\rangle = \int_{-\infty}^\infty dz\theta(z)\lambda(z)$$

and the rms spread is

$$\Delta\theta_{\mathrm{rms}} = \langle(\theta - \theta_{\mathrm{av}})^2\rangle^{1/2}$$

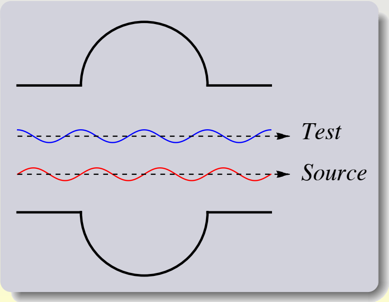
Similar to the loss factor, we define the *kick factor*,

$$\varkappa_{\mathrm{kick}} = \frac{\theta_{\mathrm{av}}}{y}\frac{\gamma mc^2}{Ne^2} = \int_{-\infty}^\infty dz\lambda(z)\int_z^\infty dz'\lambda(z')\bar{w}_t(z'-z) \qquad (4.4)$$

Wake versus impedance

One often calculates the wake making the Fourier transform of the Maxwell equations. This leads to the Fourier transforms of the wakes. With a proper normalization factor, the Fourier transform of a wake is called the *impedance*. It is also useful in stability analysis of the beams.

The longitudinal impedance has a physical meaning by itself: it is proportional to the voltage induced by a sinusoidally modulated source current on a sinusoidally modulated test one.



Both currents are sinusoidal waves moving with the speed of light, $I_s = I_{s0} e^{-i\omega(t-z/c)}$, $I_t = I_{t0} e^{-i\omega(t-z/c)}$. The impedance $Z_\ell(\omega) = V/I_{s0}$ where V is the voltage induced by the current I_{s0} on the current I_{t0}.

Impedance definition

The longitudinal Z_ℓ and transverse Z_t impedances are defined as Fourier transforms of the wakes[13]

$$Z_\ell(\omega) = \frac{1}{c}\int_0^\infty ds\, w_\ell(s) e^{i\omega s/c}\,,\; Z_t(\omega) = -\frac{i}{c}\int_0^\infty ds\, \bar{w}_t(s) e^{i\omega s/c} \qquad (4.5)$$

The integration can be actually extended into the region of negative values of z, because w_ℓ and w_t are equal to zero in that region.

Because the wakes are real, we have $Z_\ell(-\omega) = Z_\ell^*(\omega)$ and $Z_t(-\omega) = -Z_t^*(\omega)$, or

$$\begin{aligned} \mathrm{Re}\, Z_\ell(\omega) &= \mathrm{Re}\, Z_\ell(-\omega) \qquad \mathrm{Im}\, Z_\ell(\omega) = -\mathrm{Im}\, Z_\ell(-\omega) \\ \mathrm{Re}\, Z_t(\omega) &= -\mathrm{Re}\, Z_t(-\omega) \qquad \mathrm{Im}\, Z_t(\omega) = \mathrm{Im}\, Z_t(-\omega) \end{aligned}$$

[13] In principle, one can define a vectorial transverse impedance using the wake from Eq. (3.2): $\boldsymbol{Z}_t(\omega) = -ic^{-1}\int_0^\infty dz\, \boldsymbol{w}_t(z) e^{i\omega z/c}$.

Some Properties of Impedance

Impedance can also be defined in the upper half-plane of the complex variable ω where $\mathrm{Im}\,\omega > 0$. It is an analytic function there[14].
The relation between the wakes and the impedances

$$
\begin{aligned}
w_\ell(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega Z_\ell(\omega) e^{-i\omega s/c} \\
\bar{w}_t(s) &= \frac{i}{2\pi}\int_{\infty}^{\infty} d\omega Z_t(\omega) e^{-i\omega s/c}
\end{aligned}
$$

[14] This is true for classical wakes that are zero in front of the particle [and for the CSR wake in free space].

Various definitions of wakes and impedances

Other authors often introduce definitions of wake and impedance that differ from each other:

- A. Chao—uses $z = -s$ as the argument of w. His longitudinal wake $w_\ell \to W_0'$, and the transverse one $w_t = -W_1$. The impedances agree with ours. The same is for A. Wolsky, "Beam Dynamics in High Energy Particle Accelerators".
- "Handbook of Accelerator Physics and Engineering" ed. by A. Chao et al. Many articles use A. Chao's conventions for the wake and impedance.
- P. Wilson—Z_ℓ is complex conjugate of ours.
- S. Heifets, S. Kheifets (Rev. Mod. Phys, 1991) — the transverse wake has a different sign. The impedances are the same.
- S. Kheifets and B. Zotter (*Impedances and Wakes in High-Energy Particle Accelerators*, 1997)—the wake is the same, the impedance is a complex conjugate.
- E. Gianfelice, L. Palumbo. (IEEE Tr. N.S., 37, 2, 1084, (1990))—extra factor $(2\pi)^{-1}$ in Z_t.

Kramers-Kronig relations

The wake field can actually be found if only the real (or imaginary) part of the impedance is known. Indeed, for arbitrary $s \lessgtr 0$

$$
\begin{aligned}
w_\ell(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega Z_\ell(\omega) e^{-i\omega s/c} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left[\mathrm{Re}\, Z_\ell(\omega)\cos\frac{\omega s}{c} + \mathrm{Im}\, Z_\ell(\omega)\sin\frac{\omega s}{c}\right]
\end{aligned}
\tag{4.6}
$$

For $s > 0$ we have $w_\ell(-s) = 0$. Substitute $-s$ in (4.6),

$$
0 = \int_{-\infty}^{\infty} d\omega \left[\mathrm{Re}\, Z_\ell(\omega)\cos\frac{\omega s}{c} - \mathrm{Im}\, Z_\ell(\omega)\sin\frac{\omega s}{c}\right], \qquad s > 0
$$

This means that for positive s

$$
w_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\omega \mathrm{Re}\, Z_\ell(\omega)\cos\frac{\omega s}{c} = \frac{2}{\pi}\int_{0}^{\infty} d\omega \mathrm{Re}\, Z_\ell(\omega)\cos\frac{\omega s}{c}, \qquad s > 0
$$

Here we used the symmetry $\mathrm{Re}\, Z_\ell(\omega) = \mathrm{Re}\, Z_\ell(-\omega)$.

Kramers-Kronig relations

A similar derivation for the transverse wake gives

$$\bar{w}_t(s) = \frac{2}{\pi} \int_0^\infty d\omega \mathrm{Re}\, Z_t(\omega) \sin \frac{\omega s}{c}$$

Since the wake can be found from $\mathrm{Re}\, Z$, it means that there is a relation between $\mathrm{Re}\, Z$ and $\mathrm{Im}\, Z$

$$\mathrm{Re}\, Z(\omega) \to w(s) \to Z(\omega) \to \mathrm{Im}\, Z(\omega)$$

These are called the Kramers-Kronig relations.

Problem: Express $\mathrm{Im}\, Z(\omega)$ *through* $\mathrm{Re}\, Z(\omega)$ *following the approach outlined above. Answer:*

$$\mathrm{Im}\, Z(\omega) = -\frac{1}{\pi} \mathrm{P.V.} \int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Re}\, Z(\omega')}{\omega' - \omega}$$

Resonant mode impedance

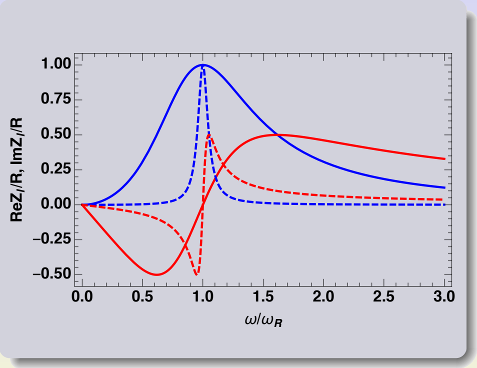
Let us calculate the impedance corresponding to the resonant wake (3.11)

$$
\begin{aligned}
Z_\ell(\omega) &= \frac{1}{c}\int_0^\infty ds\, w_\ell(s) e^{i\omega s/c} \\
&= \frac{2\varkappa}{c}\int_0^\infty ds\, e^{-\alpha s/c + i\omega s/c}\left(\cos\frac{\bar\omega s}{c} - \frac{\alpha}{\bar\omega}\sin\frac{\bar\omega s}{c}\right) \\
&= \frac{2\varkappa}{\omega_R}\left[\frac{2\alpha}{\omega_R} + i\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)\right]^{-1} \\
&= \frac{R}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)}
\end{aligned}
\tag{4.7}
$$

where $\omega_R = \sqrt{\bar\omega^2 + \alpha^2}$, $R = \varkappa/\alpha$ - the shunt impedance, $Q = \omega_R/2\alpha$. For large Q, the impedance is peaked around $\omega = \pm\omega_R$.

Resonant impedance

Resonant impedance for $Q = 1$ (solid) and $Q = 10$ (dashed). Blue lines—ReZ_ℓ, red lines—ImZ_ℓ.



In the limit of very large Q for $\omega > 0$ we can approximate $\mathrm{Re}\, Z_\ell$ by the simple equation

$$\mathrm{Re}\, Z_\ell = \frac{\pi}{2}\frac{R\omega_R}{Q}\delta(\omega - \omega_R) = \pi\varkappa\delta(\omega - \omega_R) \tag{4.8}$$

Impedance of transverse resonant wake

We can calculate the transverse impedance for the transverse resonant wake (3.13) (we drop index n here)

$$
\begin{aligned}
Z_t(\omega) &= -\frac{i}{c}\int_0^\infty ds\, \bar{w}_t(s) e^{i\omega s/c} \\
&= -\frac{2i\varkappa_t}{c}\int_0^\infty ds\, e^{-\omega_R s/2Qc + i\omega s/c} \sin\frac{\bar{\omega} s}{c} \\
&= \frac{2\varkappa_t}{\omega_R}\frac{\bar{\omega}}{\omega}\left[\frac{1}{Q\omega_R} + i\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)\right]^{-1} \\
&= \frac{\bar{\omega}}{\omega}\frac{R_t}{1 + iQ\left(\frac{\omega_R}{\omega} - \frac{\omega}{\omega_R}\right)}
\end{aligned}
\tag{4.9}
$$

where $\omega_R = \bar{\omega}/\sqrt{1 + (2Q)^{-2}}$, $R_t = 2Q\varkappa_t/\omega_R$ (R_t has dimension Ω/m). For large Q, the impedance is peaked around $\omega = \pm\omega_R$.

Energy Loss and $\mathrm{Re}\, Z_\ell$

The energy loss by a particle in a beam due to wake field is due to the real part of impedance. Let us prove this. Start from Eq. (4.1)

$$\Delta\mathcal{E}(z) = -Ne^2 \int_z^\infty dz' \lambda(z') w_\ell(z'-z). \tag{4.10}$$

Average energy change in the bunch

$$\begin{aligned}
\Delta\mathcal{E}_{\mathrm{av}} &= -\int_{-\infty}^{\infty} dz \lambda(z) \int_{-\infty}^{\infty} dz' Ne^2 \lambda(z') w_\ell(z'-z) \\
&= -Ne^2 \int_{-\infty}^{\infty} dz \lambda(z) \int_{-\infty}^{\infty} dz' \lambda(z') \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega Z_\ell(\omega) e^{-i\omega(z'-z)/c} \\
&= -\frac{Ne^2}{2\pi} \int_{-\infty}^{\infty} d\omega Z_\ell(\omega) \int_{-\infty}^{\infty} dz \lambda(z) e^{i\omega z/c} \int_{-\infty}^{\infty} dz' \lambda(z') e^{-i\omega z'/c} \\
&= -\frac{Ne^2}{2\pi} \int_{-\infty}^{\infty} d\omega Z_\ell(\omega) |\hat{\lambda}(\omega)|^2
\end{aligned}$$

where

$$\hat{\lambda}(\omega) = \int_{-\infty}^{\infty} dz \lambda(z) e^{-i\omega z/c} \tag{4.11}$$

Energy Loss and $\mathrm{Re}\, Z_\ell$

Since $\hat{\lambda}(-\omega) = \hat{\lambda}^*(\omega)$, $|\hat{\lambda}(\omega)|^2$ is an even function of ω and

$$\Delta\mathcal{E}_{\mathrm{av}} = -\frac{Ne^2}{\pi} \int_0^\infty d\omega \mathrm{Re}\, Z_\ell(\omega) |\hat{\lambda}(\omega)|^2 \tag{4.12}$$

An important property of the longitudinal impedance

$$\mathrm{Re}\, Z_\ell(\omega) \geq 0 \tag{4.13}$$

The beam loses energy at all frequencies (assuming there is no interaction of the beam with active medium, or feedback).
Note that Eq. (4.12) is the energy loss per *one particle*. If we want the energy loss for the whole beam, we multiply it by N

$$\Delta\mathcal{E}_{\mathrm{beam}} = -\frac{Q^2}{\pi} \int_0^\infty d\omega \mathrm{Re}\, Z_\ell(\omega) |\hat{\lambda}(\omega)|^2 \tag{4.14}$$

where $Q = Ne$ is the beam charge.

Energy loss for a point charge

The energy lost by the beam is equal to the energy deposited to the source of the impedance.
For a point charge, $\lambda(z) = \delta(z)$, $N = 1$ and $\hat{\lambda}(\omega) = 1$, and the energy loss is

$$\Delta\mathcal{E} = -\frac{e^2}{\pi}\int_0^\infty d\omega \mathrm{Re}\, Z_\ell(\omega)$$

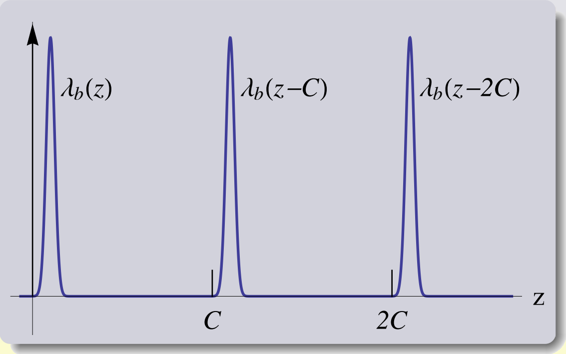
If we know the spectrum of the energy losses $\mathcal{E}_{\mathrm{sp}}(\omega)$, we can find $\mathrm{Re}\, Z_\ell(\omega)$

$$\Delta\mathcal{E} = \int_0^\infty d\omega \mathcal{E}_{\mathrm{sp}}(\omega) \to \mathrm{Re}\, Z_\ell(\omega) = -\frac{\pi}{e^2}\mathcal{E}_{\mathrm{sp}}(\omega) \qquad (4.15)$$

Using causality we can then calculate the wake through $\mathrm{Re}\, Z_\ell(\omega)$ (and find $\mathrm{Im}\, Z_\ell(\omega)$). In some cases this is the easiest way to calculate the wake.
Another method is to consider a sinusoidally modulated beam, $\lambda(z) \propto \cos(kz)$ and calculate the power loss of such modulated current. This power can be related to $\mathrm{Re}\, Z_\ell(ck)$.

Resonant heating in a ring

In a circular machine, the beam passes by each element every revolution period, so we have to generalize (4.14) for multiple turns. For this, we consider $\lambda(z)$ as a periodic function of z with the period equal to the circumference of the machine C and calculate the energy deposited over r revolution periods. We then need to carry out the integration in (4.11) over z from 0 to rC.



$$\lambda(z) = \sum_{n=0}^{r-1} \lambda_b(z - nC)$$

Resonant heating in a ring

Defining

$$\tilde{\lambda}(\omega) = \int_0^C \lambda_b(z) e^{-i\omega z/c} dz \tag{4.16}$$

we obtain

$$\hat{\lambda}(\omega) = \tilde{\lambda}(\omega) \sum_{n=0}^{r-1} e^{-i\omega n T_0} = \tilde{\lambda}(\omega) \frac{1 - \exp(-irT_0\omega)}{1 - \exp(-iT_0\omega)}, \tag{4.17}$$

where $T_0 = C/c$ is the revolution period. In the limit of large number of revolutions, $r \gg 1$, we have

$$|\hat{\lambda}(\omega)|^2 = |\tilde{\lambda}(\omega)|^2 \frac{\sin^2(rT_0\omega/2)}{\sin^2(T_0\omega/2)} \to r\omega_0 |\tilde{\lambda}(\omega)|^2 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0), \tag{4.18}$$

with $\omega_0 = 2\pi/T_0$.

Resonant heating in a ring

Substituting this expression into (4.14) and defining the power as $P = -\Delta\mathcal{E}_{\text{beam}}/rT_0$ we obtain

$$P = \frac{Q^2}{T_0^2} \sum_{n=-\infty}^{\infty} |\tilde{\lambda}(n\omega_0)|^2 \mathrm{Re}\, Z_\ell(n\omega_0). \tag{4.19}$$

This formula also works when there are many bunches in the ring—then $Q\lambda_b$ in Eq. (4.16) is the charge distribution in all these bunches. This result is important for calculation of heating in the ring of high-current accelerators.

How to compute the bunch wake using only impedance?

Assume that you know the (longitudinal) impedance as a function of frequency, $Z_\ell(\omega)$. You want to compute the $\Delta\mathcal{E}(z)$ without calculation of the wake of a point charge. Start from (4.2)

$$\begin{aligned}
\Delta\mathcal{E}(z) &= -Ne^2 \int_{-\infty}^{\infty} dz' \lambda(z') w_\ell(z' - z) \\
&= -\frac{Ne^2}{2\pi} \int_{-\infty}^{\infty} dz' \lambda(z') \int_{-\infty}^{\infty} d\omega Z_\ell(\omega) e^{-i\omega(z'-z)/c}, \\
&= -\frac{Ne^2}{2\pi} \int_{-\infty}^{\infty} d\omega Z_\ell(\omega) \hat{\lambda}(\omega) e^{i\omega z/c}
\end{aligned} \tag{4.20}$$

where

$$\hat{\lambda}(\omega) = \int_{-\infty}^{\infty} dz' \lambda(z') e^{-i\omega z'/c} \tag{4.21}$$

For a Gaussian bunch $\lambda(z) = (2\pi)^{-1/2} \sigma_z^{-1} e^{-z^2/2\sigma_z^2}$

$$\hat{\lambda}(\omega) = e^{-\omega^2 \sigma_z^2 / 2c^2} \tag{4.22}$$

Why $\int w(s)ds = 0$?

Let us prove Eq. (3.9). Use (4.12) (integrate from $-\infty$ to ∞)

$$\Delta\mathcal{E}_{\mathrm{av}} = -\frac{Ne^2}{2\pi}\int_{-\infty}^{\infty} d\omega \mathrm{Re}\, Z_\ell(\omega)|\hat{\lambda}(\omega)|^2$$

and take a very long Gaussian bunch with rms length σ_z. For a Gaussian bunch we have Eq. (4.22). For a very long bunch this is a narrow function

$$|\hat{\lambda}(\omega)|^2 = e^{-\omega^2\sigma_z^2/c^2} \approx \sqrt{\pi}\frac{c}{\sigma_z}\delta(\omega) \tag{4.23}$$

Hence, with $I_0 = Nec/\sqrt{2\pi}\sigma_z$ the peak current in the beam,

$$\Delta\mathcal{E}_{\mathrm{av}} = -\frac{Ne^2c}{2\sqrt{\pi}\sigma_z}\mathrm{Re}\, Z_\ell(0) = -\frac{1}{\sqrt{2}}eI_0\mathrm{Re}\, Z_\ell(0) \tag{4.24}$$

Why $\int w(s)ds = 0$?

But when $\sigma_z \to \infty$ we are dealing with constant current $\to$ constant magnetic field $\to$ no energy losses. Hence

$$\mathrm{Re}\, Z_\ell(0) = 0 \tag{4.25}$$

Because $\mathrm{Im}\, Z_\ell$ is an odd function of frequency $\mathrm{Im}\, Z_\ell(0) = 0$, hence $Z_\ell(0) = 0$. Using the definition of the impedance (4.5)

$$Z_\ell(\omega) = \frac{1}{c}\int_0^\infty ds\, w_\ell(s) e^{i\omega s/c}$$

we see that

$$Z_\ell(0) = \frac{1}{c}\int_0^\infty ds\, w_\ell(s) = 0$$