Lecture 5
Resistive wall wake

January 24, 2019

Lecture outline

- Skin effect and the Leontovich boundary condition.
- Parameter s_0 and the resistive wall wake.
- Longitudinal and transverse RW wake in the limit $s \gg s_0$.

Maxwell's equations in metal

To understand interaction of a beam with a metallic wall, we need to consider effects of finite conductivity, or *resistive wall* effect.
We start with quick derivation of the so called *skin effect*.
The skin effect deals with the penetration of the electromagnetic field inside a conducting medium characterized by a conductivity σ and magnetic permeability μ. We neglect the displacement current $\partial \boldsymbol{D}/\partial t$ in Maxwell's equations in comparison with $\boldsymbol{j}$:

$$\nabla \times \boldsymbol{H} = \boldsymbol{j}\,, \quad \nabla \cdot \boldsymbol{B} = 0\,, \quad \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{5.1}$$

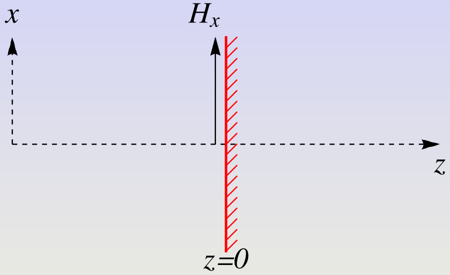
where $\boldsymbol{B} = \mu \boldsymbol{H}$. In the metal we have the relation between the current and the electric field

$$\boldsymbol{j} = \sigma \boldsymbol{E} \tag{5.2}$$

Combining all these equations, one finds the *diffusion* equation for the magnetic field $\boldsymbol{B}$:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \sigma^{-1}\mu^{-1}\nabla^2 \boldsymbol{B} \tag{5.3}$$

Skin effect



A metal occupies a semi-infinite volume $z > 0$ with the vacuum at $z < 0$. We assume that at the metal surface the x-component of magnetic field is given by $H_x = H_0 e^{-i\omega t}$. Due to the continuity of the tangential components of $\boldsymbol{H}$, H_x is the same on both sides of the metal boundary, that is at $z = +0$ and $z = -0$.

Skin effect

Seek solution inside the metal in the form $H_x = h(z)e^{-i\omega t}$. Equation (5.3) then reduces to

$$\frac{d^2 h}{dz^2} + i\mu\sigma\omega h = 0$$

with the solution $h = H_0 e^{ikz}$ and

$$k = \sqrt{i\mu\sigma\omega} = (1+i)\sqrt{\frac{\mu\sigma\omega}{2}} \equiv \frac{1+i}{\delta}$$

Note that we've chosen $\mathrm{Im}\, k > 0$ so that the field exponentially decays into the metal. The quantity δ,

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{5.4}$$

is called the *skin depth*; it characterizes how deeply the electromagnetic field penetrates into the metal, $|H_x| \propto e^{-z/\delta}$.

Skin effect

In many cases, the magnetic properties of the metal can be neglected, then $\mu = \mu_0$

$$\delta = \sqrt{\frac{2c}{Z_0 \sigma \omega}} \tag{5.5}$$

The electric field inside the metal has only y component; it can be found from the first and the last of Eqs. (5.1)

$$E_y = \frac{j_y}{\sigma} = \frac{1}{\sigma}\frac{dH_x}{dz} = \frac{ik}{\sigma} H_x = \frac{i-1}{\sigma\delta} H_x \tag{5.6}$$

The mechanism that prevents penetration of the magnetic field deep inside the metal is a generation of a tangential electric field, through Faraday's law, that drives the current in the skin layer and shields the magnetic field.
In reality the metal has finite a thickness Δ: our results are valid for $\Delta \gg \delta$.

The Leontovich boundary condition

The relation (5.6) can be rewritten in vectorial notation:

$$\boldsymbol{E}_t = \zeta \boldsymbol{H} \times \boldsymbol{n} \tag{5.7}$$

where $\boldsymbol{n}$ is the unit vector normal to the surface and directed toward the metal, and

$$\zeta(\omega) = \frac{1 - i}{\sigma \delta(\omega)} \tag{5.8}$$

Eq. (5.7) is called the Leontovich boundary condition. Remember that ζ is a function of ω — it is only applicable to the Fourier representation of the field.

Perfectly conducting metal

In the limit $\sigma \to \infty$ we have $\delta \to 0$ and $\zeta \to 0$ and we recover the boundary condition (3.3) of the zero tangential electric field on the surface of a perfect conductor. One can also show that in this limit the normal magnetic field is zero on the surface of the metal[15]:

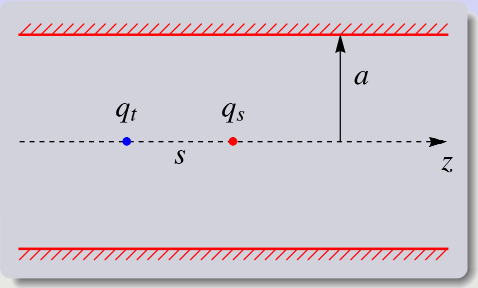
$$B_n = 0. \tag{5.9}$$

The approximation of small δ is good for calculation of EM field of short bunches (rapidly varying fields). It is not valid for a constant current ($\omega = 0$). When ω is small, the skin depth becomes much larger then the wall thickness t, $\delta \gg t$. The magnetic field penetrates through the metal, while the tangential component of the electric field is zero on the surface.

At large frequencies the conductivity begins to depend on frequency — the so called *ac conductivity*. At low temperatures there is an anomalous skin effect where (5.2) does not work.

[15] It follows from Faraday's law of induction.

Round pipe with resistive walls



We need to solve Maxwell's equations using the Lentovich boundary conditions and to find the electric field $E_z(s)$ behind the source charge to calculate the longitudinal wake. The problem is easier solved in the Fourier representation where one calculates the longitudinal impedance $Z_\ell(\omega)$.

In this problem, there is an important parameter s_0 in this problem which we now introduce using an order of magnitude estimate.

Parameter s_0

Consider a bunch of length σ_z with the peak current I propagating in the round pipe a. What is the magnetic field H_θ on the wall (this field defines E_z on the wall through the Leontovich boundary condition)? For a perfectly conducting wall this field will be the same as in vacuum (Ampere's law)

$$H_\theta = \frac{I}{2\pi a} \tag{5.10}$$

but the longitudinal electric field in the system changes the field through the Maxwell equation

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \epsilon_0 \boldsymbol{E}}{\partial t}$$

which involves the *displacement* current in z direction $\partial \epsilon_0 E_z / \partial t$. Let us estimate E_z from the boundary condition, $E_z \sim \zeta(\omega) H_\theta$. We estimate $\partial/\partial t \sim \omega \sim c/\sigma_z$. When we integrate j_z through the cross section of the pipe we get the current I. We now integrate $\partial \epsilon_0 E_z / \partial t$ through the cross section:

$$\sim a^2 \frac{c}{\sigma_z} \epsilon_0 \frac{1}{\sigma\delta} \frac{I}{a} \sim a \frac{c}{\sigma_z} \epsilon_0 \frac{1}{\sigma\sqrt{\frac{2c}{Z_0 \sigma \omega}}} I \sim a \frac{c}{\sigma_z} \epsilon_0 \frac{1}{\sigma\sqrt{\frac{2\sigma_z}{Z_0 \sigma}}} I$$

This term is of the order if I when

Round pipe with resistive walls

$$\sigma_z \sim \frac{a^{2/3}}{(Z_0\sigma)^{1/3}}$$

Here comes the parameter

$$s_0 = \left(\frac{2a^2}{Z_0\sigma}\right)^{1/3} \tag{5.11}$$

For $\sigma_z \gg s_0$ the magnetic field of the beam on the wall is very close to the vacuum one, Eq. (5.10). For $\sigma_z \lesssim s_0$ this field is suppressed by the displacement current. RW wake looks different for distances $s \gg s_0$ and $s \lesssim s_0$.

For $a = 5$ cm

Metal	Copper	Aluminium	Stainless Steel
s_0, μm	60	70	240

Round pipe with resistive walls

A. Chao calculated the longitudinal impedance valid for $a \gg \delta$,

$$Z_\ell(\omega) = \frac{Z_0 s_0}{2\pi a^2} \left(\frac{i\,\mathrm{sgn}(\kappa) + 1}{|\kappa|^{1/2}} - \frac{i\kappa}{2} \right)^{-1} \tag{5.12}$$

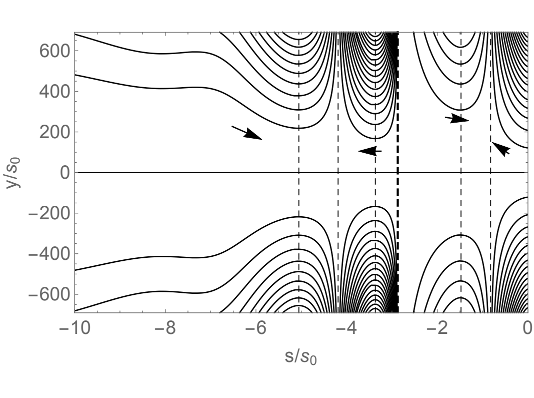
where $\kappa = \omega s_0/c$. Remarkably, this impedance depends only on the scaled frequency κ. Making the Fourier transform of the impedance, one finds the wake per unit length[16]

$$w_\ell(s) = \frac{Z_0 c}{4\pi} \frac{16}{a^2} \left(\frac{1}{3} e^{-s/s_0} \cos \frac{\sqrt{3}s}{s_0} - \frac{\sqrt{2}}{\pi} \int_0^\infty \frac{dx\, x^2}{x^6 + 8} e^{-x^2 s/s_0} \right), \qquad s > 0 \tag{5.13}$$

[Prove that the integral of this wake is equal to zero.]

[16] K. L. F. Bane and M. Sands. The Short-Range Resistive Wall Wakefields. SLAC-PUB-95-7074, Dec. 1995

Field lines



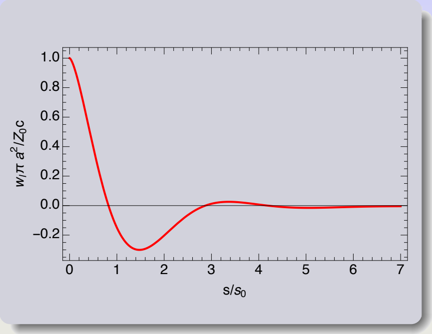
Here $-s$ is the distance behind the point charge located at $s = 0$ (courtesy of K. Bane). Note that the field changes sign 3 times and then remains accelerating at $-s \gtrsim 4.3$.

Longitudinal resistive wall wake

The wake at the origin,

$$w_\ell(0) = \frac{Z_0 c}{\pi a^2}$$

does not depend on the conductivity!



Limit $s \gg s_0$ is

$$w_\ell = -\frac{c}{4\pi^{3/2} a}\sqrt{\frac{Z_0}{\sigma s^3}} \tag{5.14}$$

σ is the conductivity. Negative wake means acceleration of the trailing charge. This limit corresponds to the approximation $\kappa \ll 1$ in the impedance,

$$Z_\ell(\omega) = \frac{Z_0 s_0}{2\pi a^2}\frac{|\kappa|^{1/2}}{i\,\mathrm{sgn}(\kappa)+1} = \frac{1}{4\pi a}\left(\frac{2Z_0|\omega|}{c\sigma}\right)^{1/2}(1 - i\,\mathrm{sgn}(\omega)) \tag{5.15}$$

Transverse resistive wall wake

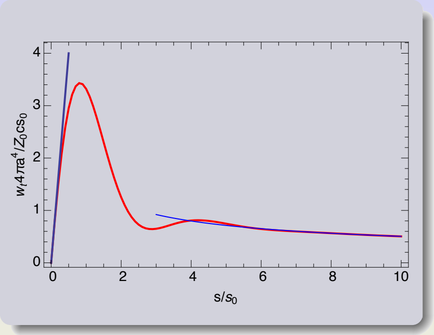
Resistive wall transverse wake for $s \gg s_0$ is

$$\bar{w}_t = \frac{c}{\pi^{3/2} a^3} \sqrt{\frac{Z_0}{\sigma s}} \qquad (5.16)$$

For, $s_0 \gtrsim s$ the wake is shown in the figure.

Slope at the origin

$$\left. \frac{d\bar{w}_t}{ds} \right|_{s=0} = \frac{2 Z_0 c}{\pi a^4}$$



The transverse impedance in the limit $s \gg s_0$ is

$$Z_t(\omega) = \frac{1 - i\,\mathrm{sgn}(\omega)}{2\pi a^3} \sqrt{\frac{2 Z_0 c}{\sigma |\omega|}} \qquad (5.17)$$

Universal values of the wake at the origin

We obtained the following results for the wake w_ℓ and the derivative $d\bar{w}_t/ds$ *at the origin* for the resistive wall:

$$w_\ell(0) = \frac{Z_0 c}{\pi a^2}$$

$$\left.\frac{d\bar{w}_t}{ds}\right|_{s=0} = \frac{2Z_0 c}{\pi a^4} \tag{5.18}$$

It turns out that these results are also valid in other situations: a metal wall covered by dielectric, a corrugated wall, a periodic sequence of round diaphragms (a model of RF structure)[17]. In all cases we talk about the limit $s \to 0$. However, the effective value of s_0 is different for different problems.

[17] A generalization for other cross sections can be found in: Baturin and Kanareykin, PRL **113**, 214801 (2014).

Resistive wall wake and a Gaussian bunch

As an example, let us calculate $\Delta\mathcal{E}_{\mathrm{av}}$ and $\Delta\mathcal{E}_{\mathrm{rms}}$ for the resistive wall wake given by Eq. (5.14) and a Gaussian distribution function,

$$\lambda(z) = \frac{1}{\sqrt{2\pi}\sigma_z} \exp\left(-\frac{z^2}{2\sigma_z^2}\right) \tag{5.19}$$

where σ_z is the rms bunch length. Note that, since w_ℓ in Eq. (5.14) is the wake per unit length of the pipe, we need to multiply the final answer by the pipe length L.

We assume $\sigma_z \gg s_0$. A direct substitution of the wake Eq. (5.14) into Eq. (4.1) gives a divergent integral when $z' \to z$. This divergence is caused by the singularity of Eq. (5.14) at $s = 0$ where it is not valid, (remember that $s \gg s_0$).

Resistive wall wake and a Gaussian bunch

One way to fix this singularity is to use the correct expression for the wake at $s \lesssim s_0$. A simpler, although more formal, approach is to represent w_ℓ as a derivative of another function (see Eq. (3.5)), $w_\ell = V'(s)$ with $V = (c/2\pi^{3/2}a)\sqrt{Z_0/\sigma s}$ for $s > 0$, and $V = 0$ for $s < 0$[18]. We then rewrite Eq. (4.1) as

$$\begin{aligned}
\Delta\mathcal{E}(z) &= -Ne^2L\int_{-\infty}^{\infty} dz'\lambda(z')\frac{dV(z'-z)}{dz} \\
&= Ne^2L\int_{z}^{\infty} dz'\frac{d\lambda(z')}{ds}V(z'-z) \\
&= \frac{Ne^2Lc\sqrt{Z_0}}{2^{3/2}\pi^2 a\sigma_z^{3/2}\sigma^{1/2}}G\left(\frac{z}{\sigma_z}\right)
\end{aligned} \tag{5.20}$$

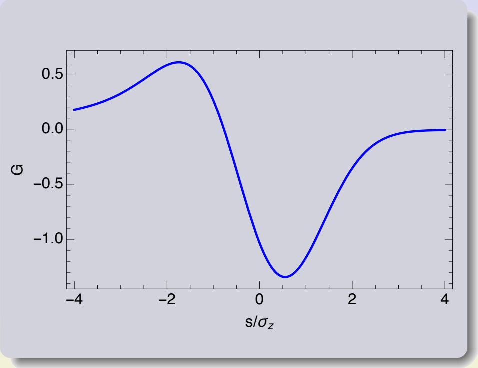
where the function $G(x)$ is

$$G(x) = -\int_x^{\infty}\frac{ye^{-y^2/2}dy}{\sqrt{y-x}}$$

[18] We should have $V(\infty) - V(-\infty) = 0$ because the area under the wake is zero.

Resistive wall wake and Gaussian bunch

Plot of the function $G(s/\sigma_z)$. The positive values of s correspond to the head of the bunch.



Particles lose energy in the head of the bunch ($s > 0$) and get accelerated in the tail ($s < 0$). On average, of course, the losses overcome the gain.

Resistive wall wake and Gaussian bunch

For the average energy loss one can find an analytical result:

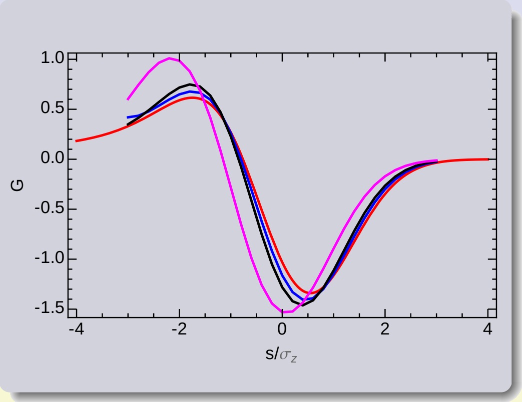
$$\Delta\mathcal{E}_{\mathrm{av}} = -\frac{\Gamma\left(\frac{3}{4}\right)}{2^{5/2}\pi^2}\frac{Ne^2c\sqrt{Z_0}L}{a\sigma_z^{3/2}\sigma^{1/2}} \tag{5.21}$$

Numerical integration of Eq. (5.20) shows that the energy spread generated by the resistive wake is approximately equal to ΔE_{av}:

$$\Delta\mathcal{E}_{\mathrm{rms}} = 1.06|\Delta\mathcal{E}_{\mathrm{av}}| \tag{5.22}$$

Calculation of the bunch wake for resistive wall

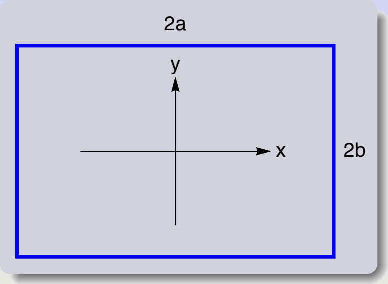
Do we make a mistake when calculate the energy loss $\Delta\mathcal{E}(z)$ using the wake in the limit $s \gg s_0$ and integrating by parts (see (5.20))? Is it better to use a more accurate wake valid for arbitrary s?



Magenta – $\sigma_z = s_0$; black – $\sigma_z = 2s_0$; blue – $\sigma_z = 3s_0$; red – this limit $s \gg s_0$.

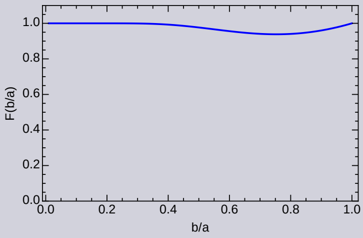
Longitudinal RW wake in a rectangular vacuum chamber

See derivations in[19].



Consider a rectangular vacuum chamber with dimensions $2a \times 2b$. We consider the limit $s \gg s_0$,

$$w_\ell = -F\left(\frac{b}{a}\right)\frac{c}{4\pi^{3/2}b}\sqrt{\frac{Z_0}{\sigma s^3}} \tag{5.23}$$

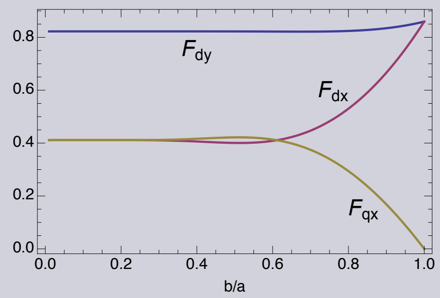


[19] Gluckstern, Zeijts and Zotter. PRE, **47**, 656 (1993)

Transverse RW wake in a rectangular vacuum chamber

When all particles have the same offset, the wake is given by Eqs. (3.10)

$$w_y(s,y) = [\bar{w}_y^d(s) + \bar{w}_y^q(s)]y$$
$$w_x(s,x) = [\bar{w}_x^d(s) + \bar{w}_x^q(s)]x$$



Again, we consider the limit $s \gg s_0$. Introduce

$$u(s) = \frac{c}{\pi^{3/2} b^3} \sqrt{\frac{Z_0}{\sigma s}}$$

(see Eq. (5.16)).

$$w_x^d(s) = F_{dx}\left(\frac{b}{a}\right) u(s) \qquad (5.24)$$
$$w_y^d(s) = F_{dy}\left(\frac{b}{a}\right) u(s)$$
$$w_x^q(s) = -w_x^q(s) = F_{qx}\left(\frac{b}{a}\right) u(s)$$

Parallel plates limit:
$F_{dx}(0) = F_{qx}(0) = \pi^2/24$,
$F_{dy}(0) = \pi^2/12$.