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Proton and Ion Linear Accelerators

1. Basics of Beam Acceleration

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Energy, Velocity, Momentum

Total energy $$E=\sqrt{(pc)^2+(mc^2)^2}=mc^2+W$$

Rest energy $$mc^2$$

Kinetic energy $$W=\sqrt{p^2c^2+m^2c^4}-mc^2=mc^2(\gamma-1)$$

Relativistic particle energy $$\gamma=\frac{mc^2+W}{mc^2}=\frac{1}{\sqrt{1-\beta^2}}$$

Particle velocity $$\vec{\beta}=\frac{\vec{\mathrm{v}}}{c}$$

Mechanical (kinetic) particle momentum $$\vec{p}=m\gamma\vec{\mathrm{v}}=mc\vec{\beta}\gamma$$

Particle velocity versus relativistic energy $$\beta=\frac{\sqrt{\gamma^2-1}}{\gamma}$$

Mechanical momentum versus velocity and relativistic energy $$\frac{p}{mc}=\beta\gamma=\sqrt{\gamma^2-1}$$

Energy, Velocity, Momentum (cont.)

	β	γ	W	cp
β	β	$\frac{\sqrt{\gamma^2-1}}{\gamma}$	$\frac{\sqrt{(1+W/E_0)^2-1}}{1+W/E_0}$	$\frac{cp/(mc^2)}{\sqrt{1+[cp/(mc^2)]^2}}$
γ	$\frac{1}{\sqrt{1-\beta^2}}$	γ	$1+W/E_0$	$\sqrt{1+\left(\frac{cp}{mc^2}\right)^2}$
W	$\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)E_0$	$E_0(\gamma-1)$	W	$mc^2\left[\sqrt{1+\left(\frac{cp}{mc^2}\right)^2}-1\right]$
cp	$mc^2\frac{\beta}{\sqrt{1-\beta^2}}$	$E_0(\gamma^2-1)^{1/2}$	$[W(2E_0+W)]^{1/2}$	cp

Some relations concerning first derivatives of relativistic factors:

$$\frac{d\beta}{d\gamma}=\frac{1}{\beta\gamma^3}\ ;\quad \frac{d(1/\beta)}{d\gamma}=-\frac{1}{\beta^3\gamma^3}\ ;\quad \frac{d(\beta\gamma)}{d\beta}=\gamma^3\ ;\quad \frac{d(\beta\gamma)}{d\gamma}=\frac{1}{\beta}\ ;$$

Logarithmic first derivatives:

$$\frac{d\beta}{\beta}=\frac{1}{\beta^2\gamma^2}\frac{d\gamma}{\gamma}=\frac{1}{\gamma(\gamma+1)}\frac{dW}{W}=\frac{1}{\gamma^2}\frac{dp}{p}\ ;\quad \frac{d\gamma}{\gamma}=(\gamma^2-1)\frac{d\beta}{\beta}=\left(1-\frac{1}{\gamma}\right)\frac{dW}{W}=\beta^2\frac{dp}{p}$$

(P. Lapostolle and M. Weiss, CERN-PS-2000-001 DR)

Vector Operations in Cartesian Coordinates

$$\nabla \psi = \frac{\partial \psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial \psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Vector Operations in Cylindrical Coordinates

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = z \qquad z = z$$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(r\,A_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r\,A_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\mathbf{z}}$$

Note that

$$\nabla\times\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & r\,A_\theta & A_z \end{vmatrix}.$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

Units

$$W = eU\ [eV],\ [electronVolt]$$

$$1\ eV = 1.6\cdot 10^{-19}[C]\ \text{x}\ 1\ [V] = 1.6\cdot 10^{-19}\ Joule$$

$$1\ \ Joule = 1\ \ Coulomb \cdot 1\ \ Volt = \frac{kg\cdot m^2}{s^2}$$

Electron energy

$$m_{electron} = 9.1\cdot 10^{-31}\,kg$$

$$c = 3\cdot 10^{8}\,m/\sec$$

$$e = 1.6\cdot 10^{-19}\,Culomb$$

$$\frac{m_{electron}c^2}{e} = 0.51092\cdot 10^{6}\,Volt$$

$$m_{electron}c^2 = 0.51092\cdot 10^{6}\,eV = 0.51092\,MeV$$

Proton energy

$$m_{proton} = 1.672\cdot 10^{-27}\,kg = 1836\,m_{electron}$$

$$m_{proton}c^2 = 938.3\,MeV$$

$$\frac{m_{proton}c^2}{e} = 938.3\cdot 10^{6}\,Volt$$

Units (cont.)

Ion Energy

Atomic mass unit (1/12 the mass of one atom of carbon-12):

1u= 1.660540 x 10^{-27} kg

$$E_a = 931.481\,MeV$$

Proton mass: 1.007276 u
Electron mass: 0.00054858 u

$$E_{ion} = 931.481 \cdot A - 0.511 \cdot Z\,[MeV]$$

A-atomic mass number

Z-number of removed electrons (ionization state)

Binding energy of removed electrons is neglected

Negative Ion of Hydrogen

H^- ion mass: 1.00837361135 u

$$E_{H^-} = E_{proton} + 2\,\mathrm{x}\,E_{electron} = 939.28\,MeV$$

Units (cont.)

Particle momentum

$$\frac{p}{mc} = \beta\gamma = \sqrt{\gamma^2 - 1} \qquad p = \frac{mc^2}{c}\sqrt{\gamma^2 - 1} \; [\frac{GeV}{c}]$$

Particle rigidity

$$B\rho = \frac{p}{q} \; [T \cdot m]$$

Example: proton beam with kinetic energy W = 3 GeV:

$$E = mc^2 + W = 3.938\, GeV \qquad \gamma = \frac{mc^2 + W}{mc^2} = 4.2 \qquad \beta = \frac{\sqrt{\gamma^2 - 1}}{\gamma} = 0.971$$

$$\frac{p}{mc} = \beta\gamma = \sqrt{\gamma^2 - 1} = 4.079 \qquad p = \frac{mc^2}{c}\sqrt{\gamma^2 - 1} = 3.82\, \frac{GeV}{c} \qquad \frac{p}{e} = B\rho = 12.7 \; T \cdot m$$

Equations of Motion

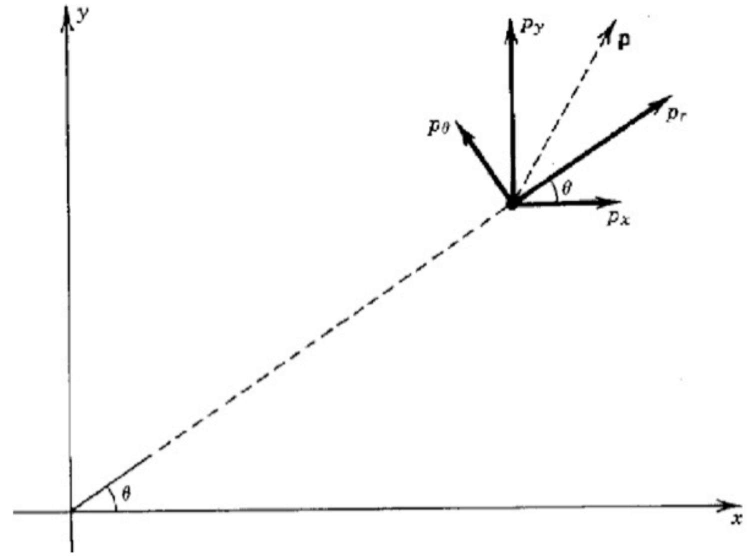
$$\frac{d\vec{x}}{dt} = \vec{\mathrm{v}} \qquad \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{\mathrm{v}} \times \vec{B})$$

Cartesian Coordinates

$$\frac{dx}{dt} = \frac{p_x}{m\gamma} \qquad \frac{dp_x}{dt} = q\left(E_x + \frac{p_y}{m\gamma}B_z - \frac{p_z}{m\gamma}B_y\right)$$

$$\frac{dy}{dt} = \frac{p_y}{m\gamma} \qquad \frac{dp_y}{dt} = q\left(E_y - \frac{p_x}{m\gamma}B_z + \frac{p_z}{m\gamma}B_x\right)$$

$$\frac{dz}{dt} = \frac{p_z}{m\gamma} \qquad \frac{dp_z}{dt} = q\left(E_z + \frac{p_x}{m\gamma}B_y - \frac{p_y}{m\gamma}B_x\right)$$



Relationship between cylindrical and Cartesian coordinates.

z-axis is directed to the reader.

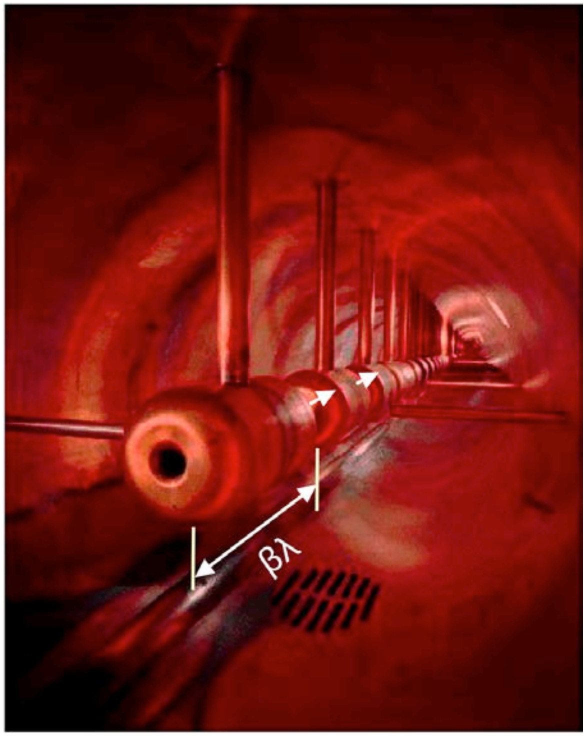
Cylindrical coordinates

$$\frac{dr}{dt} = \frac{p_r}{m\gamma} \qquad \frac{dp_r}{dt} = \frac{p_\theta^2}{m\gamma\, r} + q\left(E_r + \frac{p_\theta}{m\gamma}B_z - \frac{p_z}{m\gamma}B_\theta\right)$$

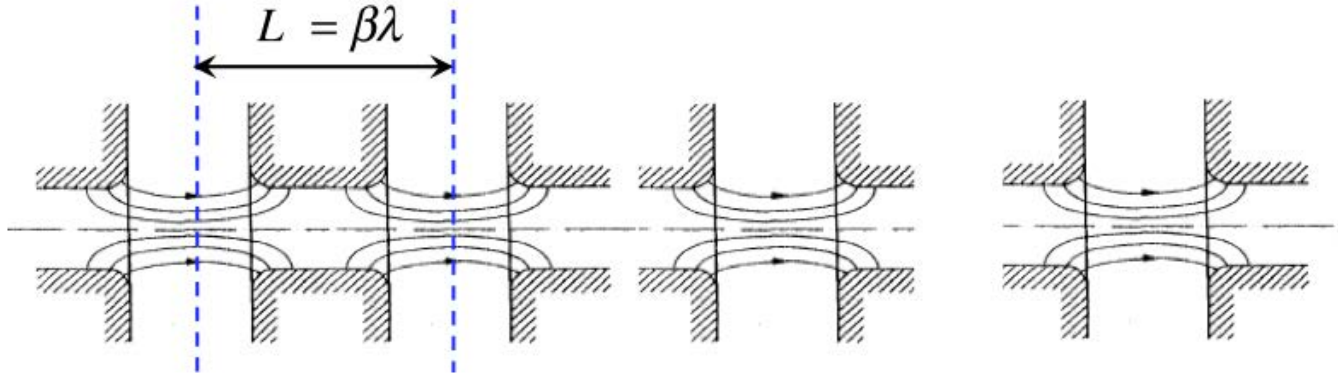
$$\frac{d\theta}{dt} = \frac{p_\theta}{m\gamma\, r} \qquad \frac{1}{r}\frac{d(r p_\theta)}{dt} = q\left(E_\theta + \frac{p_z}{m\gamma}B_r - \frac{p_r}{m\gamma}B_z\right)$$

$$\frac{dz}{dt} = \frac{p_z}{m\gamma} \qquad \frac{dp_z}{dt} = q\left(E_z + \frac{p_r}{m\gamma}B_\theta - \frac{p_\theta}{m\gamma}B_r\right)$$

Resonance Principle of Particle Acceleration



Alvarez accelerating structure



Field distribution in RF structure: $E_z(z,r,t) = E_g(z,r)\cos(\omega t)$

Time of flight between RF gaps $t_{flight} = T_{RF\ period} = \frac{1}{f}$

Distance between RF gaps $L = n\beta c T_{RF\ period} = n\beta\lambda$

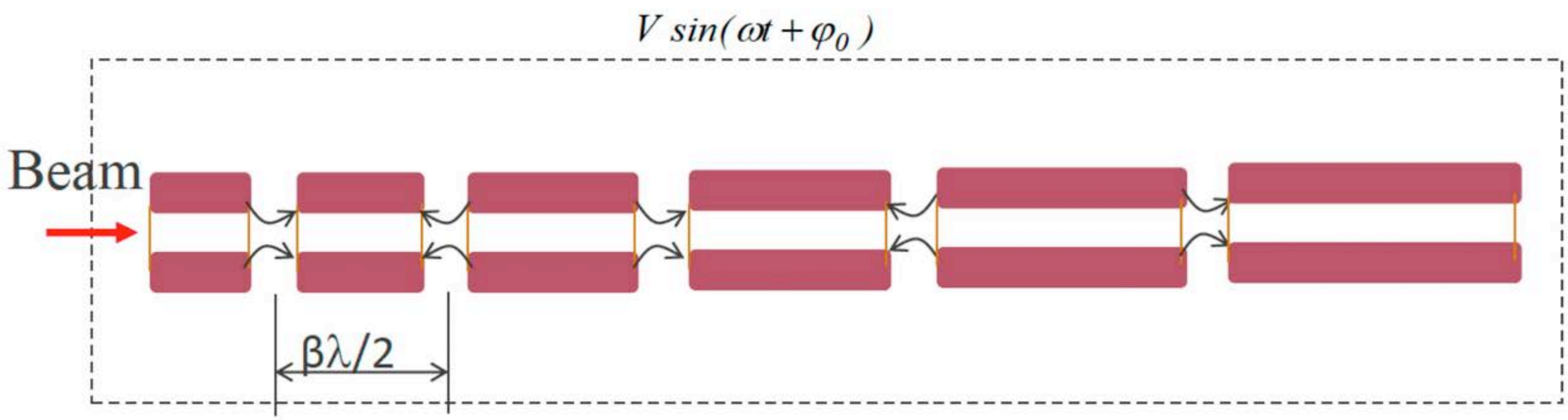
RF Frequency f

Circular RF Frequency $\omega = 2\pi f$

RF Wavelength $\lambda = \frac{c}{f}$

Acceleration in linear resonance accelerator is based on synchronism between accelerating field and particles.

Acceleration in π - Structure



Accelerating structure with π - type standing wave.

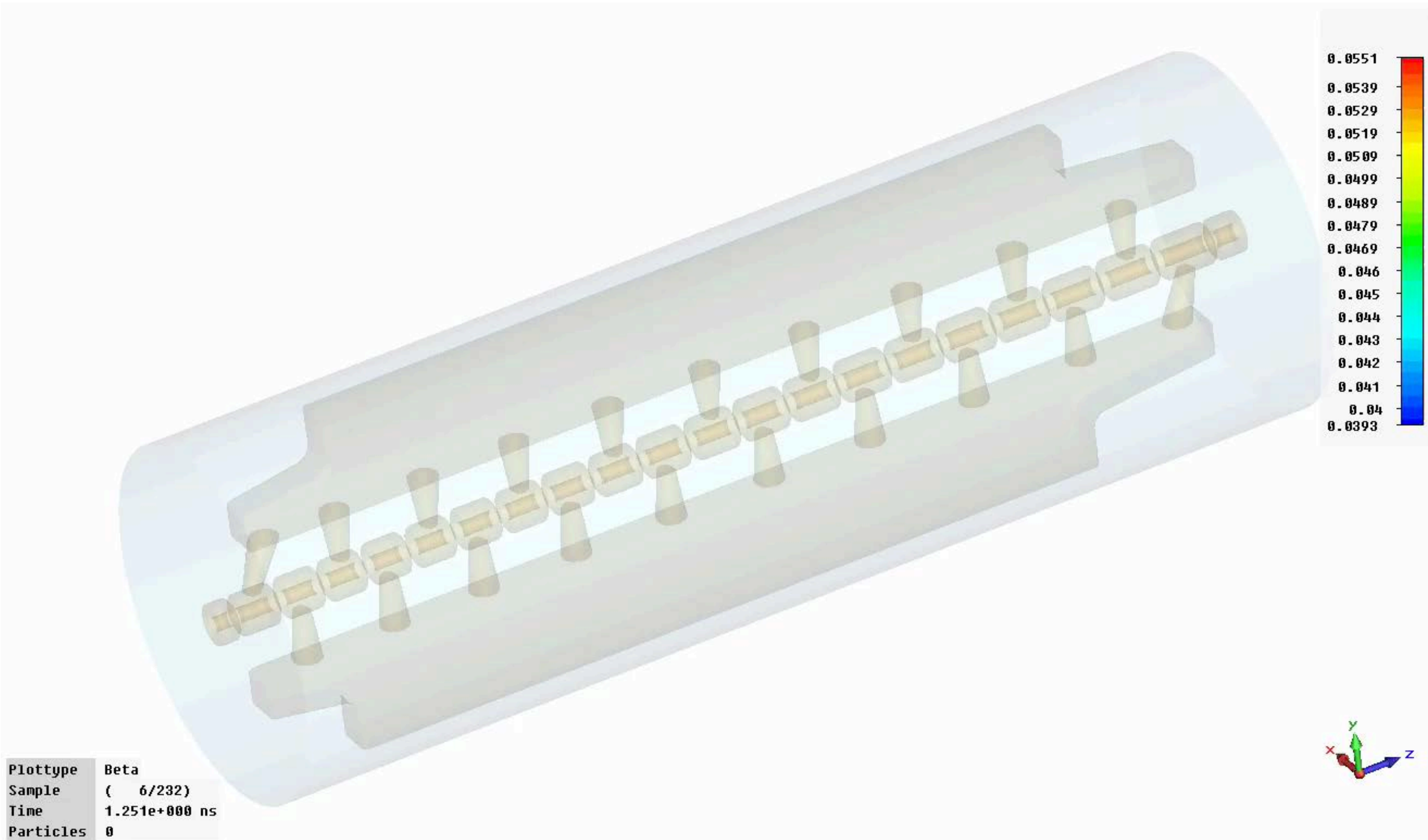
Time of flight between RF gaps of π- structure

$$t_{flight} = \frac{T_{RF\ period}}{2}$$

Distance between RF gaps of π- structure

$$L = \frac{\beta c T_{RF\ period}}{2} = \frac{\beta\lambda}{2}$$

Acceleration in π- Structure



Acceleration in π- structure (Courtesy of Sergey Kurennoy).

Induction Accelerator

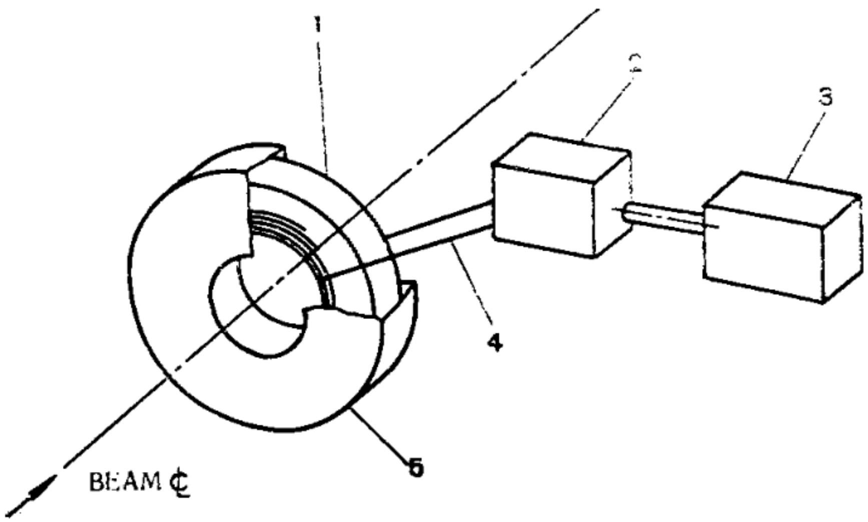
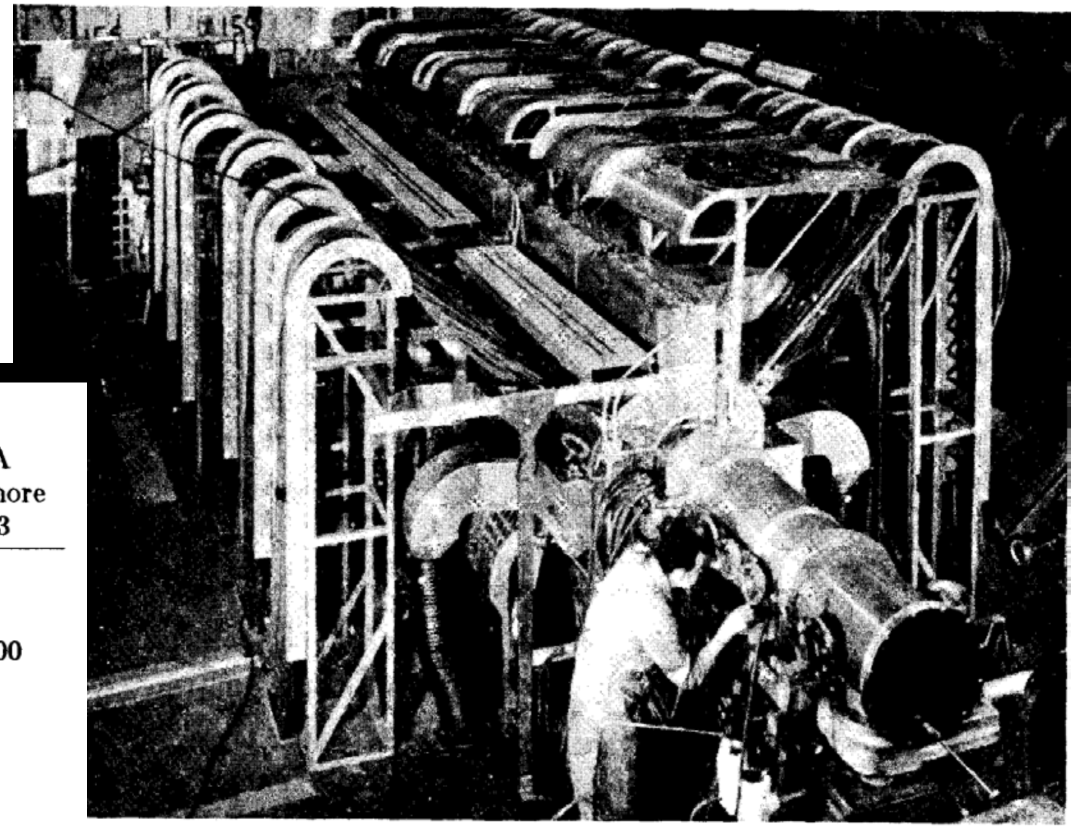


Fig. 1. Induction accelerator principle:
1 — laminated iron core; *2* — switch; *3* — pulse forming network; *4* — primary loop; *5* — secondary (case).

$$\oint \vec{E}\cdot\vec{dl} = -\frac{1}{c}\int_{s} \frac{\vec{dB}}{dt}\cdot\vec{ds},$$

Table 3. Parameters for Typical Induction Accelerators

Accelerator	Astron Injector Livermore 1963	ERA Injector Berkeley 1971	NEP 2 Injector Dubna 1971	ATA Livermore 1983
Kinetic energy, MeV	3.7	4.0	30	50
Beam current on target, A	350	900	250	10,000
Pulse duration, ns	300	2-45	500	50
Pulse energy, kJ	0.4	0.1	3.8	25
Rep rate, pps	0-60	0-5	50	5
Number of switch modules	300	17	750	200



Overhead view of the Astron accelerator as it appeared when first put into operation.

Maxwell's equations

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$rot\,\vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}$$

$$div\,\vec{D} = \rho$$

$$div\,\vec{B} = 0$$

Electric field $\vec{E}$

Electric displacement field $\vec{D} = \varepsilon_o \vec{E}$

Magnetic field $\vec{B} = \mu_o \vec{H}$

Magnetic field strength $\vec{H}$

Permittivity of free space $\varepsilon_o = 8.85 \cdot 10^{-12}\,\mathrm{F/m}$

Permeability of free space $\mu_o = 4\pi \cdot 10^{-7}\,\mathrm{H/m}$

Electromagnetic Wave Equations

In the absence of charges, $\vec{j}=0,\ \rho=0$, Maxvell's equations are

$$rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad div\,\vec{E} = 0$$

$$rot\,\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \qquad div\,\vec{B} = 0$$

speed of light in free space:

$$c = \frac{1}{\sqrt{\varepsilon_o \mu_o}} = 2.99792458\cdot 10^8\, m/\sec$$

Taking the *rot* of the *rot* equations gives:

$$rot\ \ rot\,\vec{E} = -\frac{\partial}{\partial t}(rot\,\vec{B}) = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

$$rot\ \ rot\,\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}(rot\,\vec{E}) = -\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

By using the vector identity $\quad rot\ \ rot\,\vec{A} = grad\ div\,\vec{A} - \Delta\vec{A}$

Taking into account that $div\,\vec{E}=0,\ \ div\,\vec{B}=0$ we receive wave equations:

$$\boxed{\Delta\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0} \qquad \boxed{\Delta\vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0}$$

Components of Electromagnetic Field

Most of RF cavities are excited at a fundamental mode containing three components E_z , E_r , B_θ . They are connected through Maxwell's equations, therefore it is sufficient to find solution for one component only. Taking into account condition for axial-symmetric field ($\partial/\partial\theta = 0$), wave equation for E_z component is

$$\frac{\partial^2 E_z}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial E_z}{\partial r}) - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

Radial component E_z can be determined from $div\vec{E} = 0$ as

$$div\,\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_r) + \frac{\partial E_z}{\partial z} = 0$$

which gives

$$E_r(r) = -\frac{1}{r}\int_o^r \frac{\partial E_z}{\partial z} r' dr'$$

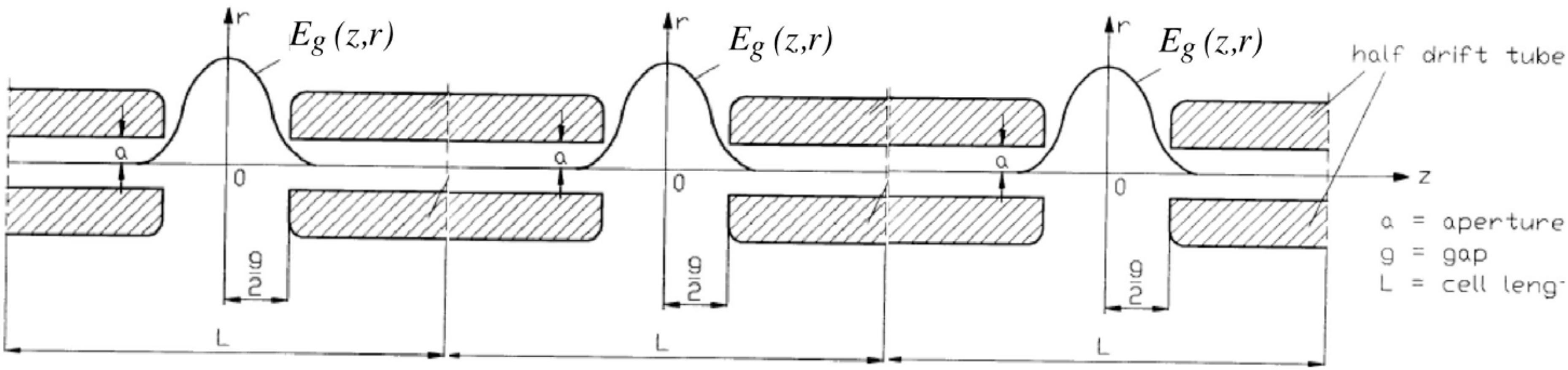
Azimuthal component of magnetic field is determined from

$$rot\,\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

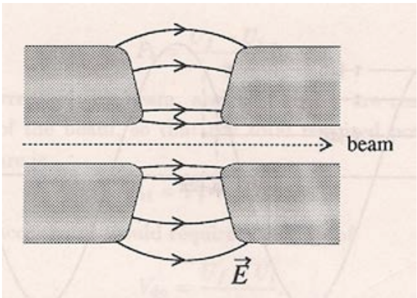
which gives

$$B_\theta = \frac{1}{c^2 r}\int_o^r \frac{\partial E_z}{\partial t} r' dr'$$

Expansion of RF Field



Periodic distribution of RF field.



Electric field lines between the ends of drift tubes.

Field in RF Gap: $E_z(z,r,t) = E_g(z,r)\cos(\omega t)$

Wave Equation for Field Distribution in RF Gap:

$$\frac{\partial^2 E_g}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_g}{\partial r}\right) + \left(\frac{\omega}{c}\right)^2 E_g = 0$$

Fourier Expansion of Field Distribution in RF Gap:

$$E_g(r,z) = A_o(r) + \sum_{m=1}^{\infty} A_m(r)\cos\left(\frac{2\pi m z}{L}\right)$$

Expansion of RF Field (cont.)

Equations for Fourier coefficients of RF gap expansion:

$$\frac{1}{r}\frac{\partial A_o(r)}{\partial r}+\frac{\partial^2 A_o(r)}{\partial r^2}+(\frac{\omega}{c})^2 A_o(r)=0, \qquad m=0$$

$$\frac{1}{r}\frac{\partial A_m(r)}{\partial r}+\frac{\partial^2 A_m(r)}{\partial r^2}-k_m^2 A_m(r)=0, \qquad m>0$$

Transverse wave number:

$$k_m=(\frac{2\pi m}{L})\sqrt{1-(\frac{L}{m\lambda})^2}$$

Solutions are Bessel functions:

$$A_o(r)=A_o J_o(\frac{r\omega}{c}), \qquad m=0$$

$$A_m(r)=A_m I_o(k_m r), \qquad m>0$$

Finally, expressions for spatial z-component $E_g\,(z,r)$

$$E_g(r,z)=A_o J_o(2\pi\frac{r}{\lambda})+\sum_{m=1}^{\infty}A_m I_o(k_m r)\cos(\frac{2\pi m z}{L})$$

Bessel Functions

Bessel functions of the order *n* are solutions $y = J_n(z)$ of differential Bessel equation:

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + (1 - \frac{n^2}{z^2})y = 0$$

Power representation of Bessel function:

$$J_n(z) = \frac{1}{n!}(\frac{z}{2})^n - \frac{1}{1!(n+1)!}(\frac{z}{2})^{n+2} + \frac{1}{2!(n+2)!}(\frac{z}{2})^{n+4} - ... = (\frac{z}{2})^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)}(\frac{z}{2})^{2k}$$

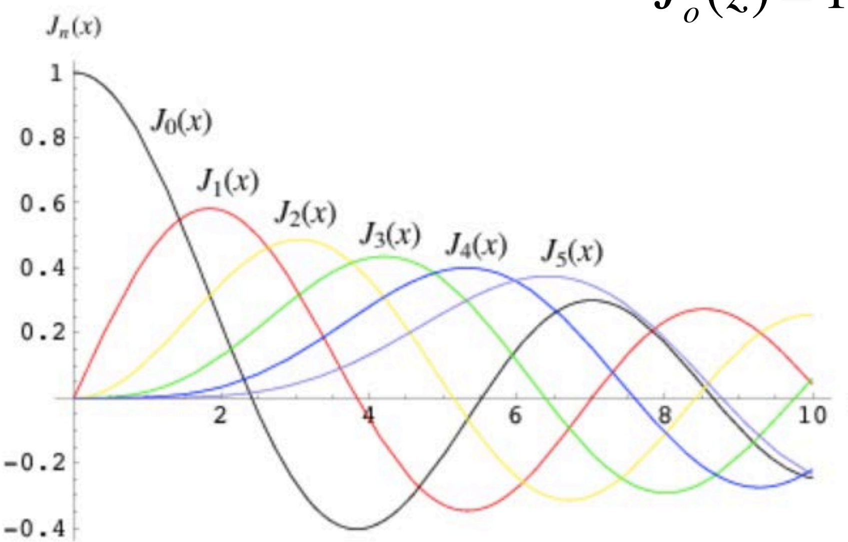
Integral representation of Bessel functions:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\ d\theta$$

Special cases for n = 0, 1:

$$J_o(z) = 1 - \frac{z^2}{4} + \frac{z^4}{64} - .....$$

$$J_1(z) = -J_o'(z) = \frac{z}{2} - \frac{z^3}{16} + ...$$



Zeros v_{nm} of Bessel function $J_n(z) = 0$.

	$m = 1$	$m = 2$	$m = 3$	$m = 4$
$n = 0$	2.405	5.52	8.654	11.792
$n = 1$	3.832	7.016	10.173	13.323
$n = 2$	5.136	8.417	11.62	14
$n = 3$	6.38	9.761	13.015	

Modified Bessel Functions

Modified Bessel functions of the *n*-th order $I_n(z) = i^{-n} J_n(iz)$ are solutions of modified Bessel differential equation:

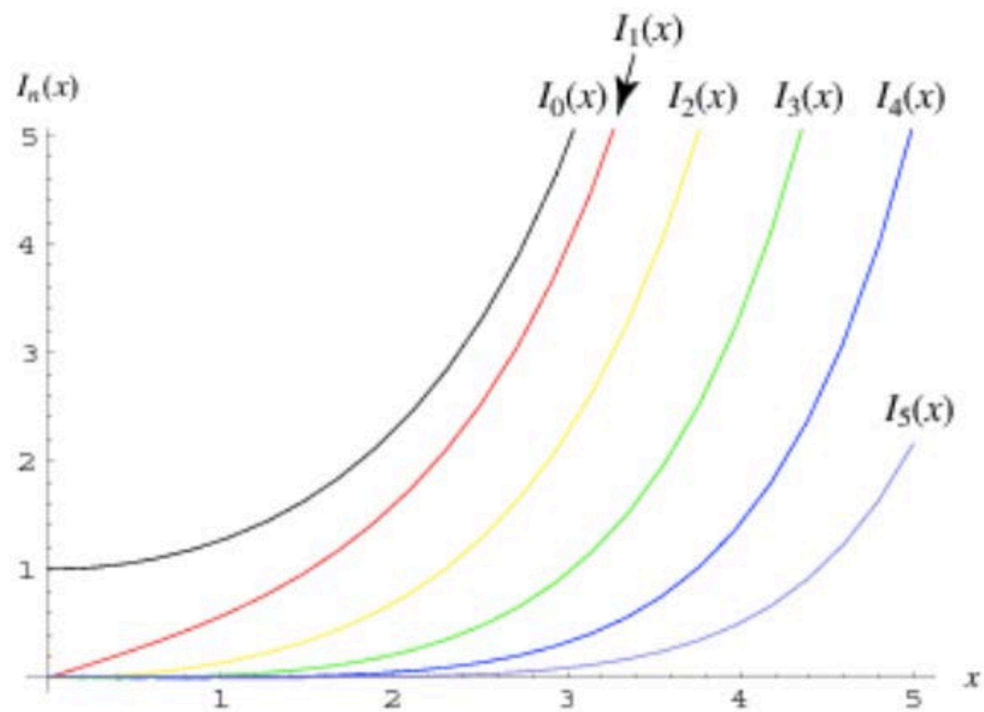
$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} - (1 + \frac{n^2}{z^2})y = 0$$

Power representation of modified Bessel functions: $I_n(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(n+k+1)} (\frac{z}{2})^{n+2k}$

Special cases for $n = 0, 1$:

$$I_o(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + \frac{z^6}{2304} + \ldots$$

$$I_1(z) = I_o^{'}(z) = \frac{z}{2} + \frac{z^3}{16} + \frac{z^5}{384} + \ldots$$



Modified Bessel functions of 1st kind, $I_n(x)$.

Integrals and Derivatives of Bessel Functions

Let $Z_n(x)$ to be an arbitrary Bessel function:

$$\frac{dZ_n(x)}{dx} = -\frac{n}{x}Z_n(x) + Z_{n-1}(x) = \frac{n}{x}Z_n(x) - Z_{n+1}(x)$$

$$\int x^{n+1} Z_n(x)\,dx = x^{n+1} Z_{n+1}(x)$$

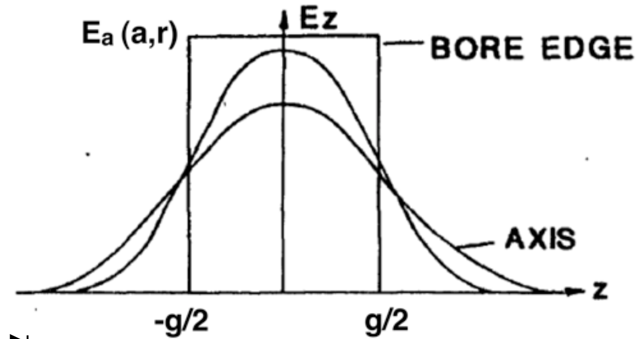
Particularly

$$Z_o^{'}(x) = -Z_1(x)$$

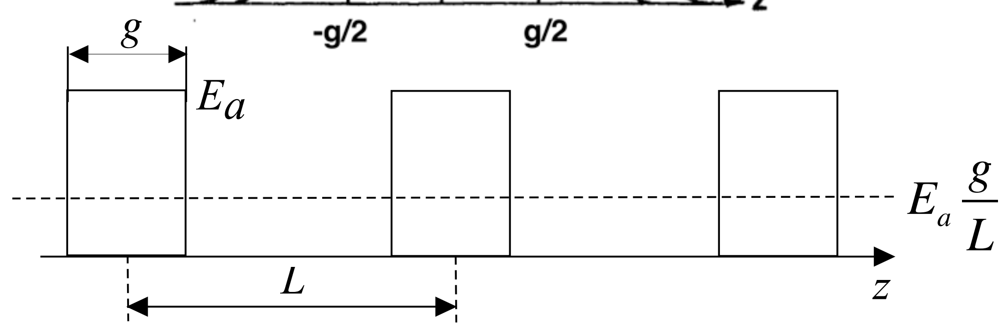
$$Z_1^{'}(x) = Z_o(x) - \frac{Z_1(x)}{x}$$

Expansion of RF Field (cont.)

To get an approximate expression for coefficients A_m , let us assume the step-function distribution of component inside RF gap of width at bore radius $r = a$



$$E_g(a,r) = \begin{cases} E_a, & 0 \le |z| \le \frac{g}{2} \\ 0, & \frac{g}{2} \le |z| \end{cases}$$



Expansion of periodic step-function

$$E_g(a,z) = E_a[\frac{g}{L} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin(\pi m\frac{g}{L})\cos(2\pi m\frac{z}{L})]$$

Field expansion in RF gap

$$E_g(r,z) = A_o J_o(2\pi\frac{r}{\lambda}) + \sum_{m=1}^{\infty} A_m I_o(k_m r)\cos(\frac{2\pi mz}{L})$$

Coefficients in field expansion:

$$A_o = \frac{E_a}{J_o(2\pi\frac{a}{\lambda})}\frac{g}{L}$$

$$A_m = \frac{2E_a}{I_o(k_m a)}\frac{g}{L}\frac{\sin(\pi m\frac{g}{L})}{\pi m\frac{g}{L}}$$

Energy Gain of Synchronous Particle in RF Gap

Equation for change of longitudinal particle momentum

$$\frac{dp_z}{dt} = qE_z(z, r, t)$$

From relativistic equations $p_z = mc\sqrt{\gamma^2 - 1}$

$$dp_z = mc^2 d\gamma / (\beta c) \qquad dW = mc^2 d\gamma$$

the equation for change of particle energy

$$\frac{dW}{dz} = qE_z(z, r, t)$$

Increment of energy of synchronous particle per RF gap

$$\Delta W_s = q \int_{-L/2}^{L/2} E_g(z) \cos \omega t_s(z) \, dz$$

When synchronous particle arrive in the center of the gap, $z = 0$, the RF phase is equal to φ_s . The time of arrival of synchronous particle in point with coordinate z

$$t_s(z) \approx \frac{\varphi_s}{\omega} + \frac{z}{\beta c}$$

or

$$\boxed{\omega t_s(z) = \varphi_s + k_z z}$$

Longitudinal wave number

$$k_z = \frac{2\pi}{\beta\lambda}$$

Energy Gain of Synchronous Particle in RF Gap (cont.)

Using equity $\cos\omega t_s = \cos\varphi_s \;\cos k_z z - \sin\varphi_s \;\sin k_z z$
the increment of synchronous particle energy per RF gap:

$$\Delta W_s = q\cos\varphi_s [\int_{-L/2}^{L/2} E_g(z)\;\cos(k_z z)dz - tg\varphi_s \int_{-L/2}^{L/2} E_g(z)\;\sin(k_z z)dz]$$

Let us multiply and divide this expression by $E_o L$, where we introduce average field E_o of the accelerating gap across accelerating period (note that E_o=A_o):

$$\boxed{E_o = \frac{1}{L}\int_{-L/2}^{L/2} E_g(z)dz} = \frac{E_a}{J_o(2\pi\frac{a}{\lambda})}\frac{g}{L} \approx E_a\frac{g}{L}$$

Effective voltage applied to RF gap: $\boxed{U = E_o L}$

The increment of synchronous particle energy per RF gap can be written as: $\boxed{\Delta W_s = qE_o TL\cos\varphi_s}$

where *transit time factor*

$$T = \frac{1}{E_o L}[\int_{-L/2}^{L/2} E_g(z)\;\cos(k_z z)dz - tg\varphi_s \boxed{\int_{-L/2}^{L/2} E_g(z)\;\sin(k_z z)dz}]$$

≈ 0

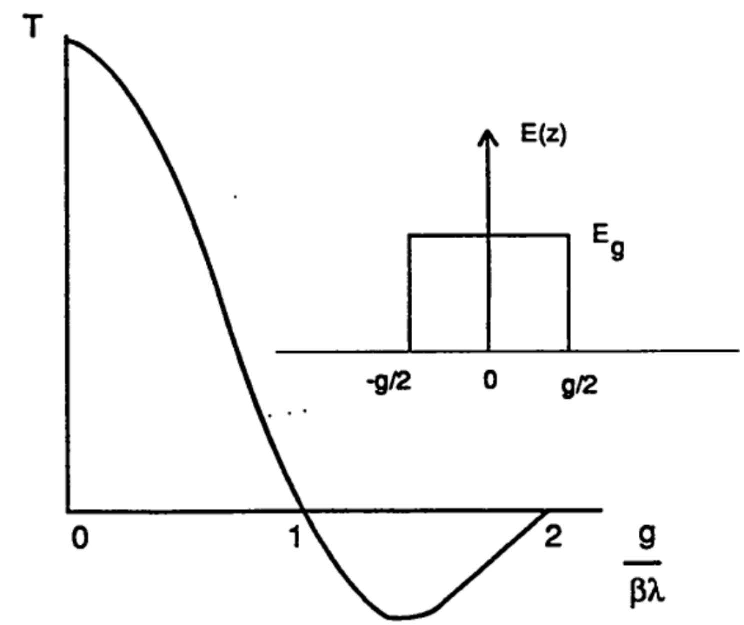
Transit Time Factor

Transit time factor indicates effectiveness of transformation of RF field into particle energy. It mostly depends on field distribution within the gap, which is determined by RF gap geometry.

Transit time factor $T = \frac{A_n}{2E_o}$, where A_n is the amplitude of *n*-th harmonics of Fourier field expansion

In most accelerators, synchronism is provided for *n* = 1, therefore:

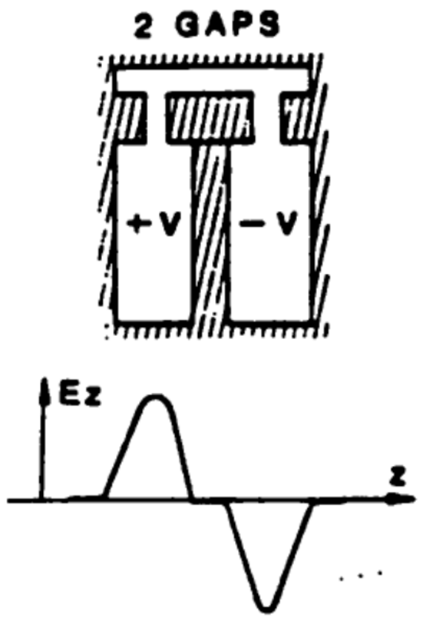
$$T = \frac{J_o(2\pi \frac{a}{\lambda})}{I_o(\frac{2\pi a}{\beta\gamma\lambda})} \frac{\sin(\frac{\pi g}{\beta\lambda})}{\frac{\pi g}{\beta\lambda}}$$



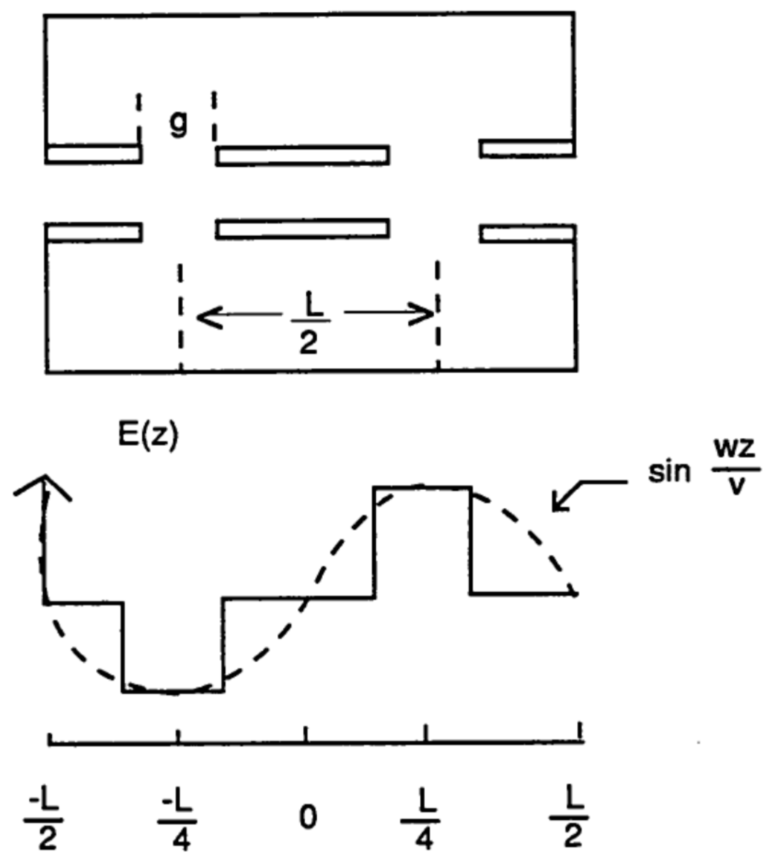
In accelerators usually aperture of the channel is substantially smaller than wavelength, $a << \lambda$, then $J_o(2\pi a / \lambda) \approx 1$, and transit time factor is

$$T = \frac{1}{I_o(\frac{2\pi a}{\beta\gamma\lambda})} \frac{\sin(\frac{\pi g}{\beta\lambda})}{\frac{\pi g}{\beta\lambda}}$$

Transit Time Factor for Two-Gap Cavity



Two gap cavity



Transit time factor for two-gap cavity

$$T = \frac{1}{I_o(\frac{2\pi a}{\beta\lambda})} \frac{\sin(\frac{\pi g}{\beta\lambda})}{\frac{\pi g}{\beta\lambda}} \sin\frac{\pi L}{2\beta\lambda}$$

Field expansion in two-gap cavity

Design of Accelerator Structure

Specify dependence of transit time factor on velocity: $T = T(\beta)$.

From equation for energy gain one can express dz_s

$$\frac{dW_s}{dz_s} = qE_o T\cos\varphi_s \quad \rightarrow \quad \boxed{dz_s = \frac{dW_s}{qE_o T\cos\varphi_s}}$$

Second equation:

$$\boxed{dt_s = \frac{dz_s}{\beta_s c}}$$

Using equation $dW_s = mc^2 \beta\gamma^3 d\beta$ we can rewrite them as

$$\boxed{dz_s = \left(\frac{mc^2}{qE_o\cos\varphi_s}\right)\frac{\beta d\beta}{T(\beta)(1-\beta^2)^{3/2}}}$$

$$\boxed{dt_s = \left(\frac{mc}{qE_o\cos\varphi_s}\right)\frac{d\beta}{T(\beta)(1-\beta^2)^{3/2}}}$$

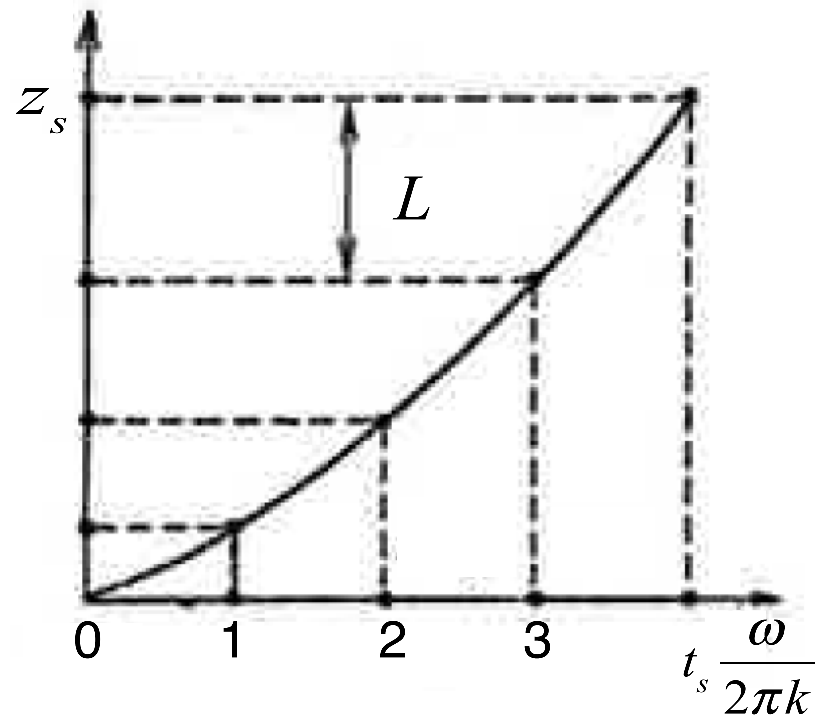
Design of Accelerator Structure (cont.)

Integration gives:

$$z_s = \left(\frac{mc^2}{qE_o \cos\varphi_s}\right)\int_{\beta_o}^{\beta} \frac{\beta d\beta}{(1-\beta^2)^{3/2} T(\beta)}$$

$$t_s = \left(\frac{mc}{qE_o \cos\varphi_s}\right)\int_{\beta_o}^{\beta} \frac{d\beta}{(1-\beta^2)^{3/2} T(\beta)}$$

Using β as independent variable, one can get parametric dependence $z_s(t_s)$. Increment in time $\Delta t_s = k(2\pi/\omega)$ corresponds to distance between centers of adjacent gaps $\Delta z_s = L$. Gap and drift tube length are determined by adjustment of the value of transit time factor $T=T(\beta, \lambda, a, g)$.



Calculation the lengths of accelerating periods.

Simplified Method of Design of Accelerator Structure

Increment of energy of synchronous particle per RF gap

$$\Delta W_s = qE_o TL\cos\varphi_s$$

Increment of energy through increment of relativistic factor

$$dW = mc^2 d\gamma$$

$$d\gamma = \beta\gamma^3 d\beta$$

Increment of velocity of synchronous particle per RF gap:

$$\beta_n \approx \beta_{n-1} + k\frac{qE_o T(\beta_s)\lambda}{mc^2\gamma_s^3}\cos\varphi_s$$

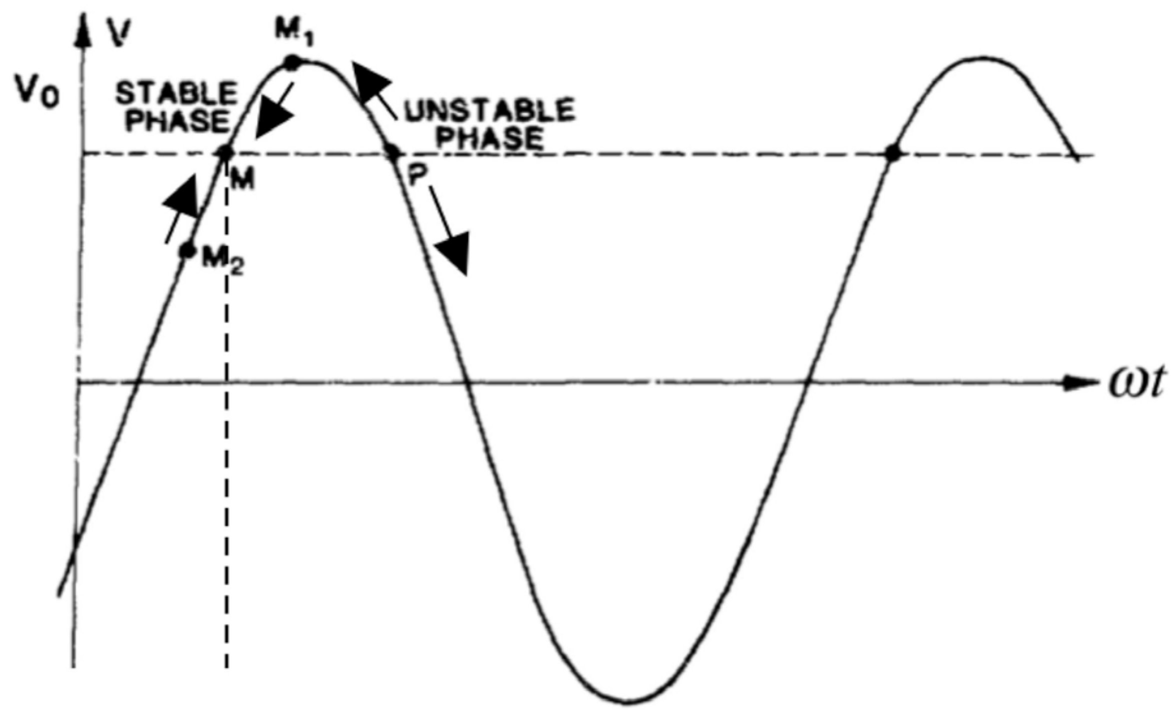
Average velocity at RF gap: $$\beta_s = \frac{\beta_n + \beta_{n-1}}{2}$$

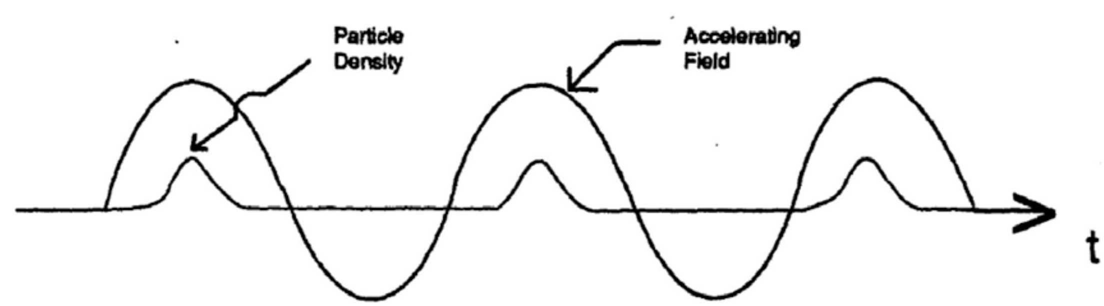
Cell length: $L = k\beta_s\lambda$ (k = 1 for 0 mode; k = 1/2 for π - mode)

Drift tube length $l = L$ - g

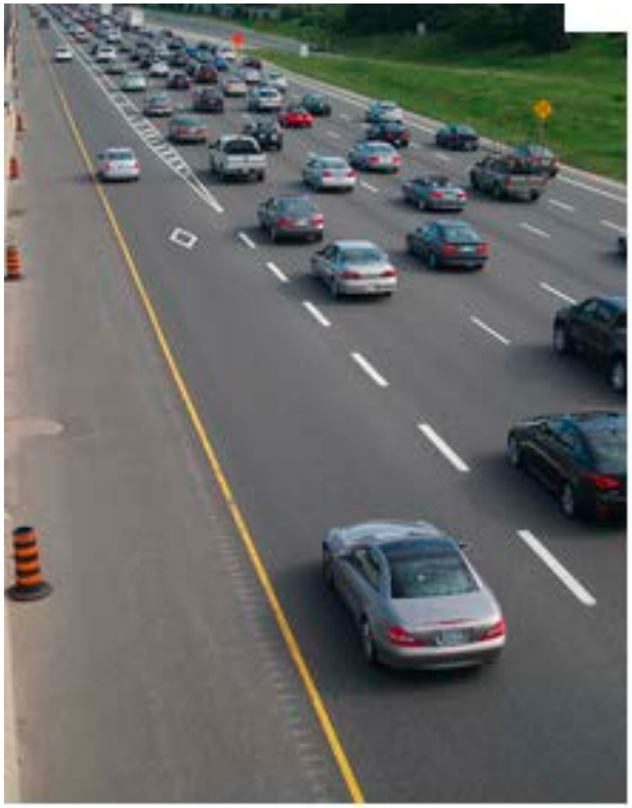
Autophasing: Stable and Unstable Phases

RF phase of synchronous particle is selected to be when the field is increasing in time. Earlier particle receive smaller energy kick than the synchronous one and will be slowing down with respect to synchronous particle. Particles, which arrive later to accelerating gap, receive larger energy gain, and will run down the synchronous particle. When non-equilibrium particles exchange their positions, this process is repeated for new particles setup, which results in stable longitudinal oscillations around synchronous particle. While synchronous particle monotonically increases it's energy, other particle perform oscillation around synchronous particle, and also increase their energy. Such principle is called resonance principle of particle acceleration.





Beam Bunching: Analogy with Traffic

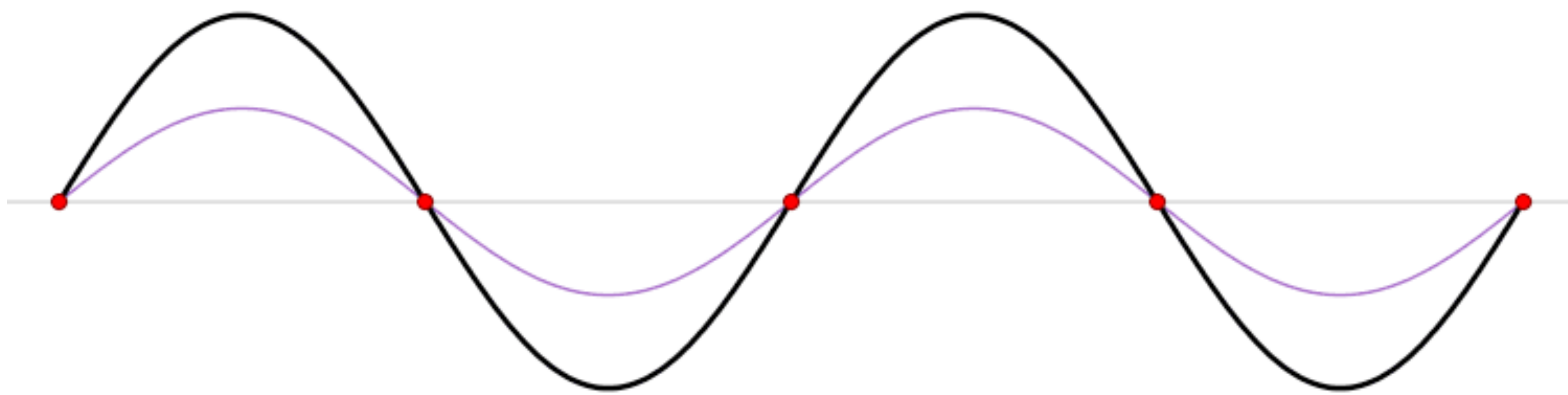


Continuous traffic

Bunched car traffic created by a traffic light

Standing Wave as a Combination of Traveling Waves

$$E_o \cos(k_z z)\cos(\omega t) = \frac{E_o}{2}[\cos(\omega t - k_z z) \quad + \quad \cos(\omega t + k_z z)]$$



Equivalent Traveling Wave

Increment of energy of arbitrary particle in RF gap

$$\Delta W = q \int_{-L/2}^{L/2} E_g(z,r)\ \cos(\omega t)\, dz$$

The RF phase at the time of arrival of arbitrary particle in point with coordinate z

$$\omega t(z) = \varphi + k_z z$$

Standing wave can be represented as combination of traveling waves:

$$\sum_{m=1}^{\infty} \cos(\frac{2\pi m z}{L}) \cos(\omega t) = \frac{1}{2}\sum_{m=1}^{\infty}\cos(\omega t - \frac{2\pi m z}{L}) + \frac{1}{2}\sum_{m=1}^{\infty}\cos(\omega t + \frac{2\pi m z}{L})$$

traveling waves in *z* – direction

traveling waves in opposite direction

Only *m* = 1 harmonic of traveling waves propagating in *z*-direction contributes to energy gain of particle. In general case *m* = *n* *(where L = nβλ).*

$$\int_{-L/2}^{L/2} \cos(\frac{2\pi z}{L} - \frac{2\pi m z}{L} + \varphi)\, dz = \begin{cases} L\cos\varphi, & m = 1 \\ 0, & m \neq 1 \end{cases}$$

$$\int_{-L/2}^{L/2} \cos(\frac{2\pi z}{L} + \frac{2\pi m z}{L} + \varphi)\, dz = 0$$

Equivalent Traveling Wave (cont.)

Increment of energy of arbitrary particle in RF gap

$$\Delta W = q\, E_o L T\, I_o(\frac{2\pi r}{\beta\gamma\lambda}) \cos\varphi$$

Taking into account equation for increment of particle energy $dW/\,dz = qE_z(z,r,t)$, the equivalent accelerating traveling wave is

$$E_z = E_o T\, I_o(\frac{k_z r}{\gamma}) \cos\varphi$$

Amplitude of equivalent traveling wave

$$E = E_o\, T$$

Electromagnetic field of equivalent traveling wave

$$E_z = E\, I_o(\frac{k_z r}{\gamma}) \cos\varphi$$

$$E_r = -\gamma E\, I_1(\frac{k_z r}{\gamma}) \sin\varphi$$

$$B_\theta = -\frac{\beta\gamma}{c} E\, I_1(\frac{k_z r}{\gamma}) \sin\varphi$$

Longitudinal Dynamics in Equivalent Traveling Wave

Phase of particle in traveling wave: $\varphi = \omega t - k_z z$

Phase velocity: φ = const $\quad d\varphi \approx \omega dt - k_z dz = 0$

$$v_{ph} = \frac{\omega}{k_z}$$

Wave number $\quad k_z(z) = \dfrac{2\pi}{\beta_{ph}(z)\lambda}$

Longitudinal equations of motion of arbitrary particle

$$\frac{d\varphi}{dz} = \frac{2\pi}{\lambda}\left(\frac{1}{\beta} - \frac{1}{\beta_{ph}(z)}\right)$$

$$\frac{dW}{dz} = qE\cos\varphi$$

Auto-phasing principle: particle with $\beta > \beta_{ph}$ is slowing down with respect to synchronous particle; particle with $\beta < \beta_{ph}$ is accelerating with respect to synchronous particle. For synchronous particle $\beta = \beta_{ph}(z)$. Dependence $\beta_{ph}(z)$ is determined by geometry of accelerating structure. Synchronous phase is selected automatically with certain value of accelerating field E:

$$\cos\varphi_s = \frac{1}{qE}\frac{dW_s}{dz}$$

With change of field E, synchronous phase is changing, and particles start oscillate around new synchronous phase.

Oscillations Around Synchronous Particle

Equations of longitudinal motion in traveling wave near axis $I_o(\frac{k_z r}{\gamma}) \approx 1$

$$\frac{dp_z}{dt} = qE\cos\varphi$$

$$\frac{dz}{dt} = \frac{p_z}{m\gamma}$$

Longitudinal momentum deviation from synchronous particle

$$p_\zeta = p_z - p_s$$

Deviation from synchronous particle

$$\zeta = z - z_s$$

Phase of particle in traveling wave:

$$\varphi = \omega t - k_z(z_s + \zeta) = \varphi_s - k_z\zeta$$

Equations of particle motion around synchronous particle

$$\frac{d\zeta}{dt} = \frac{dz}{dt} - \frac{dz_s}{dt} = d(\beta c)$$

$$d\beta = \frac{1}{\gamma^3}\frac{dp}{mc}$$

$$\boxed{\frac{dp_\zeta}{dt} = qE[\cos(\varphi_s - k_z\zeta) - \cos\varphi_s]}$$

$$\boxed{\frac{d\zeta}{dt} = \frac{p_\zeta}{m\gamma^3}}$$

Hamiltonian of Longitudinal Oscillations

Equations of motion around synchronous particle can be derived from Hamiltonian

$$H = \frac{p_\zeta^2}{2m\gamma^3} + \frac{qE}{k_z}[\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s]$$

Hamiltonian equations of motion:

$$\frac{d\zeta}{dt} = \frac{\partial H}{\partial p_\zeta} \qquad \frac{dp_\zeta}{dt} = -\frac{\partial H}{\partial \zeta}$$

Hamiltonian describes particle oscillations around synchronous particle, where parameters γ, *E*, k_z depend on longitudinal position. Let us assume that parameters γ, *E*, k_z, are changing slowly during particle oscillations. Hamiltonian with constant values of γ, *E*, k_z, is a *constant of motion*. Actually, in this case:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \zeta}\frac{d\zeta}{dt} + \frac{\partial H}{\partial p_\zeta}\frac{dp_\zeta}{dt} = \frac{\partial H}{\partial \zeta}\frac{\partial H}{\partial p_\zeta} - \frac{\partial H}{\partial p_\zeta}\frac{\partial H}{\partial \zeta} = 0$$

Time-independent Hamiltonian *coincides with particle energy (kinetic + potential)*. Equation $dH/dt = 0$ expresees conservation of energy in isolated system (conservative approximation). In this case, we get equation for phase space trajectory $p_\zeta = p_\zeta(\zeta)$ as equation

$$H(\zeta, p_\zeta) = const$$

Hamiltonian of Longitudinal Oscillations in ($\Delta W, \psi$)

Another pair of canonical variables: $\psi = \varphi - \varphi_s$, $\Delta W = W_s - W$

Phase deviation from synchronous particle $\psi = -k_z \zeta$

Inverse energy deviation from synchronous particle: $\Delta W = -\beta c p_\zeta$

Hamiltonian of energy-phase oscillations around synchronous particle:

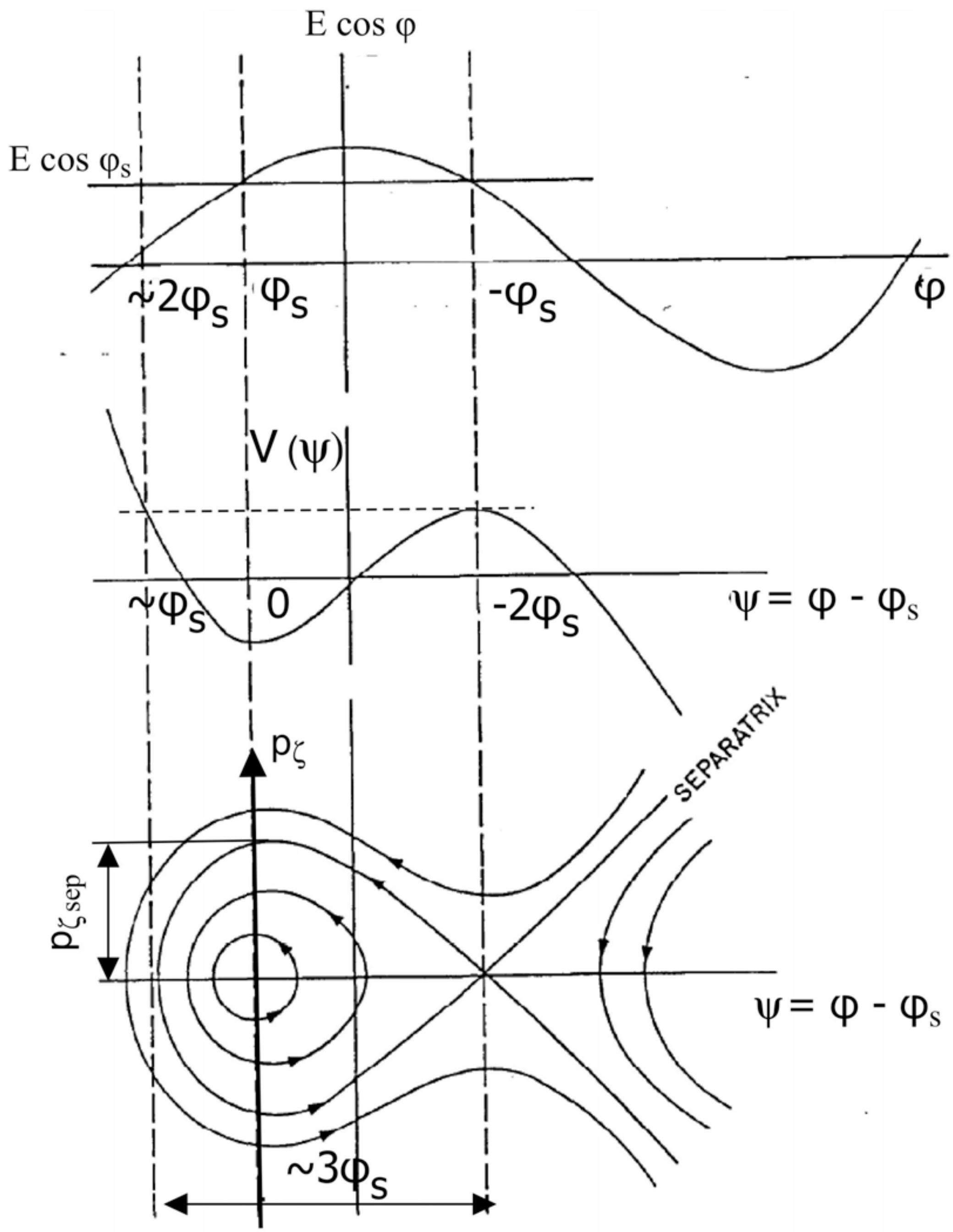
$$H = \frac{(\Delta W)^2}{2m\gamma_s^3\beta_s^2c^2}\omega + qE\beta c[\sin(\varphi_s + \psi) - \psi\cos\varphi_s]$$

Equations of motions:

$$\frac{d\Delta W}{dt} = qE\beta c[\cos\varphi_s - \cos(\varphi_s + \psi)]$$

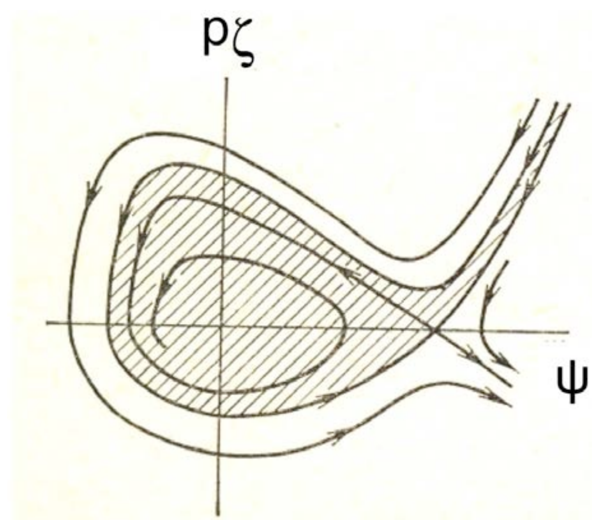
$$\frac{d\psi}{dt} = \frac{\Delta W}{m\gamma_s^3\beta_s^2c^2}\omega$$

Accelerating Field, Potential Function, and Separatrix



Potential function:

$$V(\psi) = \frac{qE}{k_z}[\sin(\varphi_s + \psi) - \psi \cos\varphi_s]$$



Separatrix of longitudinal phase space oscillations including acceleration.

Equation of Separatrix

Derivative of potential function determines two extremum points: stable point $\psi = 0$, unstable point $\psi = -2\varphi_s$.

$$\frac{dV}{d\psi} = \frac{qE}{k_z}[\cos(\varphi_s + \psi) - \cos\varphi_s] = 0$$

To be stable, potential function must have minimum in extremum point $\psi = 0$, or the second derivative hast to be positive

$$\frac{d^2V(0)}{d\psi^2} = -\frac{qE}{k_z}\sin\varphi_s > 0$$

Stability condition sin φ_s < 0

$$\boxed{\varphi_s < 0}$$

Hamiltonian, corresponding to separatrix

$$H_{sep} = H(p_\zeta = 0, \psi = -2\varphi_s)$$

$$H_{sep} = \frac{qE}{k_z}[-\sin\varphi_s + 2\varphi_s\cos\varphi_s]$$

Equation for separatrix

$$\boxed{\frac{p_\zeta^2}{2m\gamma^3} + \frac{qE}{k_z}[\sin(\varphi_s + \psi) - \psi\cos\varphi_s + \sin\varphi_s - 2\varphi_s\cos\varphi_s] = 0}$$

Phase Width of Separatrix

Phase length of separatrix Φ_s is determined from separatrix equation assuming $p_\zeta=0$

$$\sin(\varphi_s+\psi)-\psi\cos\varphi_s+\sin\varphi_s-2\varphi_s\cos\varphi_s=0$$

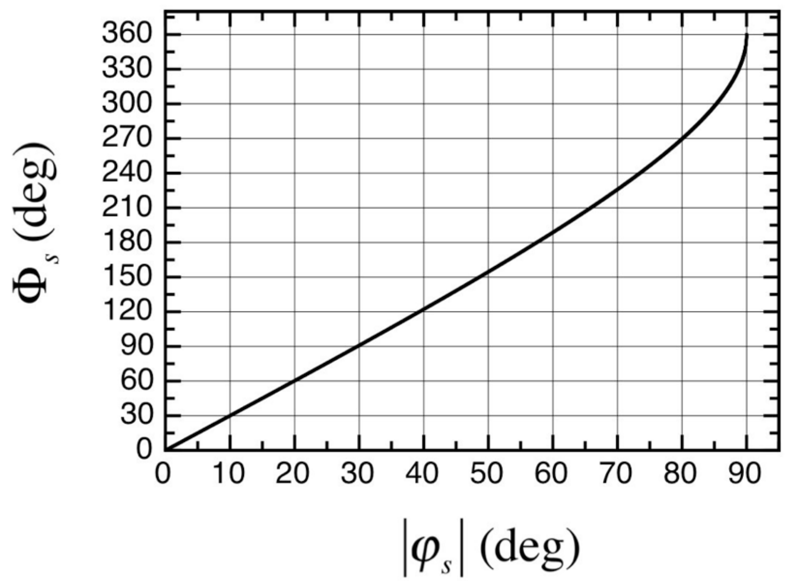
Equation has two roots $\psi_1 = -2\varphi_s$, and ψ_2. Width of separatrix is $\Phi_s = \psi_2 + 2\,|\varphi_s|$
Substitution $\psi_2 = \Phi_s - 2\,|\varphi_s|$ into upper equation gives expression for determination of phase width of separatrix:

$$tg|\varphi_s| = \frac{\Phi_s - \sin\Phi_s}{1-\cos\Phi_s}$$

For small values of synchronous phase, $tg\,\varphi_s \approx \varphi_s$ $\sin\Phi_s \approx \Phi_s - \Phi_s^3/6$
$\cos\Phi_s \approx 1-\Phi_s^2/2$ phase width of separatrix

$$\Phi_s \approx 3|\varphi_s|$$

Therefore, $\psi_2 \approx \varphi_s$



Phase width of separatrix as a function of synchronous phase.

Frequency of Linear Small Amplitude Oscillations

Equation for longitudinal oscillations

$$\frac{d^2\zeta}{dt^2} = \frac{qE}{m\gamma^3}[\cos(\varphi_s - k_z\zeta) - \cos\varphi_s]$$

For small amplitude oscillations

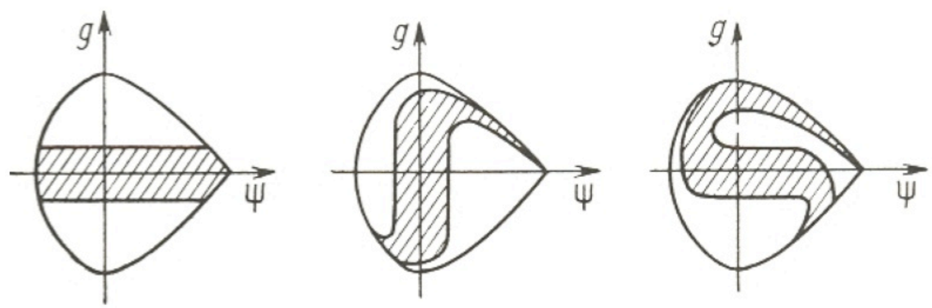
$$\cos(\varphi_s - k_z\zeta) \approx \cos\varphi_s + k_z\zeta\sin\varphi_s$$

$$\frac{d^2\zeta}{dt^2} + \left(\frac{qE\,k_z\,|\sin\varphi_s|}{m\gamma^3}\right)\zeta = 0$$

Frequency of small amplitude linear oscillations

$$\Omega = \sqrt{\frac{qEk_z|\sin\varphi_s|}{m\gamma^3}}$$

$$\frac{\Omega}{\omega} = \sqrt{\frac{qE\lambda}{mc^2}\frac{|\sin\varphi_s|}{2\pi\beta\gamma^3}}$$



Distortion of the longitudinal phase space due to nonlineartity of longitudinal forces.

At the separatrix $k_z\zeta=2\varphi_s$, frequency is zero:

$$\cos(\varphi_s - k_z\zeta) - \cos\varphi_s = 0$$

Hamiltonian of Linear Small Amplitude Oscillations

From Hamiltonian of longitudinal oscillations

$$H = \frac{p_\zeta^2}{2m\gamma^3} + \frac{qE}{k_z}[\sin(\varphi_s - k_z\zeta) + k_z\zeta\cos\varphi_s]$$

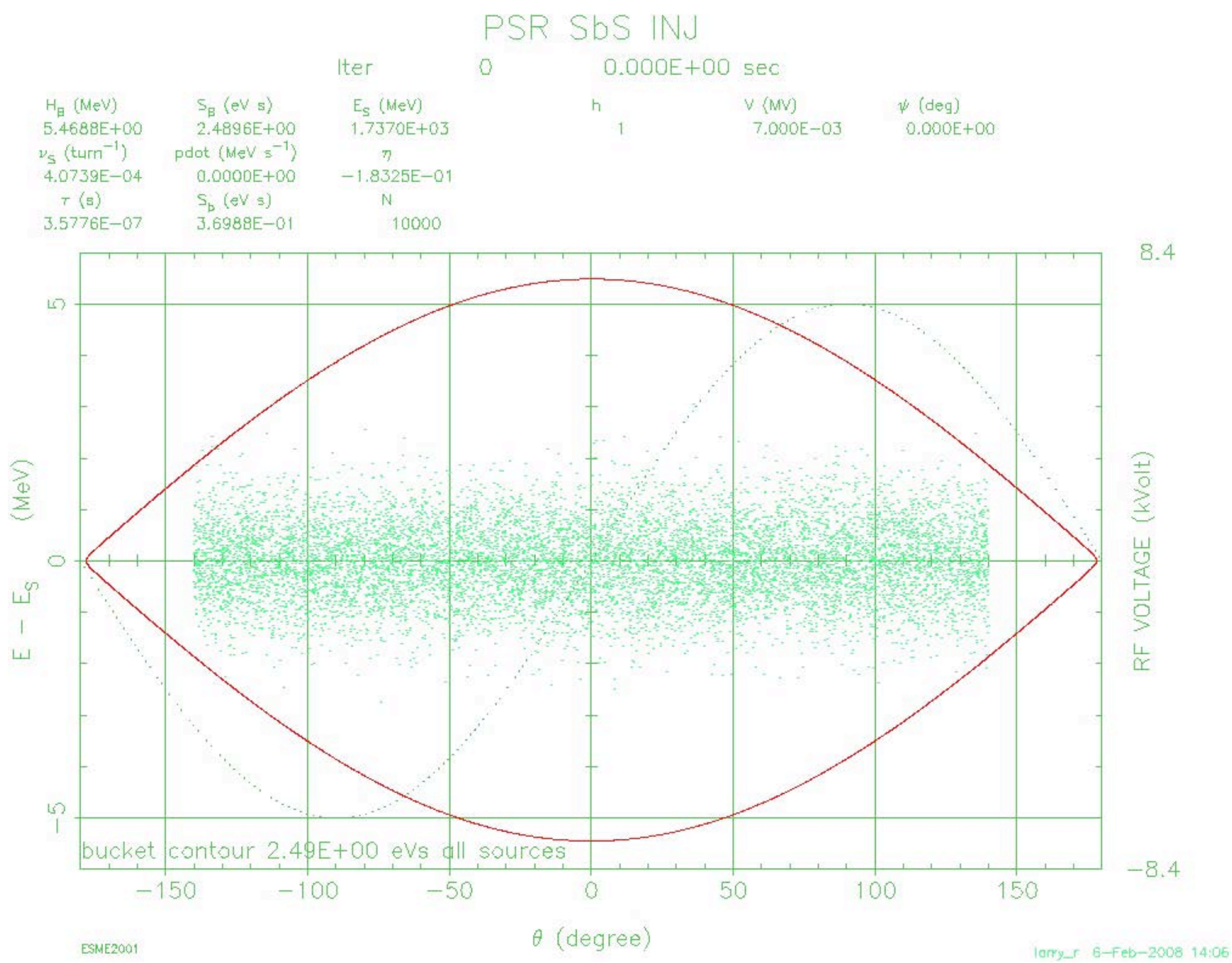
expanding trigonometric function

$$\sin(\varphi_s - k_z\zeta) \approx \sin\varphi_s - k_z\zeta\cos\varphi_s - \frac{(k_z\zeta)^2}{2}\sin\varphi_s$$

Hamiltonian of small linear oscillations:

$$\boxed{H = \frac{p_\zeta^2}{2m\gamma^3} + m\gamma^3\Omega^2\frac{\zeta^2}{2}}$$

Example of Longitudinal Oscillations



Longitudinal oscillations in RF field with φ_s = - 90^o
(Courtesy of Larry Rybarcyk).

Phase Advance of Longitudinal Oscillations

Equation of linear longitudinal oscillations

$$\frac{d^2\zeta}{dt^2} + \Omega^2\zeta = 0$$

Change variable $z = \beta c t$

$$\frac{d^2\zeta}{dz^2} + (\frac{\Omega}{\beta c})^2\zeta = 0$$

Solution of equation of longitudinal oscillations

$$\zeta = \zeta_o \cos(\frac{\Omega}{\beta c} z + \psi_o)$$

Let *S* to be a period of focusing structure. Phase advance of longitudinal oscillations per focusing period

$$\mu_{ol} = \frac{\Omega}{\beta c} S = \sqrt{2\pi(\frac{qE\lambda}{mc^2})\frac{|\sin\varphi_s|}{\beta\gamma^3}}\ (\frac{S}{\beta\lambda})$$

Phase advance of longitudinal oscillations per accelerating period

$$\mu_{oa} = \frac{\Omega}{\beta c} L = \sqrt{2\pi(\frac{qE\lambda}{mc^2})\frac{|\sin\varphi_s|}{\beta\gamma^3}}\ (\frac{L}{\beta\lambda})$$

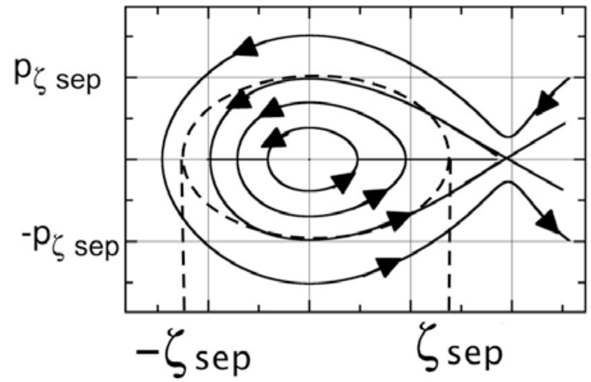
For Alvarez structure $L = \beta\lambda$, for π – mode structures $L = \beta\lambda/2$

Longitudinal Acceptance

Longitudinal acceptance is a phase space area of stable oscillations available for the beam (*area of separatrix*). Let us determine longitudinal acceptance using elliptical approximation to separatrix.

The half - width of separatrix in momentum is determined from separatrix equation assuming $\psi = 0$:

$$\frac{p_{\zeta sep}}{mc} = 2\beta\gamma^3 \frac{\Omega}{\omega}\sqrt{1 - \frac{\varphi_s}{tg\varphi_s}}$$



Elliptical approximation of separatrix

Hamiltonian $$H = \frac{p_\zeta^2}{2m\gamma^3} + m\gamma^3\Omega^2\frac{\zeta^2}{2}$$

is constant along elliptical trajectory.

Maximal value of ζ at ellipse is $$m\gamma^3\Omega^2\frac{\zeta_{sep}^2}{2} = \frac{p_{\zeta sep}^2}{2m\gamma^3}$$ or $$\zeta_{sep} = 2\frac{\beta c}{\omega}\sqrt{1 - \frac{\varphi_s}{tg\varphi_s}}$$

Taking approximation $tg\,\varphi_s \approx \varphi_s + \varphi_s^3/3$

$$\sqrt{1 - \frac{\varphi_s}{tg\varphi_s}} \approx \frac{\varphi_s}{\sqrt{3}}$$

Effective length of separatrix

$$\Phi_{s\,eff} = 2\pi\frac{(2\zeta_{sep})}{\beta\lambda} = 4\sqrt{1 - \frac{\varphi_s}{tg\varphi_s}} \approx \frac{4|\varphi_s|}{\sqrt{3}}$$

Longitudinal Acceptance (cont.)

Area of separatrix ellipse is $\pi\, \zeta_{sep}\, (p_{sep} / mc)$. Phase space area of acceptance is determined as a product of ellipse semi-axis:

$$\varepsilon_{acc} = \zeta_{sep} \frac{p_{sep}}{mc} = \frac{2}{\pi} \lambda \beta^2 \gamma^3 (\frac{\Omega}{\omega})(1 - \frac{\varphi_s}{tg\varphi_s})$$

The value of π is not included in the value of acceptance, but is included in units of acceptance ($\pi\, m\, radian$), or, more often ($\pi\ \ cm\, mrad$).

Using approximation $1 - \frac{\varphi_s}{tg\varphi_s} \approx \frac{\varphi_s^2}{3}$, normalized longitudinal acceptance

$$\boxed{\varepsilon_{acc} = \frac{2}{3\pi} \beta^2 \gamma^3 (\frac{\Omega}{\omega})\, \varphi_s^2\, \lambda}$$

Often longitudinal acceptance and beam emittance are determined in phase plane (φ - φ_s, W-W_s) in units ($\pi\, keV\, deg$).

Relationship between phase and longitudinal coordinate and between energy and momentum

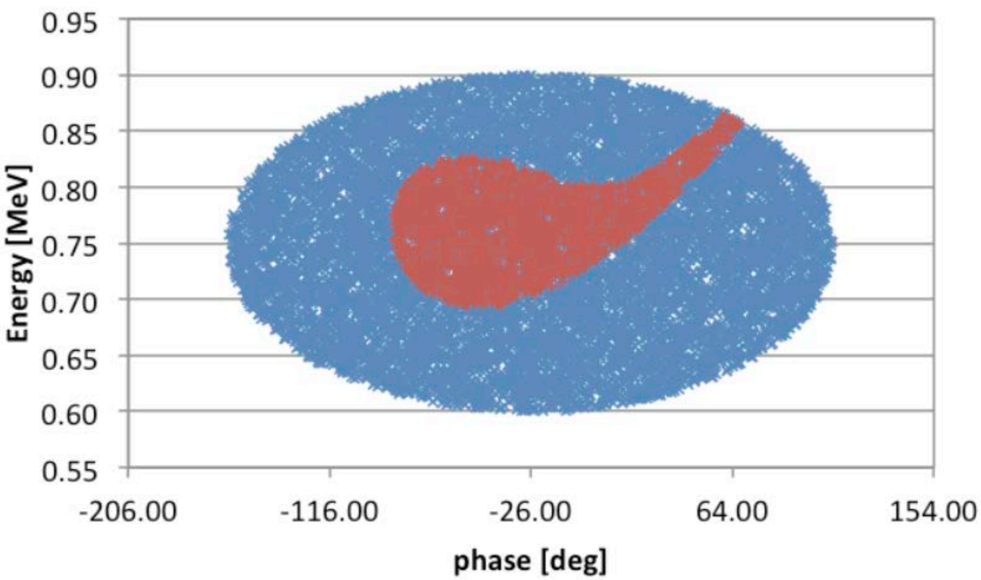
$$\Delta\varphi = 360^o \frac{\zeta}{\beta\lambda}$$

$$\Delta W = mc^2 \beta (\frac{\Delta p}{mc})$$

Transformation of longitudinal phase space area in different units:

$$\boxed{\breve{\varepsilon}_{acc}\ [\pi \cdot keV \cdot \deg] = \varepsilon_{acc}\ [\pi \cdot m \cdot rad] \frac{360^o}{\lambda[m]}\ \ mc^2[keV]}$$

Longitudinal Acceptance: Example



LANL DTL Acceptance (red)

Accelerating gradient $E=E_oT$	1.6 MV/m
Synchronous phase φ_s	-26^o
Wavelength, λ	1.49 m
Energy	750 keV
Velocity, β	0.04

Longitudinal frequency:

$$\frac{\Omega}{\omega} = \sqrt{\frac{qE\lambda}{mc^2}\frac{|\sin\varphi_s|}{2\pi\beta\gamma^3}} = 0.0665$$

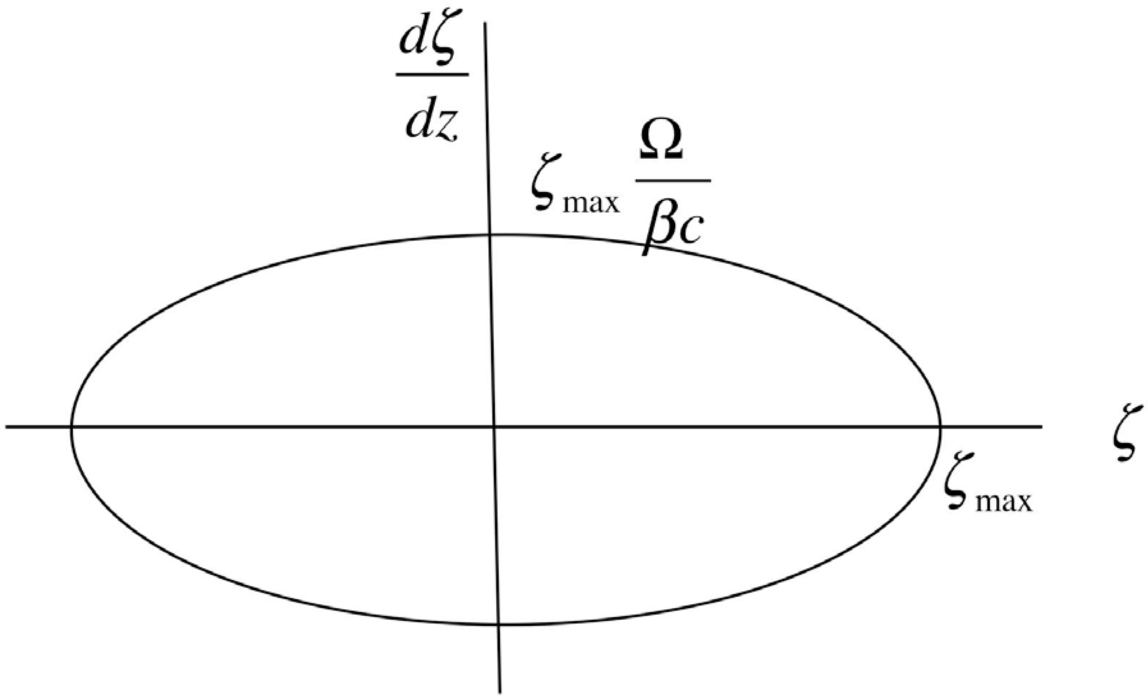
DTL Longitudinal acceptance:

$$\varepsilon_{acc} = \frac{2}{\pi}\lambda\beta^2\gamma^3(\frac{\Omega}{\omega})(1-\frac{\varphi_s}{tg\varphi_s}) = 7.17\cdot10^{-6}\pi\, m\, rad = 1.62\ \pi\ MeV\deg$$

Unnormalized Longitudinal Beam Emittance

Longitudinal oscillations:

$$\frac{d^2\zeta}{dz^2} + \left(\frac{\Omega}{\beta c}\right)^2 \zeta = 0$$



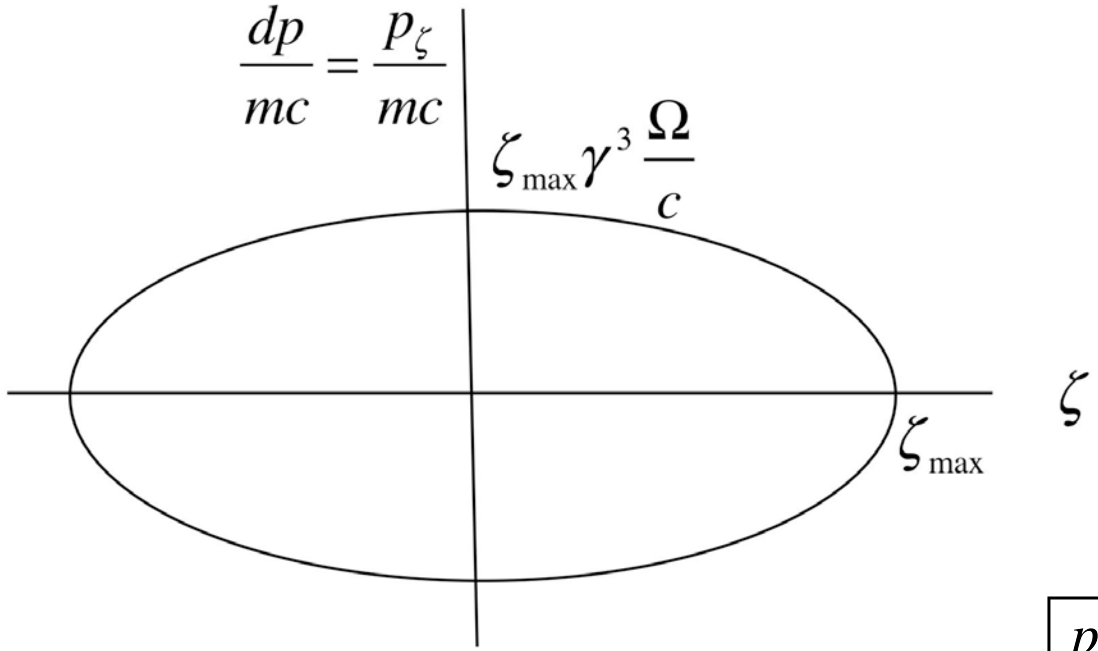
$$\zeta = \zeta_{max}\cos\left(\frac{\Omega}{\beta c}z + \psi_o\right)$$

$$\frac{d\zeta}{dz} = -\zeta_{max}\frac{\Omega}{\beta c}\sin\left(\frac{\Omega}{\beta c}z + \psi_o\right)$$

Unnormalized longitudinal emittance of matched beam:

$$\ni_z = \zeta_{max}^2 \frac{\Omega}{\beta c}$$

Normalized Longitudinal Beam Emittance



$$\frac{d\zeta}{dt} = d(\beta c)$$

$$d\beta = \frac{1}{\gamma^3}\frac{dp}{mc}$$

$$\frac{d\zeta}{dz} = \frac{d(\beta c)}{\beta c} = \frac{1}{\beta\gamma^3}\frac{dp}{mc}$$

$$\boxed{\frac{p_{\zeta\max}}{mc} = \zeta_{\max}\gamma^3\frac{\Omega}{c} = 2\pi(\frac{\zeta_{\max}}{\lambda})(\frac{\Omega}{\omega})\gamma^3}$$

Normalized longitudinal emittance of matched beam:

$$\boxed{\varepsilon_z = \beta\gamma^3 \ni_z = \zeta^2_{\max}\gamma^3\frac{\Omega}{c} = 2\pi(\frac{\zeta^2_{\max}}{\lambda})(\frac{\Omega}{\omega})\gamma^3}$$

Adiabatic Damping of Longitudinal Oscillations

Previous analysis was performed in conservative approximation assuming accelerator parameters are constant along the machine. Consider now effect of acceleration on longitudinal oscillations. Equations of motion for small oscillations around synchronous particle.

Hamiltonian of linear oscillations

$$H = \frac{p_\zeta^2}{2m\gamma^3} + m\gamma^3\Omega^2 \frac{\zeta^2}{2}$$

Along phase space trajectory $H = const$. Let us divide expression for Hamiltonian by H. Phase space trajectory is an ellipse

$$\frac{p_\zeta^2}{p_{\zeta\,max}^2} + \frac{\zeta^2}{\zeta_{max}^2} = 1$$

Semi-axis of ellipse

$$p_{\zeta\,max} = \sqrt{2Hm\gamma^3} \qquad \zeta_{max} = \frac{1}{\Omega}\sqrt{\frac{2H}{m\gamma^3}}$$

The value of Hamiltonian, *H*, is the energy of particle oscillation around synchronous particle. Product of semi-axis of ellipse, gives the value of phase space area comprised by a particle performing linear longitudinal oscillations. Largest phase space trajectory comprises longitudinal beam emittance:

$$\varepsilon_z = \frac{p_{\zeta\,max}}{mc}\zeta_{max} = \frac{2H}{mc\Omega}$$

Adiabatic Damping of Longitudinal Oscillations (cont.)

If parameters of accelerator are changing adiabatically along the channel, the value of beam ellipse in phase space is conserved according to theorem of adiabatic invariant. In this case, energy of particle oscillation around synchronous particle, H , is proportional to frequency of longitudinal oscillation, Ω:

$$H \sim \Omega$$

Adiabatic change of parameters means that parameters are changing slowly during one oscillation period of $2\pi/\Omega$.

The semi-axes of beam ellipse are changing as

$$\zeta_{\max} = \sqrt{\frac{\varepsilon_z c}{\gamma^3 \Omega}} \sim \frac{1}{\gamma^{3/2}\Omega^{1/2}} \qquad \frac{p_{\zeta \max}}{mc} = \sqrt{\frac{\gamma^3 \varepsilon_z \Omega}{c}} \sim \gamma^{3/2}\Omega^{1/2}$$

Adiabatic Damping of Longitudinal Oscillations (cont.)

Many accelerators are designed keeping the constant values of equivalent traveling wave, E, and synchronous phase φ_s. In this case, longitudinal oscillation frequency drops as

$$\Omega \sim \frac{1}{\beta^{1/2}\gamma^{3/2}}$$

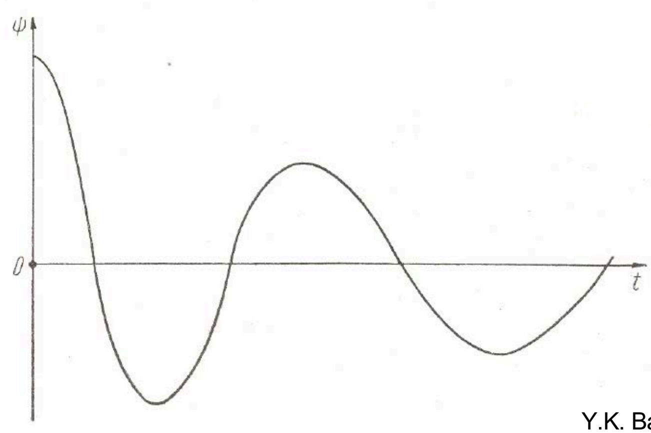
Semi-axes of beam ellipse at phase plane are changing as

$$\zeta_{\max} \sim \frac{\beta^{1/4}}{\gamma^{3/4}} \qquad p_{\zeta\,\max} \sim \frac{\gamma^{3/4}}{\beta^{1/4}}$$

Phase length of the bunch and relative momentum spread drop as

$$\Delta\psi \sim \frac{1}{(\beta\gamma)^{3/4}}$$

$$\frac{\Delta p}{p_s} \sim \frac{1}{\beta^{5/4}\gamma^{1/4}}$$



Adiabatic Phase Damping

Longitudinal Beam Phase Space $\Delta W \Delta\phi = \text{Constant}$

Beam Energy Spread $\Delta W = \text{Constant x } (\beta\gamma)^{3/4}$

Beam Phase Width $\Delta\phi = \dfrac{\text{Constant}}{(\beta\gamma)^{3/4}}$

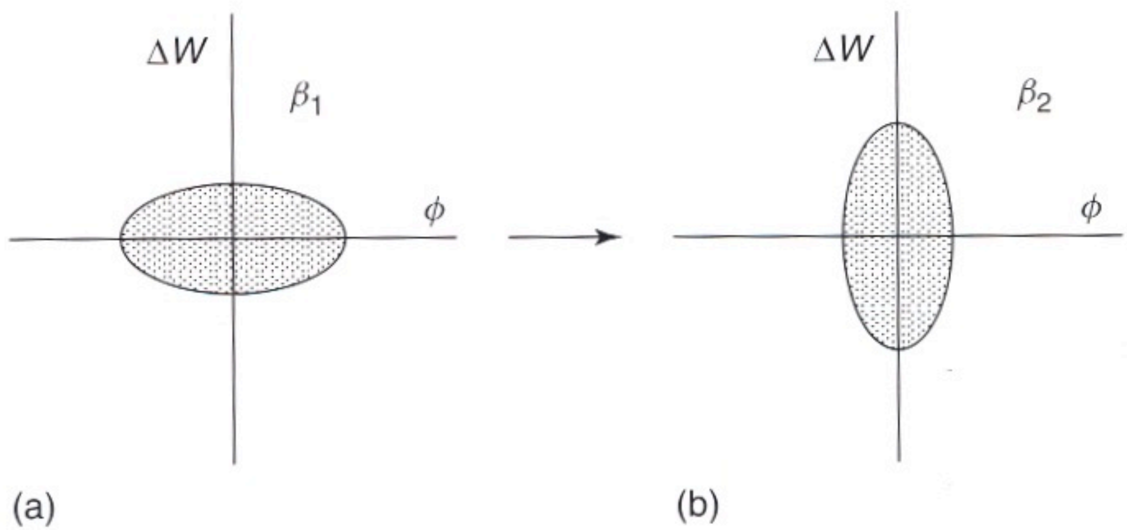
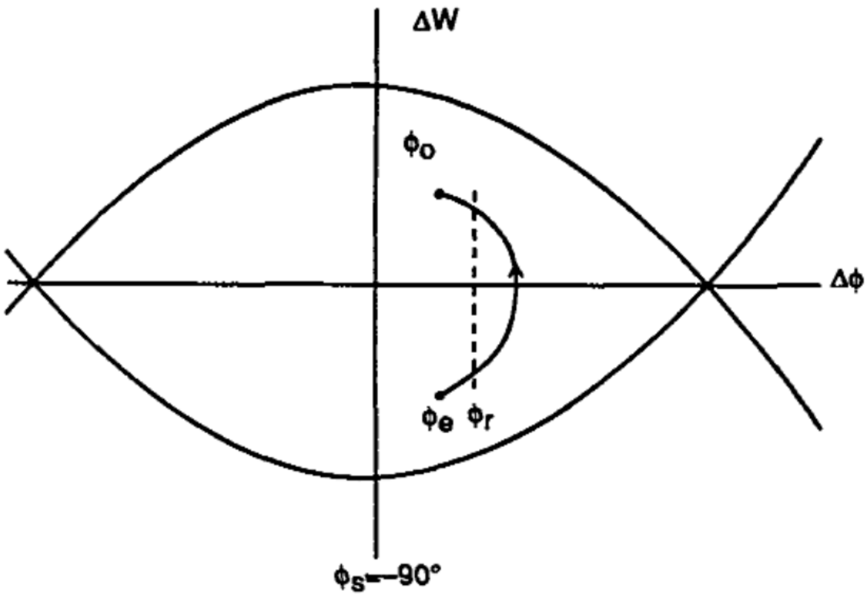


Figure 6.8 Phase damping of a longitudinal beam ellipse caused by acceleration. The phase width of the beam decreases and the energy width increases, while the total area remains constant.

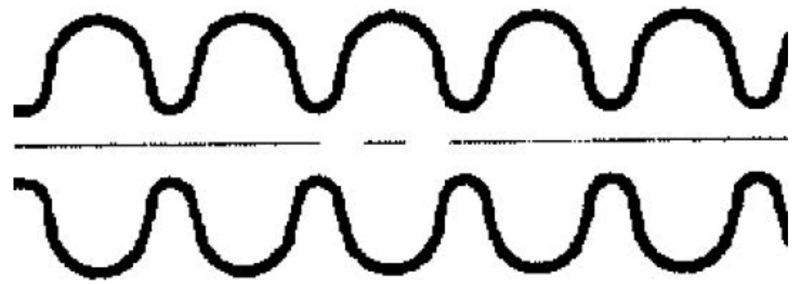
Acceleration in Sections with Constant β

LANSCE high-energy linear accelerator.

Acceleration in Sections with Constant β (cont.)



Phase space trajectory in structure with constant length cell.



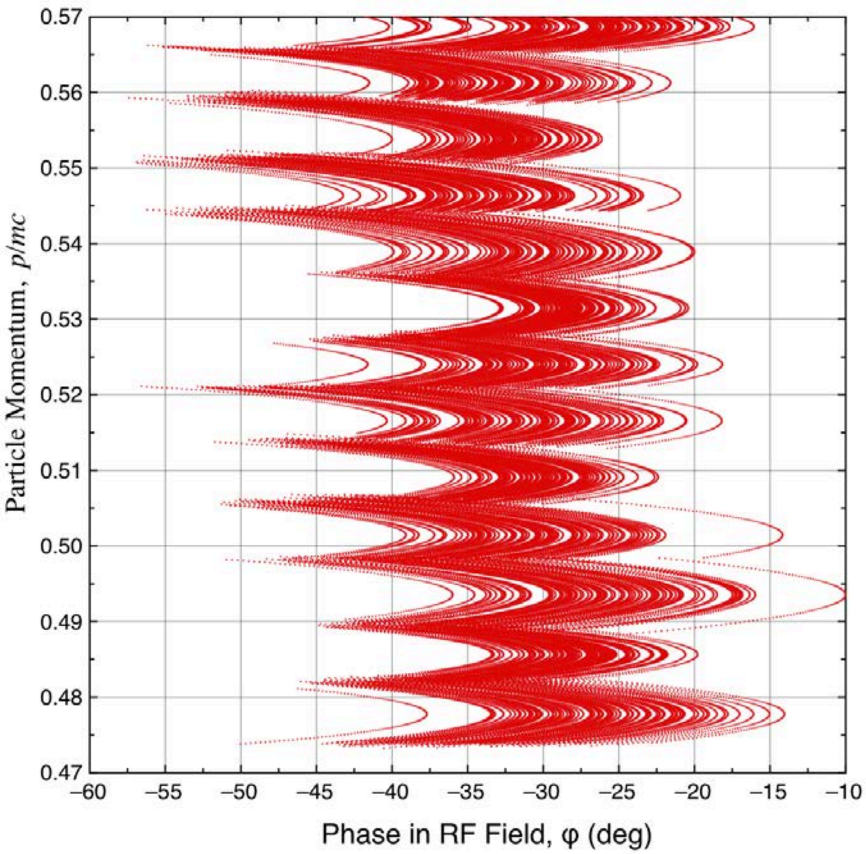
Accelerator structure with constant length cell.

Because cell lengths are equal, actual synchronous phase in each structure is $\varphi_s = -90^o$. Energy gain per tank (for π - structure):

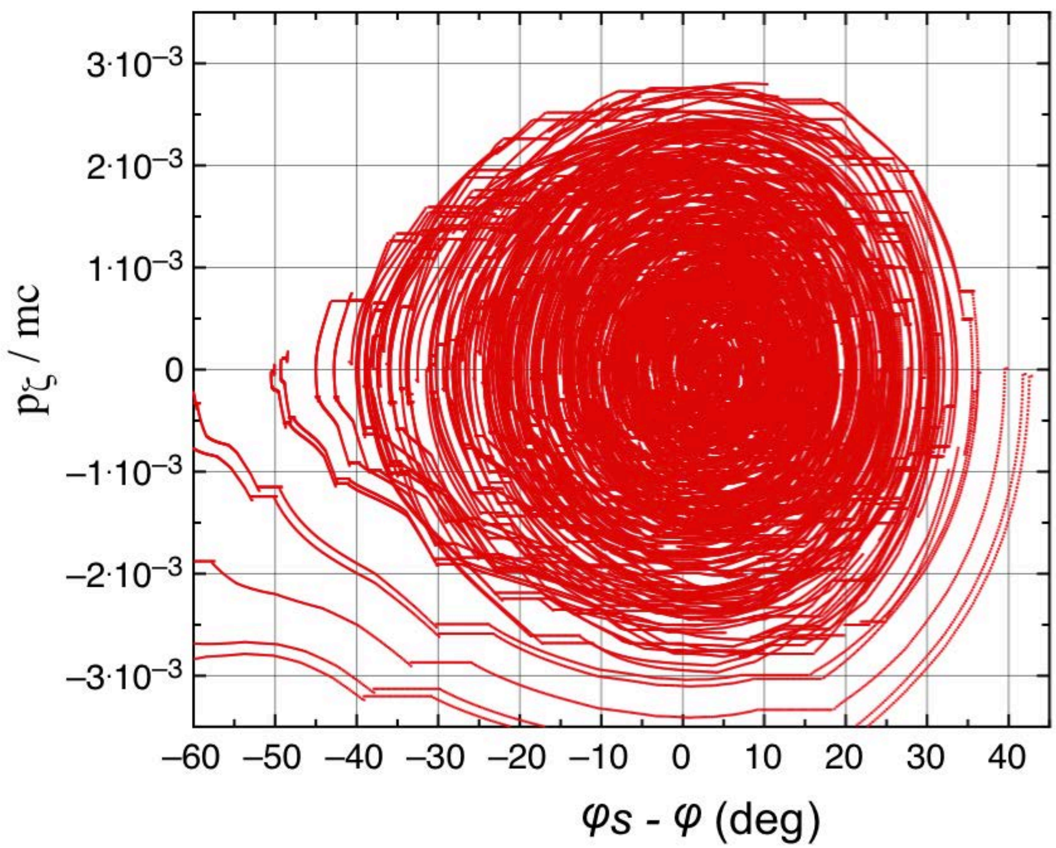
$$\Delta W_{ref} = qE_o T \cos\varphi_{ref} N_{cell} \frac{\beta\lambda}{2}$$

Acceleration in Multiple Sections with Constant β

Dynamics in RF field of multiple section with constant β



Dynamics around synchronous particle



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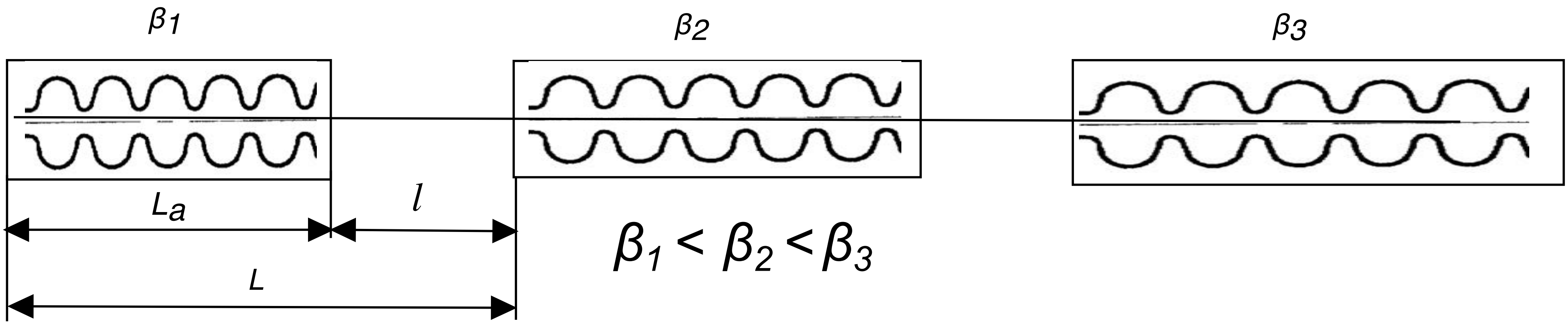
Acceleration in Multiple Sections with Constant β (cont.)



Synchronous phase $\varphi_s \approx \varphi_{ref}$

Phase advance of longitudinal oscillations
In single tank
$$\mu_{oa} = \sqrt{2\pi\left(\frac{qE\lambda}{mc^2}\right)\frac{|\sin\varphi_s|}{\beta\gamma^3}\left(\frac{L_a}{\beta\lambda}\right)}$$

Effective accelerating gradient $\tilde{E} = E\frac{L_a}{L_a + l}$

Effective phase advance of longitudinal oscillations
per accelerating period *L*
$$\tilde{\mu}_{oa} \approx \sqrt{2\pi\left(\frac{qE\lambda}{mc^2}\right)\left(\frac{L_a}{L_a + l}\right)\frac{|\sin\varphi_s|}{\beta\gamma^3}\left(\frac{L_a + l}{\beta\lambda}\right)} = \mu_{oa}\sqrt{1 + \frac{l}{L_a}}$$

Dynamics in Sections with β_s = 1

In accelerating sections with β_s = 1 there is no synchronous particles.

Equation for change of particle momentum:

$$\frac{dp}{dt} = qE\cos\varphi$$

Equation for change of particle phase $\varphi = \omega t - k_z z$

$$\frac{d\varphi}{dt} = \omega - \frac{2\pi}{\lambda}\beta c$$

Wave number for β_s = 1

$$k_z = \frac{2\pi}{\beta_s \lambda} = \frac{2\pi}{\lambda}$$

Introducing dimensionless momentum $p_\varphi = \frac{p}{mc}$ we can write:

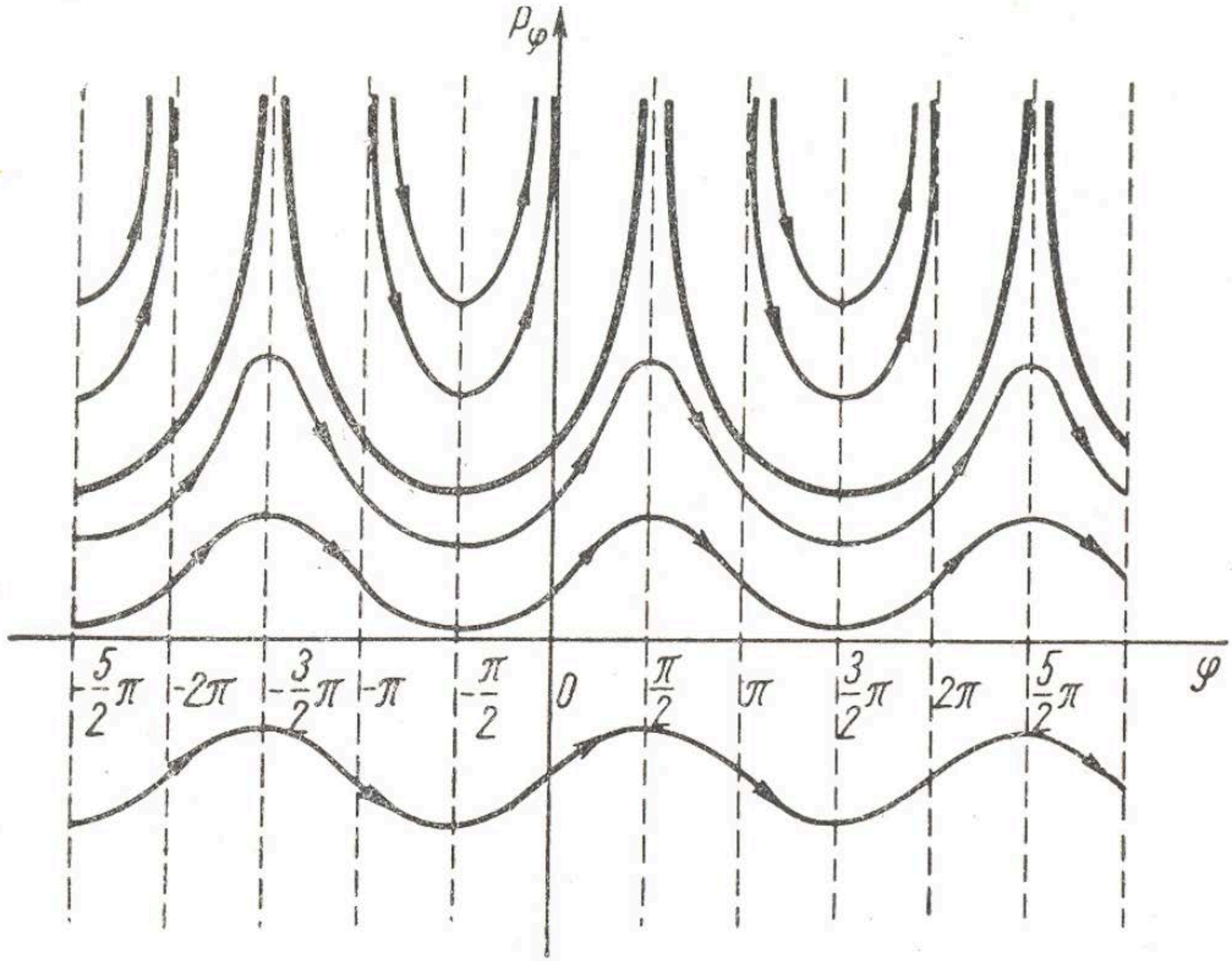
$$\frac{dp_\varphi}{d\varphi} = \frac{qE\lambda}{2\pi mc^2} \frac{\cos\varphi}{(1 + \frac{p_\varphi}{\sqrt{1+p_\varphi^2}})}$$

Integration gives:

$$\boxed{C = \sqrt{1+p_\varphi^2} - p_\varphi + (\frac{qE\lambda}{2\pi mc^2})\sin\varphi}$$

where C is the constant of integration.

Phase Space Trajectories for $\beta_s = 1$



Minimal Energy of Particles Accelerated in Wave with $\beta_s = 1$

Accelerated particles: $p_\varphi \to \infty$ $\qquad C = (\frac{qE\lambda}{2\pi mc^2})\sin\varphi$

If $C > \frac{qE\lambda}{2\pi mc^2}$, p_φ is finite and particles are not accelerated until infinity

For accelerated particles $C \le \frac{qE\lambda}{2\pi mc^2}$

Therefore boundary of acceleration is determined by $C = \frac{qE\lambda}{2\pi mc^2}$

$$\frac{qE\lambda}{2\pi mc^2}(1-\sin\varphi) = \sqrt{1+p_\varphi^2} - p_\varphi$$

Minimal value $p_{\varphi\, min}$ is determined by $\varphi = -\pi/2$, or $\sin\varphi = -1$

$$p_{\varphi\min} = \frac{1-4(\frac{qE\lambda}{2\pi mc^2})^2}{4(\frac{qE\lambda}{2\pi mc^2})}$$

For indefinite acceleration of protons in wave with E = 5 MV/m, λ = 1 m, $p_{\varphi\, min}$ = 294, or minimal kinetic energy W_{min}= 275 GeV.

Beams with lower energies can be accelerated in finite length section with β_s= 1 within $-\pi/2 < \varphi < \pi/2$.

RF Cavities Tuning: Threshold Field

The increment of energy that the equilibrium particle receives during each acceleration period is determined by the increase in the period length and, therefore, is determined by the design od accelerator:

$$\Delta W_s = eE_o TL \cos\varphi_s = const$$

The threshold field at which the equilibrium phase is still real ($\cos\varphi_s = 1$) is

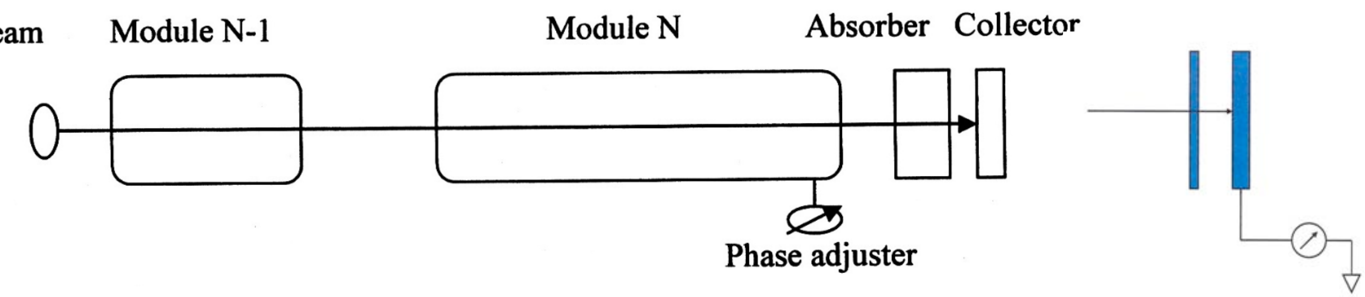
$$E_{th} = \frac{\Delta W_s}{eTL}$$

Accelerating field must be $E_o \geq E_{th}$

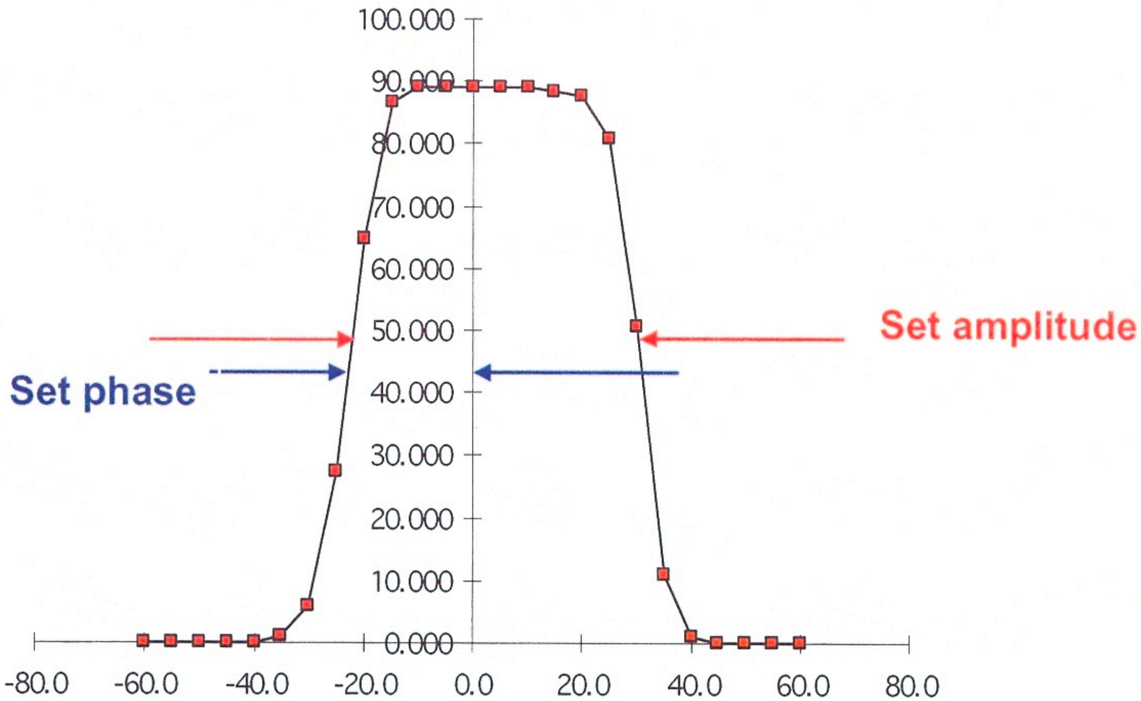
Synchronous phase is $\cos\varphi_s = \frac{E_{th}}{E_o}$

The threshold field is determined through measurement of width of energy capture region as a function of field in resonator. This is done by measurement of dependence of accelerated beam current versus injection energy. The threshold field is determined by extrapolating of the energy width of capture region to zero value.

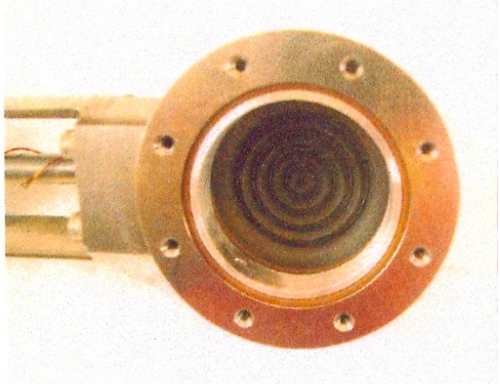
Phase Scans to Set the Phase and Amplitude of RF Linac

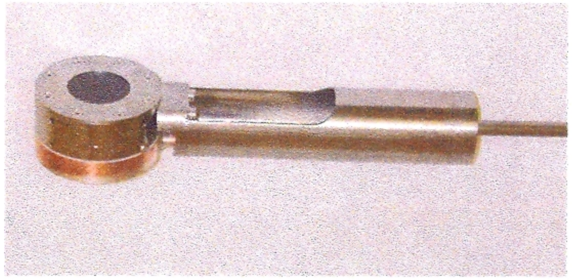


Schematic of the phase scan measurement setup. At LANL linac there are 4 absorber/collectors at 40, 70, 100, and 121 MeV.



Result of phase scan





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Phase Scans to Set the Phase and Amplitude of RF Linac

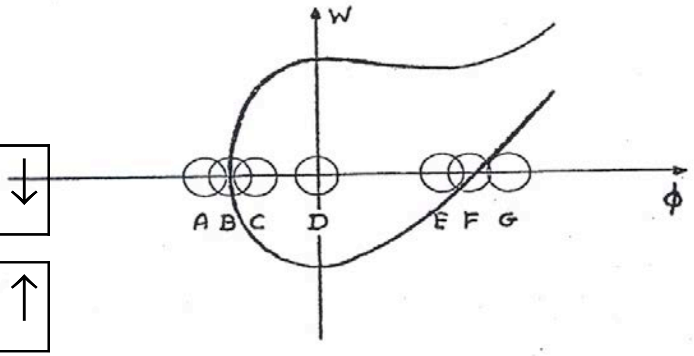
Energy gain of synchronous particle per gap is constant

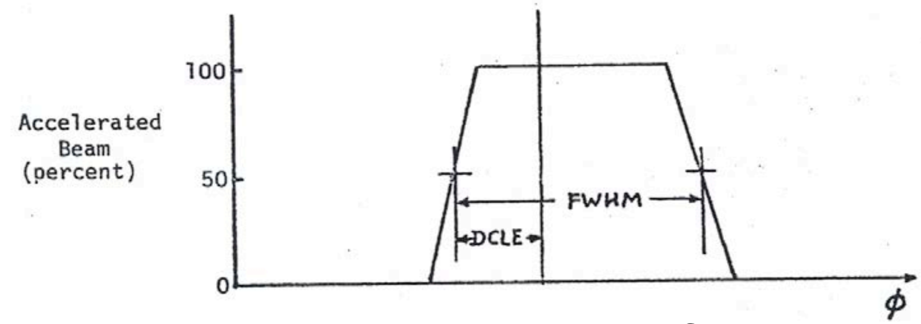
$$\Delta W_s = eEL\cos\varphi_s = const$$

Decrease of accelerating field results in decrease of phase width of separatirx (and vise versa)

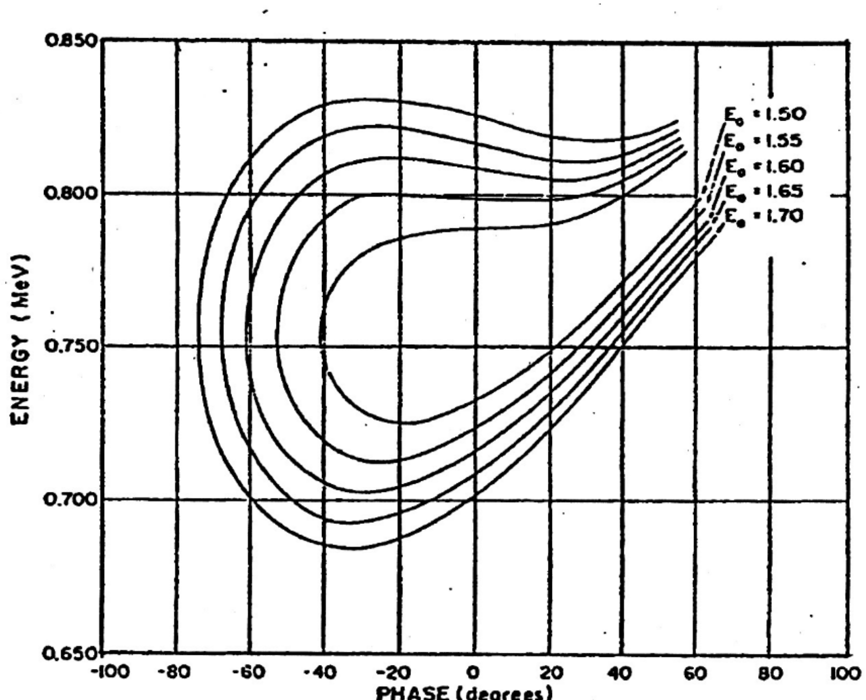
$$E\downarrow \rightarrow \cos\varphi_s \uparrow \rightarrow \varphi_s \downarrow \rightarrow \Phi_{sep} \approx 3\varphi_s \downarrow$$

$$E\uparrow \rightarrow \cos\varphi_s \downarrow \rightarrow \varphi_s \uparrow \rightarrow \Phi_{sep} \approx 3\varphi_s \uparrow$$

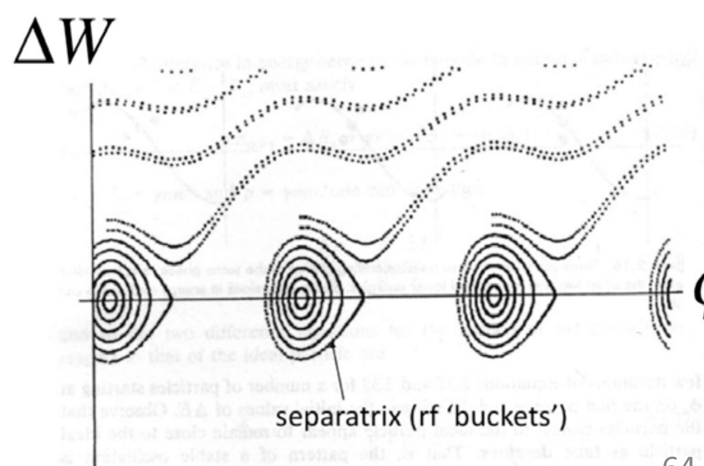




Accelerated beam as a function of beam phase

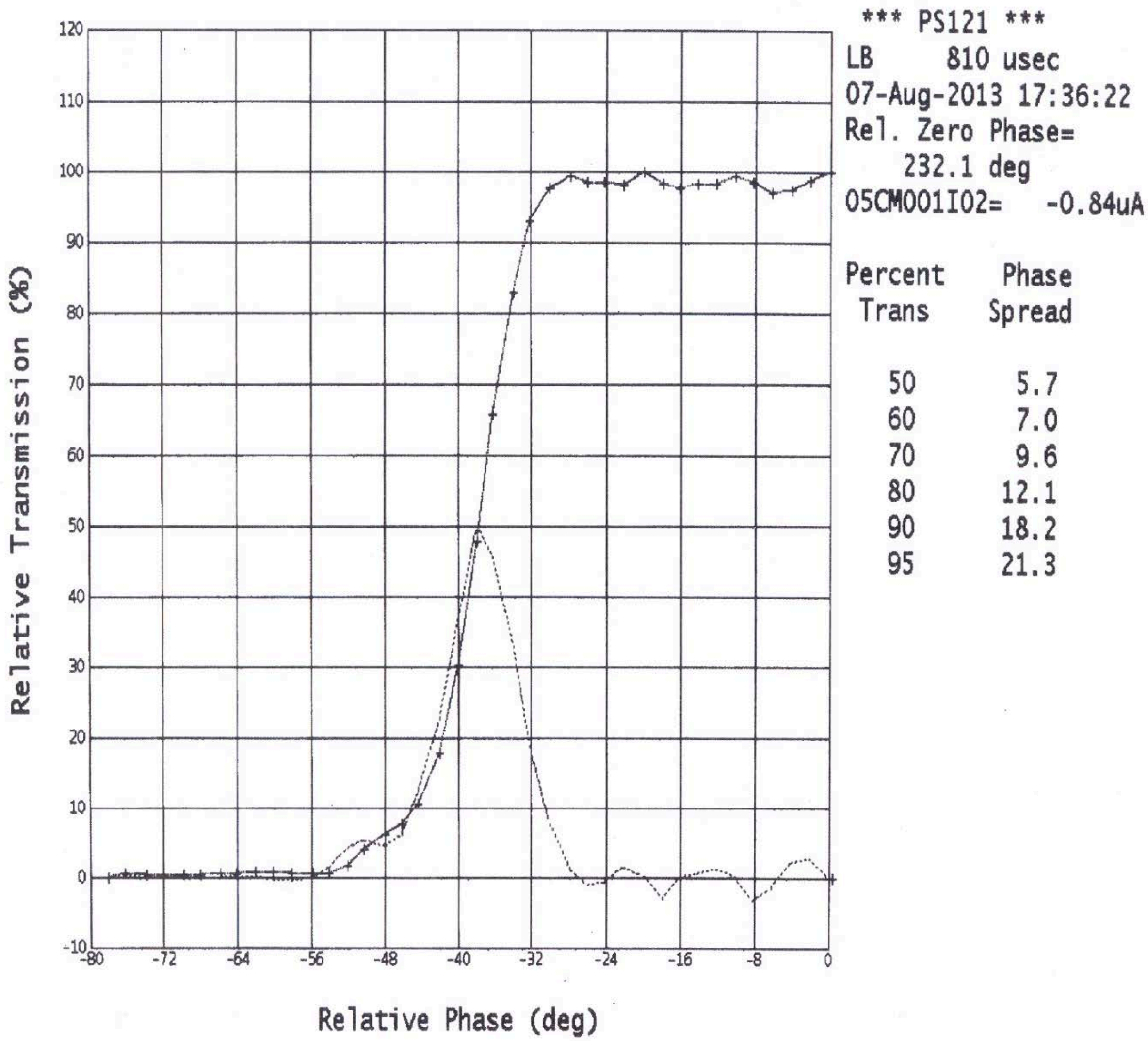


Longitudinal acceptance of RF linac for 5 different average axial field amplitudes.



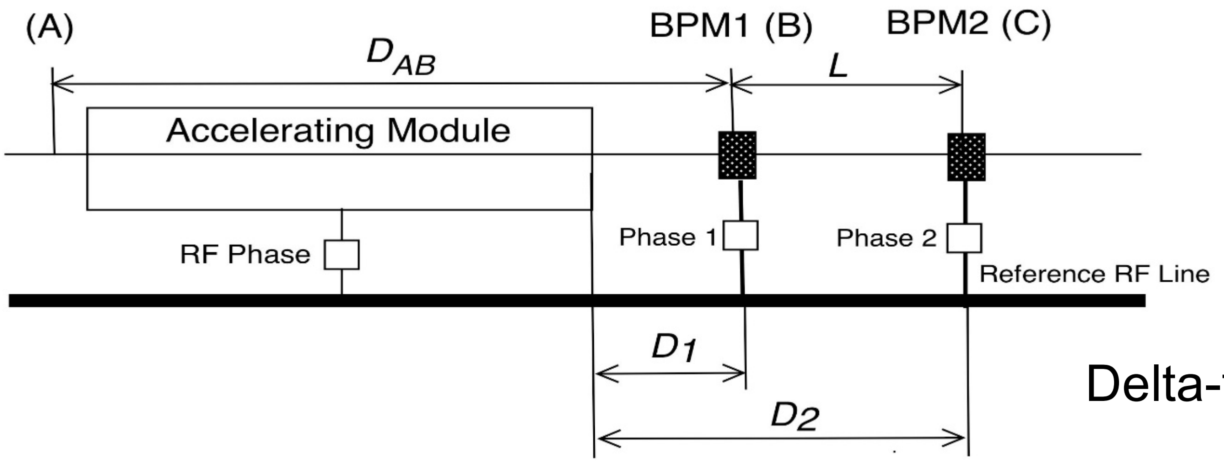
Sequence of RF buckets

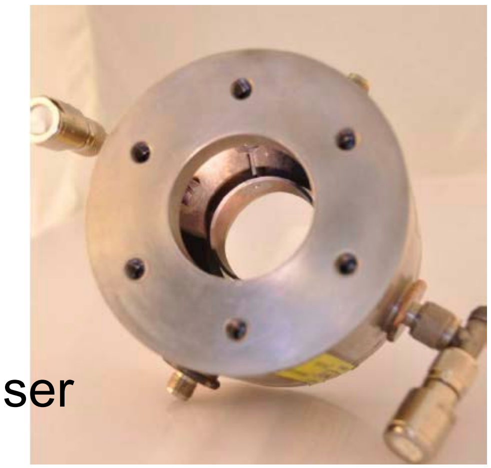
Determination of Bunch Length Using Phase Scan



LANL Phase Scan at the energy of 121 MeV.

Delta-t Tuning Procedure





Delta-t tranduser

Time-of-flight of the beam centroid from location A to B and from A to C: t_{AB}, t_{AC}

Change in $t_{AB,}$ t_{AC} values when accelerating module is switched from off to on are

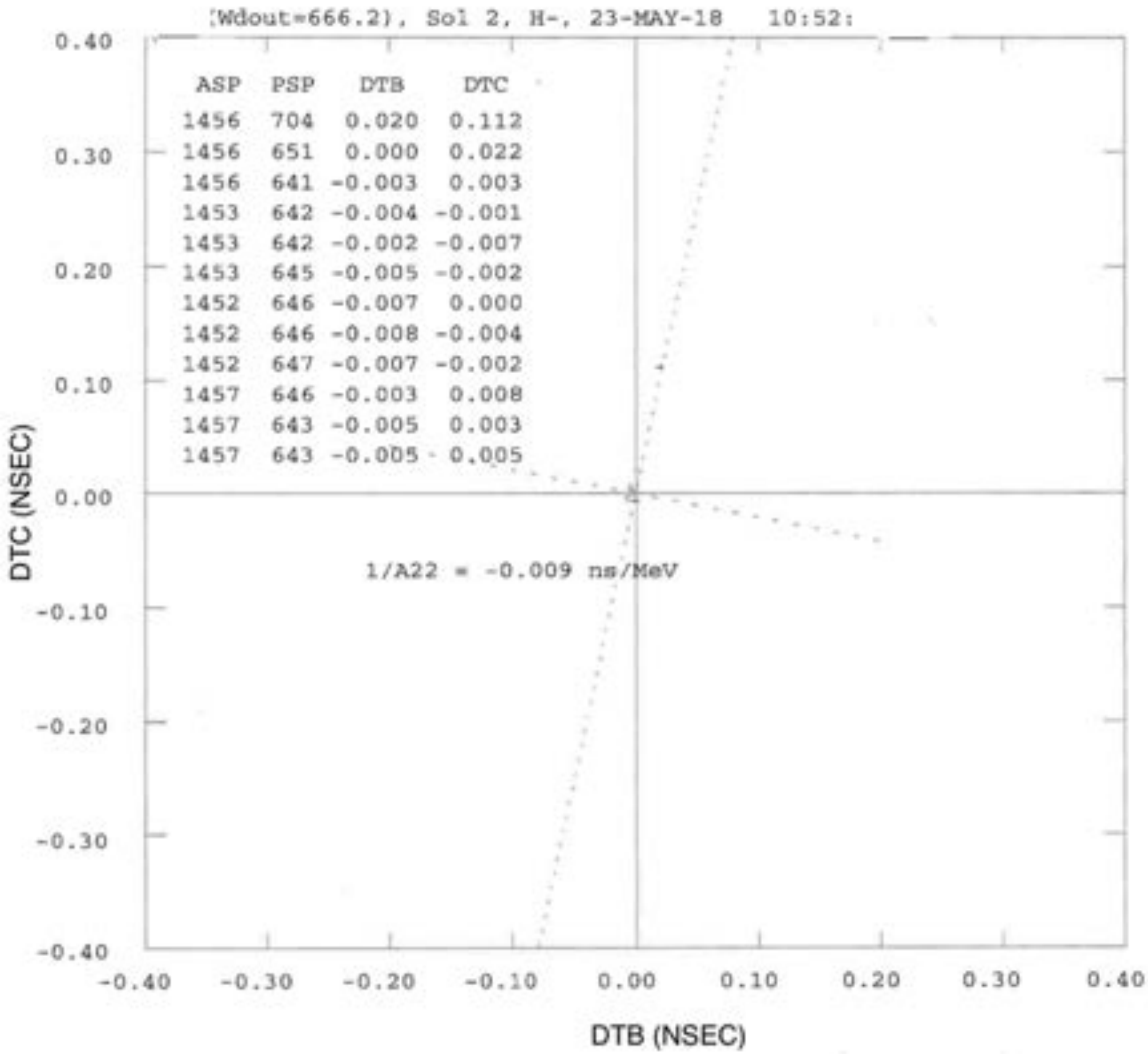
$$t_B = t_{AB,off} - t_{AB,on} \qquad t_C = t_{AC,off} - t_{AC,on}$$

Deviation of values $t_{B,}$ t_C from design values:

$$\Delta t_B = -\frac{D_{AB}}{mc^3(\beta\gamma)_A^3}\Delta W_A - \frac{\Delta\varphi_B - \Delta\varphi_A}{\omega} - \frac{D_1}{mc^3}\left[\frac{\Delta W_A}{(\beta\gamma)_A^3} - \frac{\Delta W_B}{(\beta\gamma)_B^3}\right]$$

$$\Delta t_C = \Delta t_B - \frac{D_2 - D_1}{mc^3}\left[\frac{\Delta W_A}{(\beta\gamma)_A^3} - \frac{\Delta W_B}{(\beta\gamma)_B^3}\right]$$

Delta-t Tune



Output of delta-t program displaying search of amplitude (ASP) and phase (PSP) while minimizing values of (DTB) and (DTC).

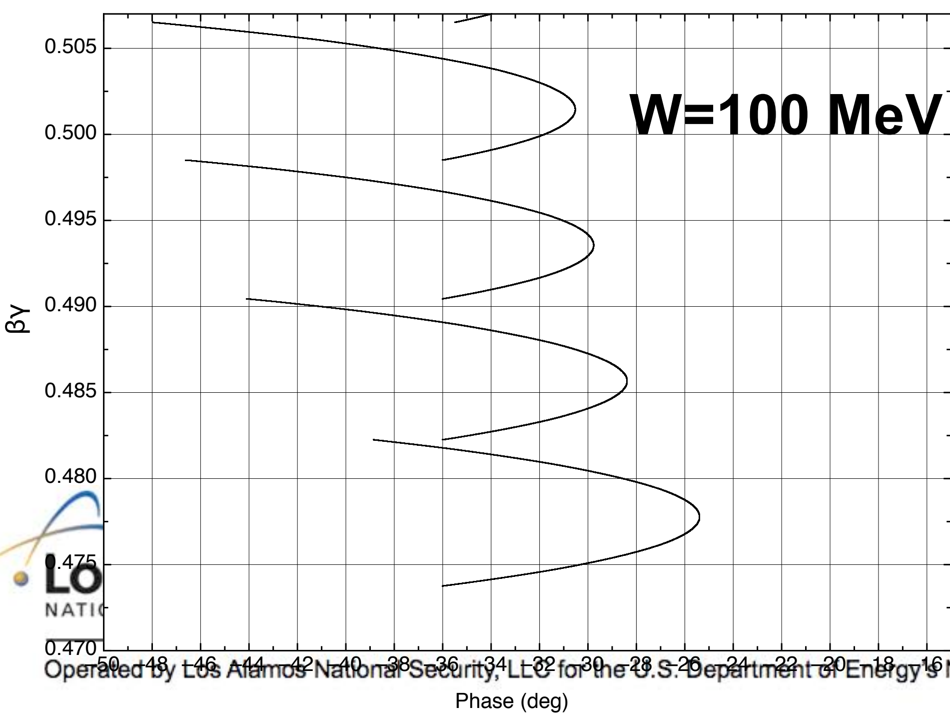
Delta-t Tuning Issues

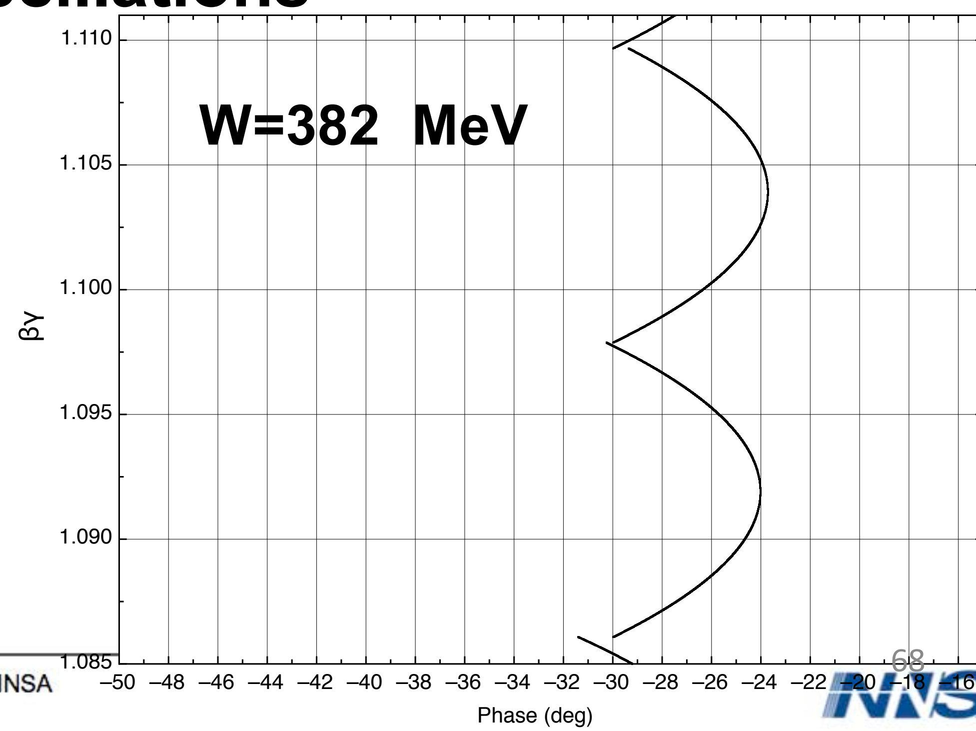
Delta-t tuning procedure works well only when particles perform significant longitudinal oscillations within RF tanks. If longitudinal oscillations are "frozen", then combination of Δt_B, Δt_c can be obtained with infinitely large number of combinations of (E, φ_s).

In linac phase advance of longitudinal oscillation per module drops as

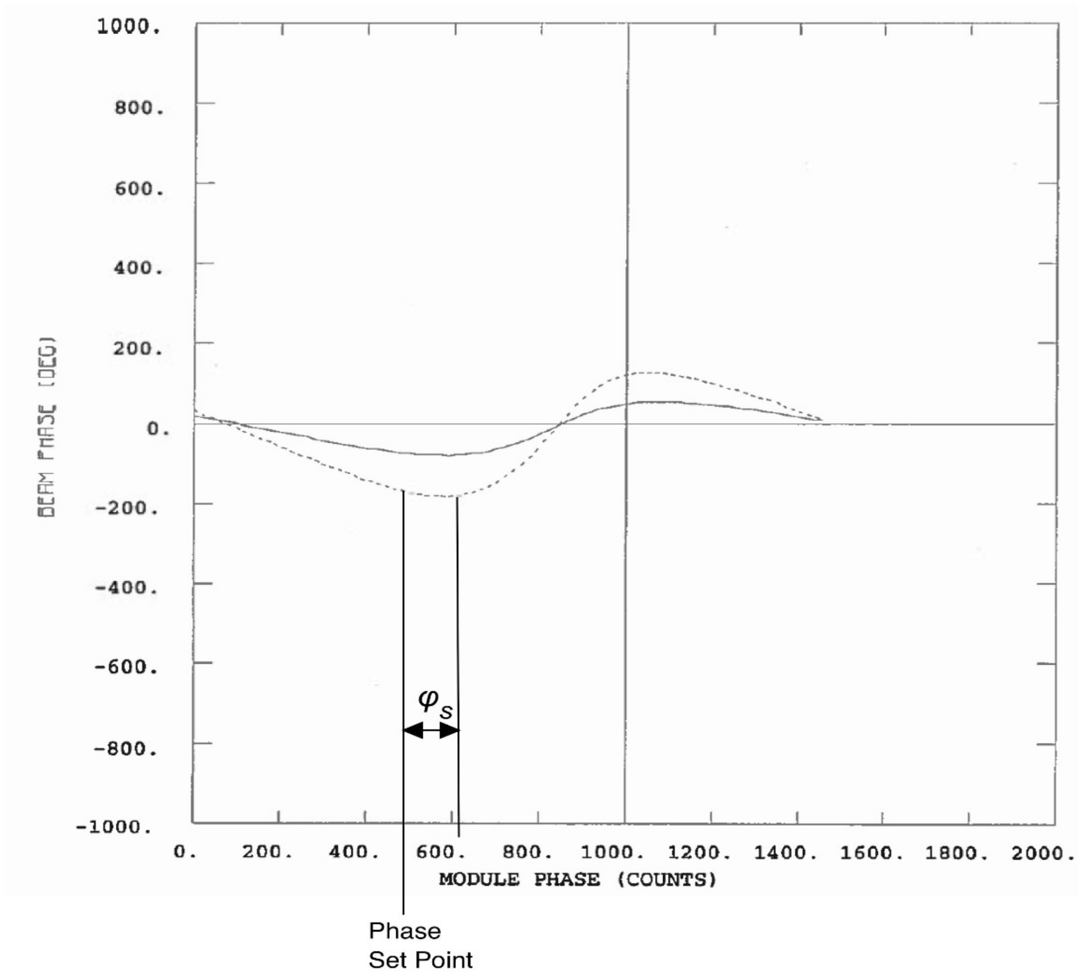
$$\mu_{oa} \sim \sqrt{\frac{E\left|\sin\varphi_s\right|}{(\beta\gamma)^3}}$$

Phase Oscillations



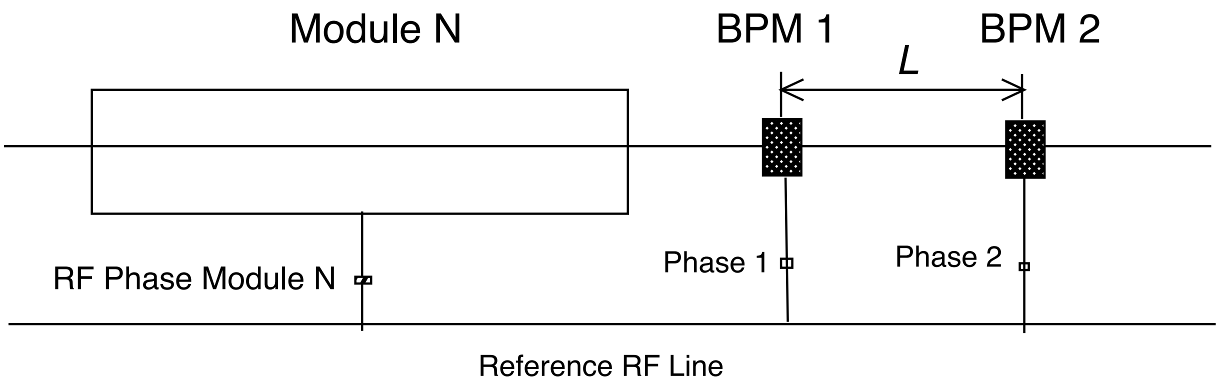


Phase Scans



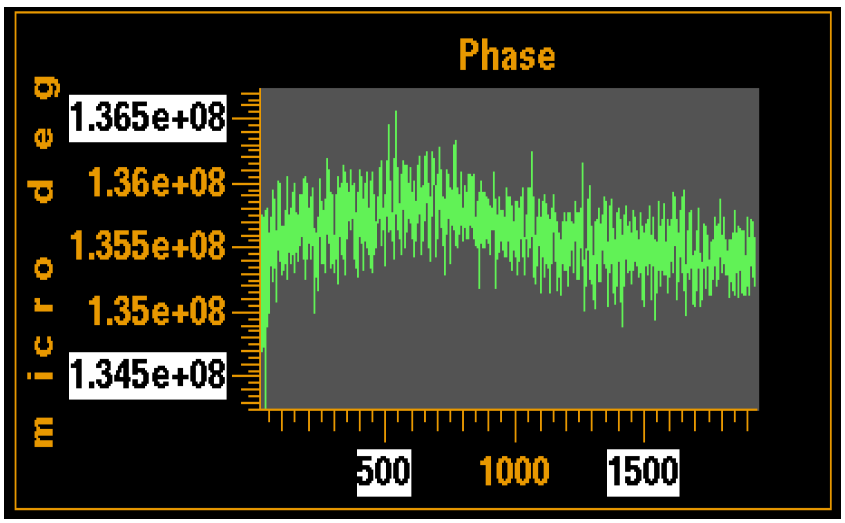
Phase scan: measurement of time of arrival of the beam to downstream pickup loop versus RF phase of the accelerating module.

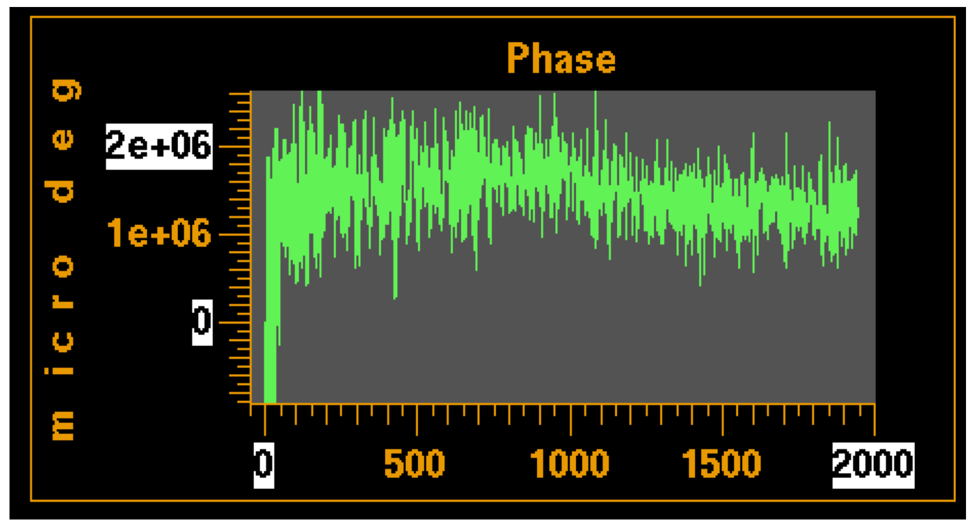
Measurement of Beam Energy by Difference in BPM RF Phases



Beam velocity

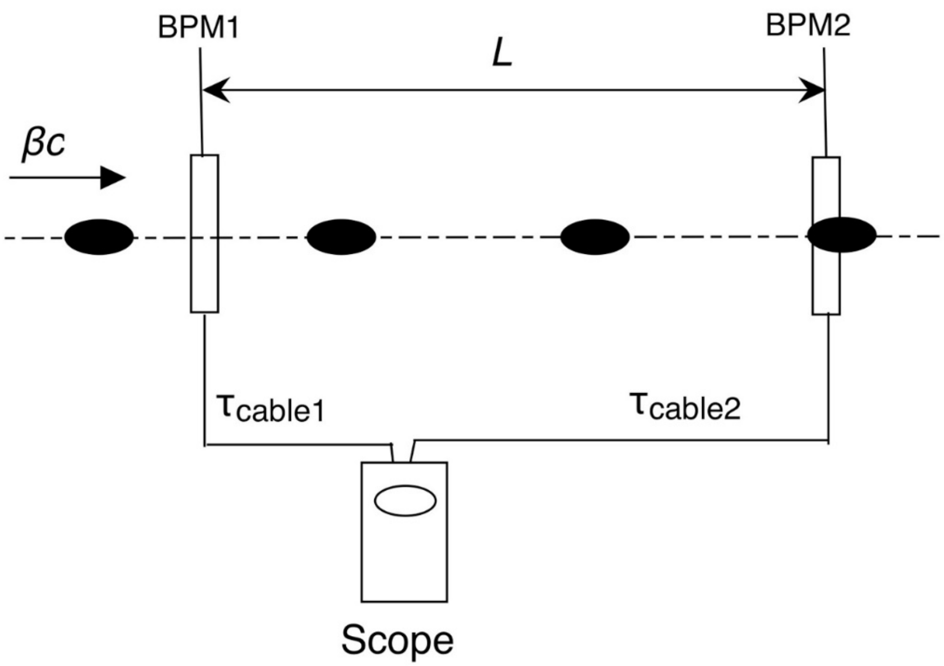
$$\beta = \frac{L}{\lambda\left(N + \frac{\varphi_{loop2} - \varphi_{loop1} + \Delta\varphi_{corr}}{2\pi}\right)}$$





Beam RF phases measured at delta-t loops.

Time-Of-Flight Measurement of Absolute Beam Energy



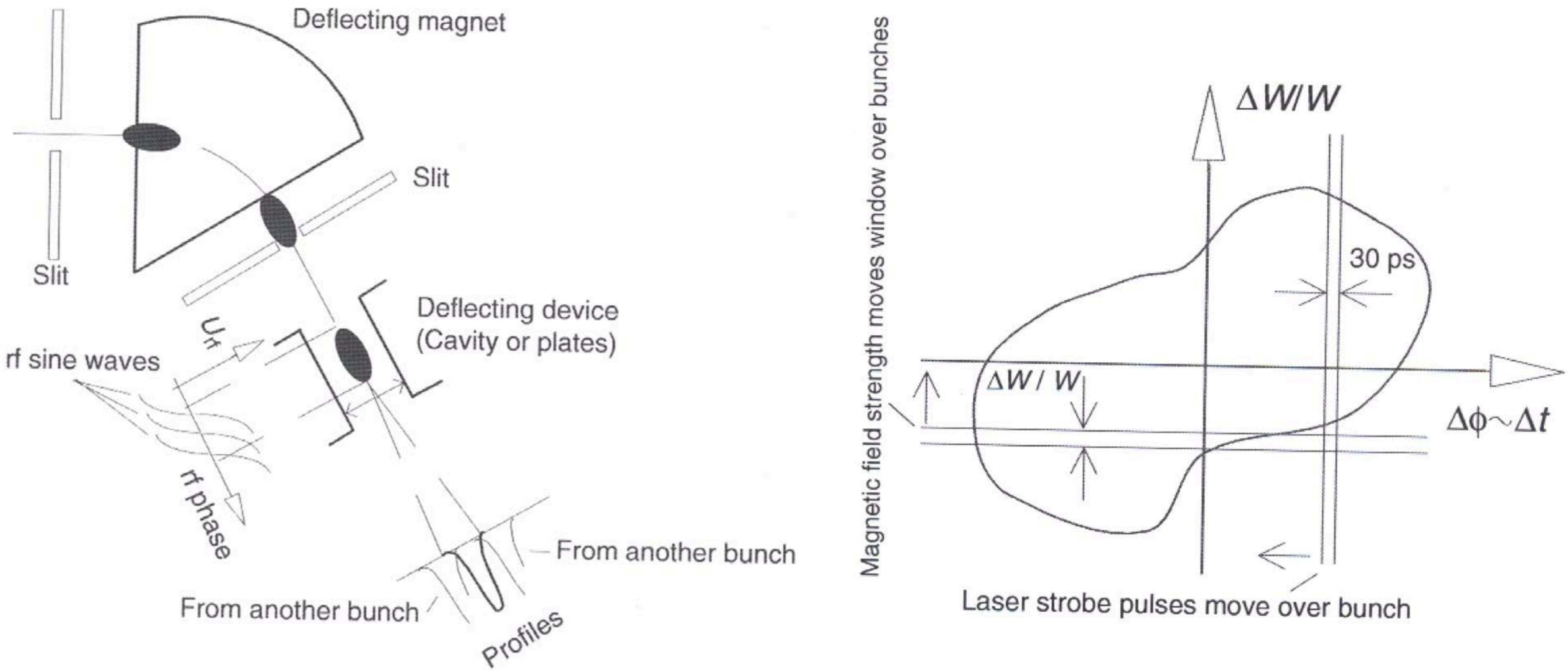
Beam velocity

$$\beta = \frac{L}{c[t-(\tau_{cable2}-\tau_{cable1})]}$$

Beam energy

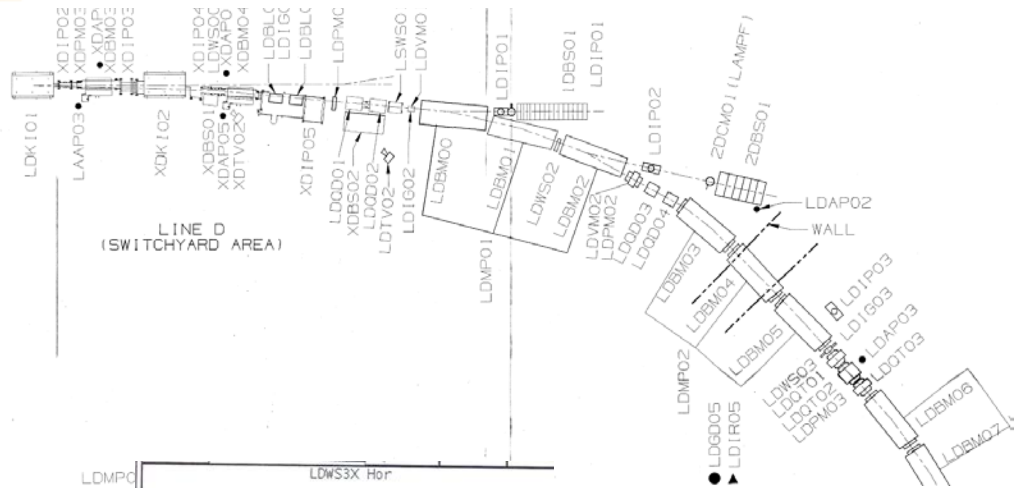
$$W = mc^2\left(\frac{1}{\sqrt{1-\beta^2}}-1\right)$$

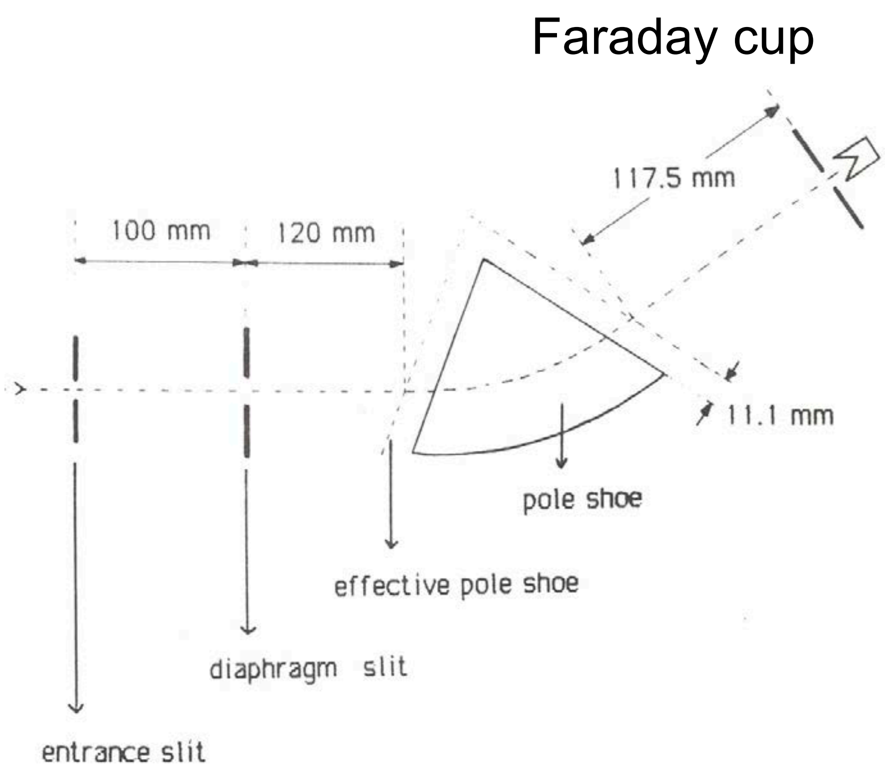
Longitudinal Beam Emittance Measurement



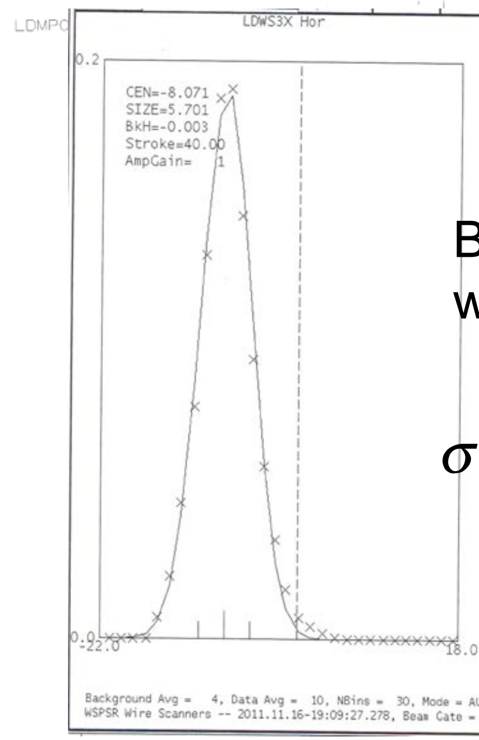
Measurement of Beam Energy Spread

High-dispersive part of 800 MeV beamline





Magnetic energy analyzer



Beam size in point with high dispersion:

$$\sigma_{tot} = \sqrt{\sigma + (\eta_{disp} \frac{\sigma_p}{p})^2}$$

Beam energy- spread-dependent wire scan

Bunch Shape Monitor (A. Feschenko, PAC2001)

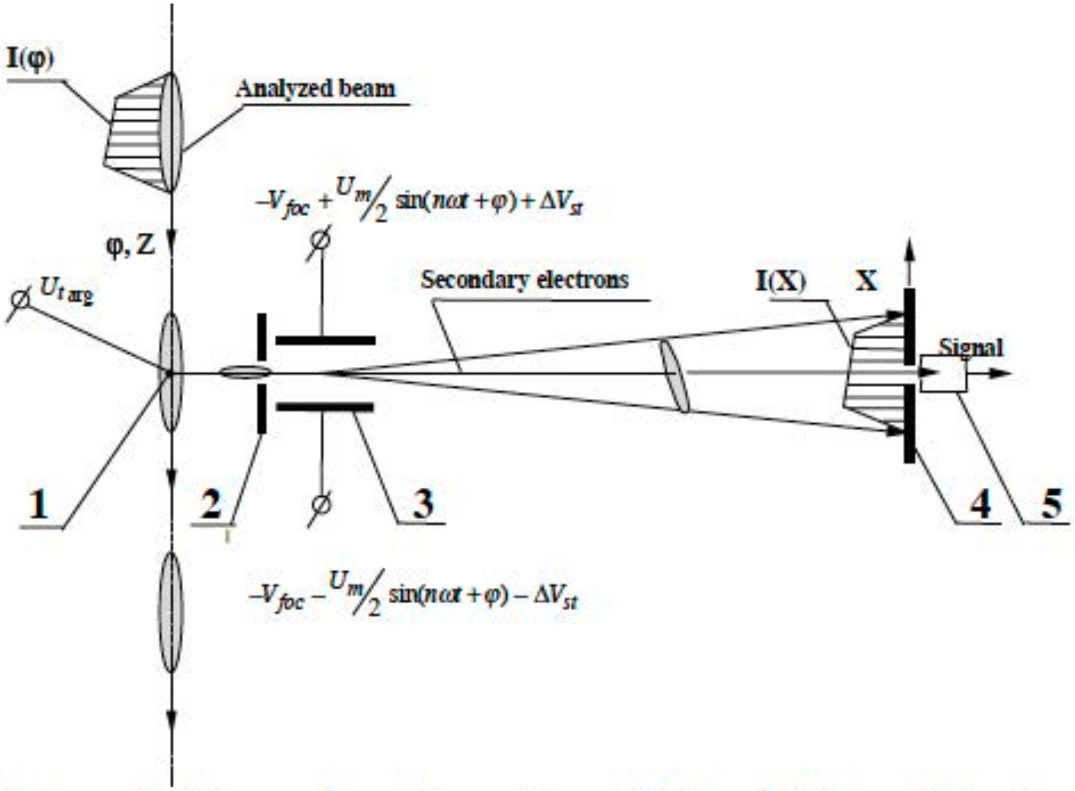


Figure 1: General configuration of Bunch Shape Monitor (1 –wire target, 2-input collimator, 3-deflector, 4-output collimator, 5-electron collector).

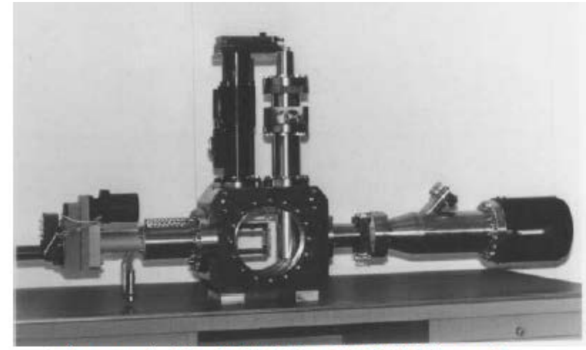


Figure 4: 3D-BSM for CERN Linac-2.

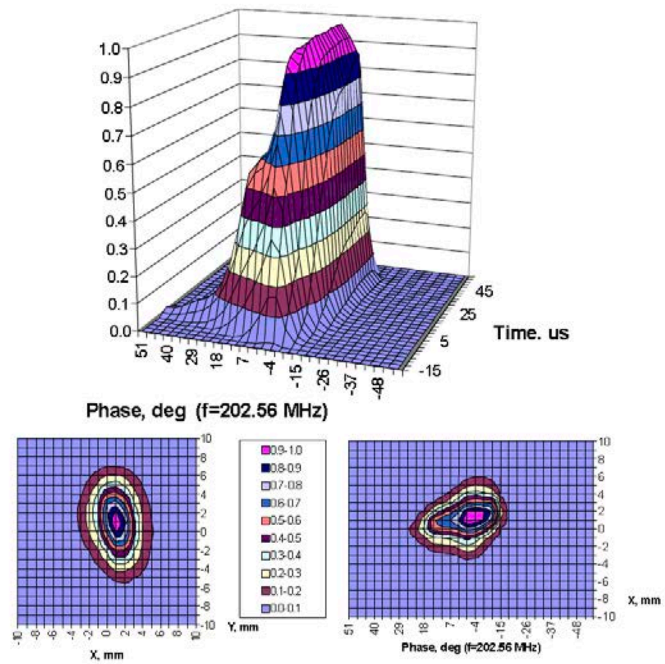


Figure 14: Behaviour of bunch shape in time, beam cross-section and longitudinally-transversal distribution measured at the exit of CERN Linac-2 with the 3D-BSM.

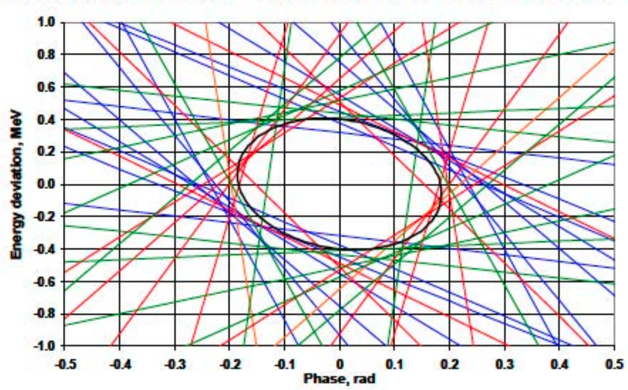


Figure 7: Bunch boundaries transformed to the entrance of CCL#1 and an equivalent phase ellipse.