## 0.1 Approaches to Solving the T-BMT Equation

The dynamics of the spin vector $\vec{S}$ of a charged particle having mass $m$ and charge $q$ in the laboratory frame is described by the Thomas-BMT (T-BMT) equation,

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = \frac{q}{\gamma m}\vec{S} \times \left((1+G\gamma)\vec{B}_{\perp} + (1+G)\vec{B}_{\parallel}\right). \tag{1}$$

Here $\gamma$ denotes the relativistic Lorentz factor, $G$ the particle's anomalous gyromagnetic ratio, $\vec{B}_{\perp}$ and $\vec{B}_{\perp}$ respectively the magnetic field components perpendicular and parallel to the particle's velocity. In addition, we have assumed the absence of electric fields. We can transform this equation by expanding about a reference orbit described by a Frenet-Serret coordinate system. See figure 1. Thus we have

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}s} = \frac{\hat{s}}{\rho}, \quad \frac{\mathrm{d}\hat{s}}{\mathrm{d}s} = -\frac{\hat{x}}{\rho}, \quad \text{and} \quad \frac{\mathrm{d}\hat{z}}{\mathrm{d}s} = 0, \tag{2}$$

Particle motion can be parameterized in this coordinate system as

$$\vec{r} = \vec{r}_o(s) + x\hat{x} + z\hat{z}, \tag{3}$$

where $\vec{r}_o(s)$ is the reference orbit, and $\hat{s} = \mathrm{d}\vec{r}_o/\mathrm{d}s$.

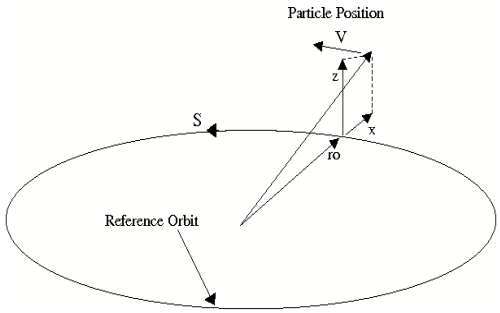


Figure 1: The curvilinear coordinate system for particle motion in a circular accelerator. The unit vectors $\hat{x}$, $\hat{s}$ and $\hat{z}$ are the transverse radial, longitudinal, and transverse vertical basis vectors; and $\vec{r}_0(s)$ is the reference orbit.

1) Show that in the absence of solenoid magnets and vertical bends, the T-BMT equation can, to first order in the coordinates $(x, s, z)$, be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}\vec{S}}{\mathrm{d}s} &\approx v\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)\vec{S}\times\vec{F} \approx \vec{S}\times\vec{F} \\
F_x &= -(1+G\gamma)z'' \\
F_s &= (1+G\gamma)z'/\rho - (1+G)\left(\frac{z}{\rho}\right)' \\
F_z &= x''(1+G\gamma) - \frac{1+G\gamma}{\rho}
\end{aligned}$$

Here you will use the fact that $B\rho = \gamma m v/e$ and approximate $\frac{\mathrm{d}t}{\mathrm{d}s} \approx 1/v$ since the other terms will introduce orders of $x, s, z$ and $1/\rho$ beyond our approximation. To accomplish this, first express the perpendicular and parallel magnetic field components in the forms

$$\vec{B}_\perp = \frac{1}{v^2}(\vec{v} \times \vec{B}) \times \vec{v}, \quad \vec{B}_\parallel = \frac{1}{v^2}(\vec{v} \cdot \vec{B})\vec{B}. \tag{4}$$

Then use the Lorentz force equation to express $\vec{v} \times \vec{B}$, hence also $\vec{B}_\perp$ in terms of $\mathrm{d}\vec{v}/\mathrm{d}t$:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{q}{m\gamma}\vec{v} \times \vec{B}, \tag{5}$$

$$\vec{B}_\perp = \frac{m\gamma}{qv^2}\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \times \vec{v}. \tag{6}$$

In addition, make use of the fact that

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{v}{\left[\left(1 + \frac{x}{\rho}\right)^2 + x'^2 + z'^2\right]^{1/2}}, \tag{7}$$

where $\rho$ denotes the radius of curvature of the local Frenet-Serret coordinate system. Now use (2) together with (4) through (7) in the T-BMT equation (1) to obtain the first-order result (4). In the present context, "first-order" means drop all terms of second order and higher in $x$, $z$, $1/\rho$ and their derivatives. Also remember to make use of the initial assumption that solenoids and vertical bends are absent.

Accounting for the derivatives of the $x, s, z$ basis vectors of $\vec{S}$ we can obtain,

$$\begin{aligned}
\frac{\mathrm{d}S_x}{\mathrm{d}s} &= S_s(F_z + 1/\rho) - F_s S_z \\
\frac{\mathrm{d}S_s}{\mathrm{d}s} &= -S_x(F_z + 1/\rho) + F_x S_z \\
\frac{\mathrm{d}S_z}{\mathrm{d}s} &= S_x F_s - F_x S_s
\end{aligned}$$

If we change to a basis vector which rotates with the beam $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ we obtain,

$$\begin{aligned}
&\frac{\mathrm{d}\vec{S}}{\mathrm{d}s} = \vec{f} \times \vec{S}, \\
&f_1 = (1 + G\gamma)z'', \\
&f_2 = -(1 + G\gamma)z'/\rho + (1 + G)\left(\frac{z}{\rho}\right)', \\
&f_3 = -(1 + G\gamma)x'' + \frac{G\gamma}{\rho}.
\end{aligned} \tag{8}$$

2) Define a two-component *spinor* $\Psi$ such that the $j$-th component of the spin vector is given by

$$S_j = \langle\Psi|\sigma_j|\Psi\rangle = \Psi^\dagger\sigma_j\Psi. \tag{9}$$

Here $\Psi$ denotes a classical vector with components $u$ and $d$, which are related to the three components of $\vec{S}$ by

$$\begin{aligned} S_1 &= u^*d + ud^* \\ S_2 &= -\mathrm{i}(u^*d - ud^*) \\ S_3 &= |u|^2 - |d|^2. \end{aligned} \tag{10}$$

Using $\vec{\sigma} = (\sigma_x, \sigma_s, \sigma_z)$, the vector of the Pauli spin matrices, and $\zeta(s) = -f_1 + \mathrm{i}f_2$, show that equation 8 can be transformed into,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}s} = -\frac{\mathrm{i}}{2}(\vec{\sigma}\cdot\vec{f})\Psi = -\frac{\mathrm{i}}{2}H\Psi = -\frac{\mathrm{i}}{2}\begin{pmatrix} \frac{G\gamma}{\rho} & -\zeta(s) \\ -\zeta(s)^* & -\frac{G\gamma}{\rho} \end{pmatrix}\Psi, \tag{11}$$

where we have dropped the first term in $f_3$, $-(1+G\gamma)x''$, since it is small compared to $G\gamma$.

3) Using the differential relation $\mathrm{d}\theta = \mathrm{d}s/\rho$, one may transform (11) so as to make $\theta$ the independent variable. Then, considering the effects of a *single* resonance, wherein $\zeta(\theta)$ becomes $\epsilon_K e^{-\mathrm{i}K\theta}$, the T-BMT equation becomes

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\theta} = -\frac{\mathrm{i}}{2}\begin{pmatrix} G\gamma & -\epsilon_K e^{-\mathrm{i}K\theta} \\ -\epsilon_K^* e^{\mathrm{i}K\theta} & -G\gamma \end{pmatrix}\Psi. \tag{12}$$

At this point, assume $G\gamma =$ constant, and transform the spinor equation (12) into the *resonance precessing frame* by defining

$$\Psi_K(\theta) = e^{\frac{\mathrm{i}}{2}K\theta\sigma_z}\Psi(\theta). \tag{13}$$

Show that

$$\frac{\mathrm{d}\Psi_K}{\mathrm{d}\theta} = \frac{\mathrm{i}}{2}\begin{pmatrix} K - G\gamma & \epsilon_K \\ \epsilon_K^* & G\gamma - K \end{pmatrix}\Psi_K. \tag{14}$$

4) The code `SpinTrack.cc` integrates the single-resonance T-BMT equation (14) with two orthogonal snakes located at $\theta = 0$ and $\theta = \pi$, with an axis of rotation oriented at angle $\phi = \pm\pi/4$, as is the case in RHIC. The code takes as input the file Resonance.in:

```
k0:      452
w0R,w0I: 0.0  0.0
k1:      453
w1R,w1I: 0.0  0.0
k2:      393
w2R,w2I: 0.432733 0.112896
k3:      392
w3R,w3I: 0.0  0.0
```

```
k4:      394
w4R,w4I: 0.0  0.0
k5:      422
w5R,w5I: 0.0  0.0
k6:      423
w6R,w6I: 0.0  0.0
```

In this version of the code, we integrate the single-resonance T-BMT equation by taking the matrix exponential of equation 14. [In this case, our single resonance is located at $K = \texttt{k2} = 393 + Q_y = 422.67$ ($Q_y = 29.67$), with *strength* $\epsilon_K = 0.432\,733 + \mathrm{i}\,0.112\,896$). The code's default initial value for $G\gamma_0$ is 414.8, with an acceleration rate of $\frac{\mathrm{d}G\gamma}{\mathrm{d}\theta} = 3.741\,18 \times 10^{-6}\,\mathrm{rad}^{-1}$ (as in RHIC). The default number of turns is $\texttt{NT} = 670\,000$.

- Compile the code:
  ```
  mpic++ SpinTrack.cc -o SpinTrack
  ```

- With Resonance.in in your directory, run the code using
  ```
  mpiexec -n 1 ./SpinTrack
  ```
  This will generate a file called:
  ```
  TBTAmp1.000Tau5.000Qs8.900CV2.00Q29.670R0.dat
  ```
  After some header information, it lists the turn number, value of G$\gamma$, vertical component of the spin vector $S_y$, and the estimated Envelope of the Spin vector given by the following equation:

$$\begin{aligned} \delta &= K - G\gamma \\ \lambda &= \sqrt{\delta^2 + |\epsilon_k|^2} \\ b &= \frac{|\epsilon_K|}{\lambda}\,\sin\frac{\lambda\pi}{2} \\ S_{\mathrm{env}} &= 1 - 8b^2(1-b^2) \end{aligned} \tag{15}$$

  Identify the locations of the nodes and anti-nodes.

- Now run the same code using the command
  ```
  mpiexec -n 1 ./SpinTrack \
    nstrobe= 1 Ggam0= 420.0 NT= 250000
  ```
  In this case, you are now starting around an anti-node and turning off stroboscopic averaging to orient your initial spin vector. What is different about this tracking? How do you think you could compute the final polarization in this case?

- Run it again turning back on stroboscopic averaging by using:
  ```
  mpiexec -n 1 ./SpinTrack Ggam0= 420.0 NT= 250000
  ```
  How does the spin vector behave now?

- Run the code again, now using more particles:

  ```
  mpiexec -n 20 ./SpinTrack Ggam0= 420.0 NT= 250000
  ```

  Now the code should produce 20 files one for each particle where the complex phase of the spin resonance is distributed evenly over 0 to $2\pi$. The "R0", "R1" indicate the particle number for each file. The python script `AvgFiled.py` lists all the `TBT*.dat` files in the current directory and then reads them in calculating an average for the vertical Spin component. Run it using:

  ```
  python AvgFiled.py > YourFileName.dat
  ```

  It generates a file listing turn number, G$\gamma$, and average vertical Spin. Notice how the average trajectory converges.