SPIN@USPAS Summer 2021
Graduate "Spin Dynamics" Homework

HOME WORK 1 / ex. 2.1: LOW ENERGY SPIN ROTATOR
SOLUTION

1/ Relationship between E_Y and B_Z for a straight electron trajectory.

$$E_Y = v\, B_Z \tag{1}$$

2/

$$d\vec{v}/ds = \vec{v}\,' = \vec{v}\times\vec{B}/B\rho \;\;\Rightarrow\;\; \text{velocity precession} = BL/B\rho \tag{2}$$

with $\vec{B}$ the precession vector. By analogy,

$$\vec{S}\,' = \vec{S}\times\vec{\omega}/B\rho \;\;\Rightarrow\;\; \text{spin precession } \theta_{sp} = \omega L/B\rho \tag{3}$$

with $\vec{\omega}$ the spin precession vector.

The spin angular momentum results from two torques (Thomas-BMT equation), one due to the magnetic field, one due to the electric field. The respective precession vectors can be expressed under the form

$$\begin{aligned} {}^{B}\vec{\omega}_{sp} &= (1+a)\vec{B}_{\|} + (1+a\gamma)\vec{B}_{\perp} \\ {}^{E}\vec{\omega}_{sp} &= \gamma\left(a+\frac{1}{1+\gamma}\right)\frac{\vec{E}\times\vec{v}}{c^2} \end{aligned} \tag{4}$$

with $\vec{B}_{\|}$ the projection of $\vec{B}$ on the velocity direction and $\vec{B}_{\perp} = \vec{B} - \vec{B}_{\|}$ normal to the latter. The anomalous magnetic moment for the electron is $a = 1.15965\times 10^{-3}$. With $E_Y = v\,B_Z$ and in particular $\vec{B}\perp\vec{v}$ always, one has

$$\vec{\omega} \;=\; {}^{B}\vec{\omega}_{sp} + {}^{E}\,\vec{\omega}_{sp} = (1+a\gamma)\vec{B}_{\perp} + \gamma\left(a+\frac{1}{\gamma+1}\right)\frac{\vec{E}\times\vec{v}}{c^2} \tag{5}$$

Taking $\vec{z}$ unitary vector along Z, accounting in addition for $\vec{E}\perp\vec{v}$, one has

$$\vec{B} = B_Z\vec{z}, \quad \vec{E}\times\vec{v} = -vE_Y\vec{z} = -v^2B_Z\vec{z}, \quad \vec{\omega} = \omega\,\vec{z} \tag{6}$$

With these ingredients ω can be substituted in the θ_{sp} expression above to yield

$$\theta_{\text{sp.th}} = \underbrace{(1+a\gamma)\frac{B_ZL}{B\rho}}_{\text{from }\tilde{\text{B}}} \underbrace{-\gamma\left(a+\frac{1}{\gamma+1}\right)\beta^2\frac{B_ZL}{B\rho}}_{\text{from }\;\tilde{\text{E}}} = 30^o \tag{7}$$

3/ $B_Z = \dfrac{\dfrac{B\rho\,\theta_{\text{sp}}}{L}}{\left[1+a\gamma-\gamma\left(a+\dfrac{1}{\gamma+1}\right)\beta^2\right]} = 0.00407378\,\text{T}$

and
$E_Y = vB_Z = 982939\,\mathrm{V/m}$

4/ Mis-set field values found are $\mathrm{E_Y} = -19\mathrm{E}5\ [\mathrm{V/m}]$ and $\mathrm{B_Z} = 8\mathrm{E}-03\ [\mathrm{T}]$.

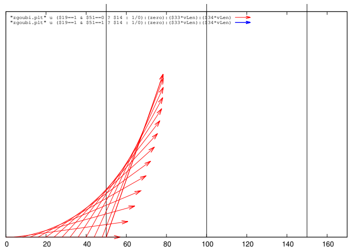
4.a/ Fragments of zgoubi.plt header (which tells column contents, see also Sec. 8.3 in the Users' Guide) and first two steps data lines, followed by fragments of the last two steps data lines:

```
# TRAJECTORIES - STORAGE FILE, 30-03-2021 04:34:40. WFSegment_FIT.INC.dat
#  1 ,  2   , 3 ,  4 , 5 ,  6 , 7 ,  8  ,  9   , 10  ,  11, 12,  13, 14, 15  , 16  , 17, 18 ,   19, 20  ,  21 , 22, 23, 24, 25,   26  ,  27  ,   28  , 29 ,  30 ,
# KEX,  Do-1, Yo,  To, Zo,  Po, So,   to,   D-1, Y-DY,   T,  Z,   P,  S, time, beta, DS, KART,  IT, IREP, SORT,  X, BX, BY, BZ,   RET,   DPR,    PS,   SXo,  SYo,
# int, float, cm, mrd, cm, mrd, cm, mu_s, float,   cm, mrd, cm, mrd, cm, mu_s,  v/c, cm,  int, int,  int,   cm, cm, kG, kG, kG, float, float, float, float,float,fl
    1   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00
 0.0000000000000000E+00   0.0000000000000000E+00   0.0000000000000000E+00   0.0000000000000000E+00   0.0000000000000000E+00   0.0000000000000000E+00   0.0000000000
    1   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00
-1.4784942357382036E-04  -4.3398188439933176E-03  -2.1686995248122773E+00  -1.6689455010073568E-17  -8.3454250394809348E-15   3.9999999999999996E+00   1.6578294181
...............................
    1   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00
-2.1386295082320372E-02  -6.2534744642443774E-01  -2.2913949973127291E+01  -2.6406724835381783E-15  -1.0687486763609849E-13   5.0005049538361718E+01   2.0779853354
    1   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00   0.00000000E+00
-2.1386295082320372E-02  -6.2534744642443774E-01  -2.2913949973127291E+01  -2.6406724835381783E-15  -1.0473455411037868E-13   5.0005049538361718E+01   2.0779853354
```

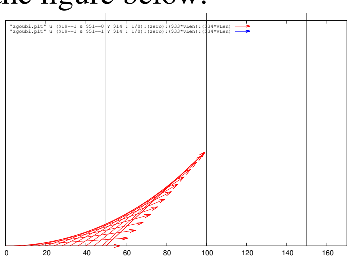
As read from the last line in zgoubi.plt, the final relative momentum D, Y coordinate and T (horizontal) angle with these mis-set field values are

$$\mathrm{D}{-2.1386295082320372\mathrm{E}{-02}}, \quad \mathrm{Y} = -6.2534744642443774\mathrm{E}{-01}\ [\mathrm{cm}], \quad \mathrm{T} = -2.2913949973127291\mathrm{E}{+01}\ [\mathrm{mrad}]$$

whereas zeros is expected. The resulting spin motion is sketched in the figure below:



4.b Proper settings: $\mathrm{E_Y} = 982939\,\mathrm{V/m}, \quad \mathrm{B_Z} = 0.00407378\,\mathrm{T}$
The resulting spin motion is sketched in the figure below:

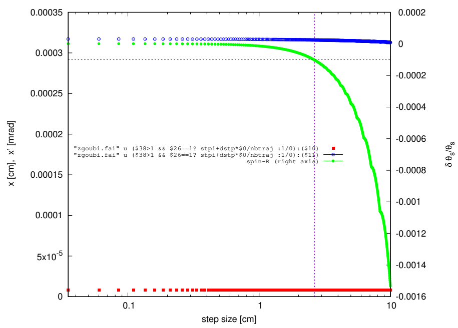


4.c Input data file for a step size scan:

```
WFSegment.
'OBJET'
2.31147953865                                              ! Rigidity of a 350 keV electron.
2
3  1                 ! 3 electrons, same initial coordinates, for spin matrix computation by SPNPRT.
0.  0. 0. 0. 0. 1. 'o'
0.  0. 0. 0. 0. 1. 'o'
0.  0. 0. 0. 0. 1. 'o'
1 1 1
'PARTICUL'
POSITRON

'SPNTRK'                                          ! Allows getting the rotation of all 3 spin components
4                                ! (they are computed independently), for matrix computation by SPNPRT.
1. 0. 0.
0. 1. 0.
0. 0. 1.

'MARKER'  #S_WFSegment_EBth
'WIENFILT'
20
0.5 -982939 0.00407378  1                                  ! Theoretical E (V/m) and B (T) field values.
0. 0. 0.    ! Hard-edge entrance face, XE=lambdaE_e=lambdaB_e=0. Substitue 20. 5. 5. for fringe fields.
0.2401  1.8639  -0.5572  0.3904 0. 0.                        ! Fringe field coefficients in Enge model.
```

```
0.2401  1.8639  -0.5572  0.3904 0. 0.
0. 0. 0.        ! Hard-edge exit face, XS=lambdaE_s=lambdaB_s=0. Substitue 20. 5. 5. for fringe fields.
0.2401  1.8639  -0.5572  0.3904 0. 0.
0.2401  1.8639  -0.5572  0.3904 0. 0.
.2                ! Integration step size. It may be given different values, depending on the exercise.
1. 0. 0. 0.
'MARKER'  #E_WFSegment_EBth

'FAISCEAU'                                              ! Log local particle coordinates in zgoubi.res.
'FAISTORE'
zgoubi.fai                                              ! Log local particle coordinates in zgoubi.fai.
1
'REBELOTE'
400 0.1 0 1                        ! Repeat the previous sequence, 400 times, and prior to each repeat,
1                                                                   ! change value of one parameter,
WIENFILT 80 0.01:10.                          ! namely, number 80 (integration step size) in WIENFILT.

'SYSTEM'
2
gnuplot <./gnuplot_scanTraj.gnu
okular ./gnuplot_scanTraj.eps &
'END'
```

Resulting graph:

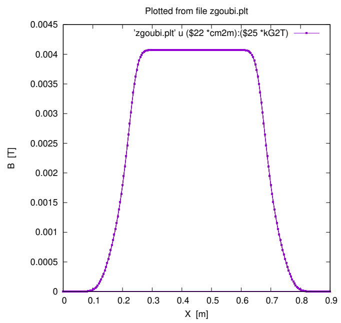


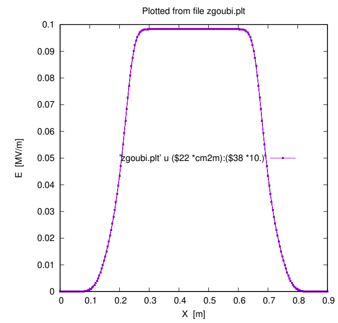
5/ Input data file to FIT EY and BZ:

```
WFSegment.
'OBJET'
2.31147953865                                                         ! Rigidity of a 350 keV electron.
2
3  1                         ! 3 electrons, same initial coordinates, for spin matrix computation by SPNPRT.
0.  0. 0. 0. 0. 1. 'o'
0.  0. 0. 0. 0. 1. 'o'
0.  0. 0. 0. 0. 1. 'o'
1 1 1
'PARTICUL'
POSITRON

'SPNTRK'                                              ! Allows getting the rotation of all 3 spin components
4                                   ! (they are computed independently), for matrix computation by SPNPRT.
1. 0. 0.
0. 1. 0.
0. 0. 1.

'MARKER'  #S_WFSegment_EBth
'WIENFILT'
20
0.5 -982939 0.00407378  1                                  ! Theoretical E (V/m) and B (T) field values.
0. 0. 0.     ! Hard-edge entrance face, XE=lambdaE_e=lambdaB_e=0. Substitue 20. 5. 5. for fringe fields.
0.2401  1.8639  -0.5572  0.3904 0. 0.                         ! Fringe field coefficients in Enge model.
0.2401  1.8639  -0.5572  0.3904 0. 0.
0. 0. 0.        ! Hard-edge exit face, XS=lambdaE_s=lambdaB_s=0. Substitue 20. 5. 5. for fringe fields.
0.2401  1.8639  -0.5572  0.3904 0. 0.
0.2401  1.8639  -0.5572  0.3904 0. 0.
.2               ! Integration step size. It may be given different values, depending on the exercise.
1. 0. 0. 0.
'MARKER'  #E_WFSegment_EBth

'FAISCEAU'                                              ! Log local particle coordinates in zgoubi.res.
'FAISTORE'
zgoubi.fai                                              ! Log local particle coordinates in zgoubi.fai.
1
'REBELOTE'
400 0.1 0 1                        ! Repeat the previous sequence, 400 times, and prior to each repeat,
1                                                                   ! change value of one parameter,
WIENFILT 80 0.01:10.                          ! namely, number 80 (integration step size) in WIENFILT.

'SYSTEM'
2
gnuplot <./gnuplot_scanTraj.gnu
okular ./gnuplot_scanTraj.eps &
'END'
```

New field values: -982883.82 and 4.07355283E-03

Resulting graphs:





6/ Input data file to scan fringe extent and re-FIT EY and BZ:

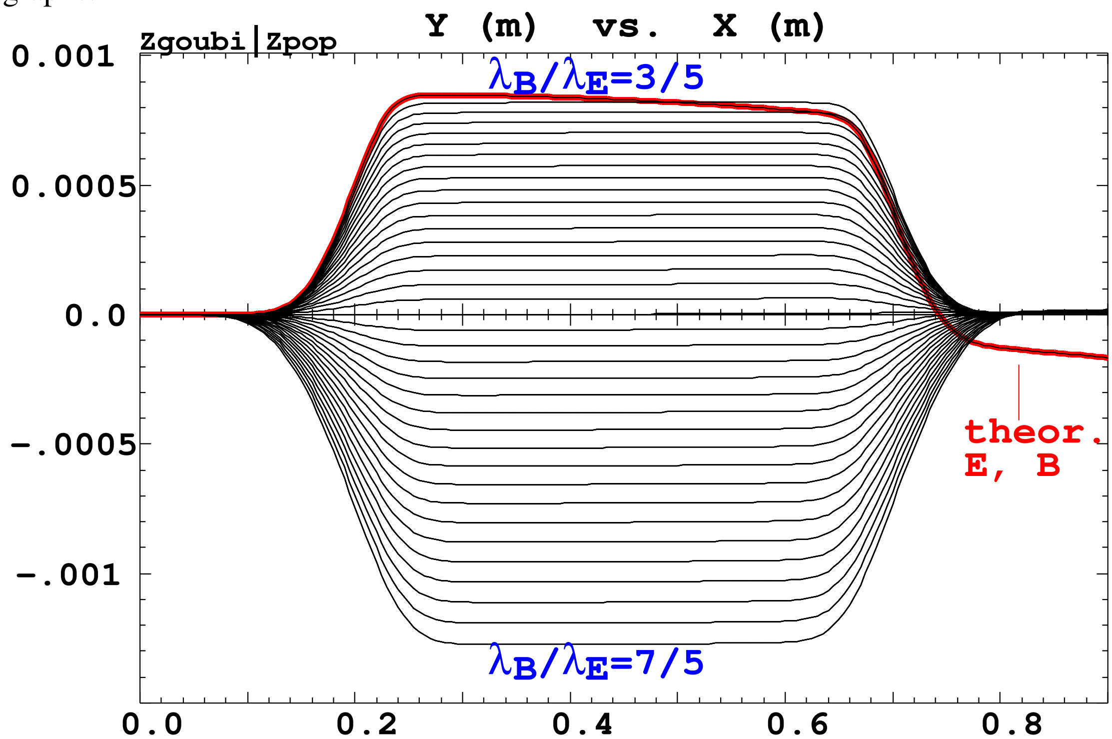
```
WFSegment_FIT.INC.dat
! In addition to the comments on the right, use the Users' Guide.
'OBJET'
2.31147953865                                                     ! Rigidity of a 350 keV electron.
2
3  1                                     ! 3 electrons are needed for spin matrix computation by SPNPRT.
0.  0. 0. 0. 0. 1. 'o'                   ! Initial coordinates all zero, except for relative regidity 1.
0.  0. 0. 0. 0. 1. 'o'
0.  0. 0. 0. 0. 1. 'o'
1 1 1
'PARTICUL'
POSITRON

'SPNTRK'                          ! Set the initial spin components, proper for spin matrix computation.
4                                                                 !
1. 0. 0.                                                                                ! Initial SX=1.
0. 1. 0.                                                                                ! Initial SY=1.
0. 0. 1.                                                                                ! Initial SZ=1.

'INCLUDE'
1
./WFSegment_EBTheor.inc[#S_WFSegment_FF:#E_WFSegment_FF]

 'FIT2'                       ! The numbering of optical elements to be used is to be rad in zgoubi.res.
 2
 5 11 0 .8                                                  ! Keyword 5: WIENFILT; paremeter 11: E.
 5 12 0 .8                                                  ! Keyword 5: WIENFILT; paremeter 12: B.
 5 1E-15                                                       ! Six constraints; penalty is 1e-15:
 3    1 2 #End 0. 1. 0                                                                 ! Y=0 at end.
 3    1 3 #End 0. 1. 0                                                                 ! T=0 at end.
 10.2 1 1 #End 0.52359877559 1. 0                                             ! 30 deg spin rotation,
 10   1 4 #End 1. 1. 0                                                  ! |S|=1, for all 3 particles.
 10   2 4 #End 1. 1. 0
 10   3 4 #End 1. 1. 0

'REBELOTE'
37 0.1 0 1                    ! NPASS is of the form int*(7[cm]-3[cm])+1 to allow for lambdaB/lambdaE=1.
2
WIENFILT 22 3.:7.                                        ! vary lambda_B at entrance EFB from 3 to 7 cm.
WIENFILT 52 3.:7.                                            ! vary lambda_B at exit EFB from 3 to 7 cm.

'FAISCEAU'                                                   ! Log trajectory coordinates in zgoubi.res.
'SPNPRT'  MATRIX                                   ! Log spin coordinates and spin matrix in zgoubi.res.
'SYSTEM'
2
gnuplot <./gnuplot_Zplt_sYZ.gnu
gnuplot < ./gnuplot_scanEB_FFratio.gnu
'END'
```

gnuplot files:

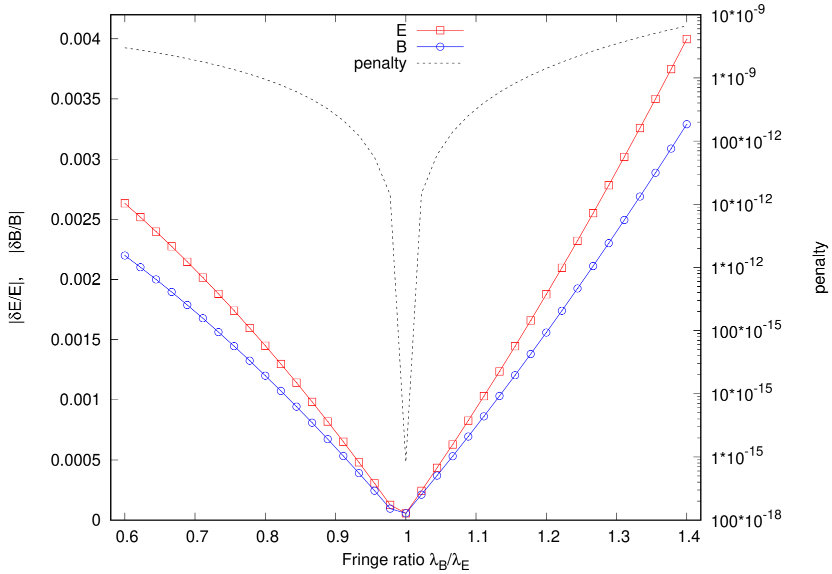
```
#gnuplot_Zplt_sYZ.gnu
set title "Plotted from file Zplt "
set key maxcol 1; set key t r
set xtics mirror; set ytics mirror
set size ratio 1
set xlabel 's  [m]'; set ylabel 'Y  [m]'
cm2m = 0.01; kG2T= 0.1; MeV2eV = 1e6; c = 2.99792458e8

plot \
   'zgoubi.plt' u ($14 *cm2m):($10 *cm2m) w p ps .4 notit ,\
   'zgoubi.plt' u ($14 *cm2m):($10 *cm2m) w p ps 0

     set terminal postscript eps color  enh
       set output "gnuplot_Zplt_sY.eps"
       replot
       set terminal X11
       unset output
pause 2
exit


# gnuplot_scanEB_FFratio.gnu
set xlab "Fringe ratio {/Symbol l}_B/{/Symbol l}_E"; set ylab "|{/Symbol d}E/E|,     |{/Symbol d}B/B|"; set y2lab "penalty"
set k c t spacin 1; set xtics mirror; set ytics nomirror; set y2tics; set logscale y2; set format y2 "%.0s*10^{%T}"
# grep status of FIT variables, in zgoubi.res:
system "grep '   1    11  ' zgoubi.res |  cat > grep.res4E.out"
system "grep '   2    12  ' zgoubi.res |  cat > grep.res4B.out"
```

```
system "grep 'penalty value ' zgoubi.res |  cat > grep.penalty.out"
# Parameters of the lambda_B/lambda_E scan, to determine the horizontal scale:
NPASS = 37; lmdaBi = 3.; lmdaBf = 7.; lmdaE = 5.; dlmda = (lmdaBf-lmdaBi)/(NPASS-1); Eth= -982939.444; Bth= 4.0737834E-03

plot  [.58:1.42] [:.0042] \
"grep.res4E.out" u ($0>0? (lmdaBi+ dlmda*($0-1))/lmdaE :1/0):(abs(($6-Eth)/Eth)) axes x1y1 w lp pt 4 lc rgb "red" tit "E" ,\
"grep.res4B.out" u ($0>0? (lmdaBi+ dlmda*($0-1))/lmdaE :1/0):(abs(($6-Bth)/Bth)) axes x1y1 w lp pt 6 lc rgb "blue" tit "B" ,\
"grep.penalty.out" u ($0>0? (lmdaBi+dlmda*($0-1))/lmdaE :1/0):($5) axes x1y2 w l dt 2 lc rgb "black" tit "penalty" ; pause 1

 set terminal postscript eps blacktext color enh    # size 8.3cm,4cm "Times-Roman" 12
 set output "gnuplot_scanEB_FFratio.eps"
 replot
 set terminal X11
 unset output

pause .2
exit
```

Resulting graphs:



A scan of the on-momentum reference trajectories across the Wien filter, with varying fringe field extent ratio λ_B/λ_E. The reference trajectories have zero coordinates at entrance by hypothesis, and at exit (together with maintaining $\theta_s = 30\,\mathrm{deg}$) as a result of the FIT procedure which adjusts E and B amplitudes. A graph obtained using zpop, which reads stepwise particle data from zgoubi.plt: menu 7; 1/1 to open zgoubi.plt; 2/[8,2] to select Y versus distance



Variation of the E_Y and B_Z fields (relative to their hard-edge model values), necessary to recover (i) exact 30^o spin precession, (ii) straight trajectory, when the λ_B/λ_E ratio is varied (B field fringe extent is varied, while E fringe extent maintained constant in this case). The fit "penalty" quantifies the distance to these two constraints, steadily small here.

7/ Spinor formulation

The spinor rotation matrix is $(\vec{\Omega}\cdot\vec{\sigma})\phi$ with $\Omega\phi$ the spin rotation angle.
The 2x2 transfer matrix is

$$T = e^{\frac{i}{2}(\vec{\Omega}\cdot\vec{\sigma})\phi} = I\,\cos\frac{\Omega\phi}{2} + i\,(\frac{\vec{\Omega}}{\Omega}\cdot\vec{\sigma})\,\sin\frac{\Omega\phi}{2}$$

$\vec{\Omega} = \begin{pmatrix}\Omega_X\\ \Omega_Y\\ \Omega_Z\end{pmatrix} \equiv \begin{pmatrix}0\\ 0\\ \Omega_Z\end{pmatrix}$ as both $\vec{E}$ and $\vec{B}$ resut in a precession around the vertical (Z) axis.

Thus $\vec{\Omega}\cdot\vec{\sigma} = \Omega_Z\,\sigma_Z,\ |\Omega_Z| = |\Omega|,\ T = e^{\frac{i}{2}\Omega_Z\,\phi\,\sigma_Z} = I\,\cos\frac{\Omega_Z\phi}{2} + i\,\sigma_Z\,\sin\frac{\Omega_Z\phi}{2}$

$$= \cos\frac{\Omega_Z\phi}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + i\sin\frac{\Omega_Z\phi}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$= \begin{pmatrix} e^{i\frac{\Omega_Z\phi}{2}} & 0\\ 0 & e^{-i\frac{\Omega_Z\phi}{2}}\end{pmatrix}$$

With spin precession angle $\Omega_Z\phi = \pi/2$, $\frac{\Omega_Z\phi}{2} = \pi/4$, one gets $e^{\pm i\frac{\Omega_Z\phi}{2}} = \frac{1\pm i}{\sqrt{2}}$, thus

$$T = \begin{pmatrix} \frac{1+i}{\sqrt{2}} & 0\\ 0 & \frac{1-i}{\sqrt{2}}\end{pmatrix}$$

Transform to real 3D space using $\vec{S} = \begin{pmatrix} S \\ S_y \\ S_z \end{pmatrix} \equiv \psi^\dagger \, \vec{\sigma} \psi$, this yields the expected $\dfrac{\pi}{2}$ angle Z-rotation matrix

$$M_{\mathrm{x}} \; = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Apply to initial $\vec{S}_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, this yields the expected

$$\vec{S}_f = M \, \vec{S}_i = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$