# SPIN@USPAS Summer 2021
# Graduate “Spin Dynamics” Homework

## HOME WORK 2 / ex. 2.2: SYNCHRONIZED TORQUE SOLUTION

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Actually the 3 energies concerned will be treated simultaneously, using a single simulation input data file.

1/ Finding the closed orbits for 200 keV, 108.412 MeV and 370.082556 MeV.
From $R = B\rho/B_0$, with R the $Y_0$ coordinate under OBJET, $B_0 = 5$ kG the field value, one gets,

$$\mathrm{Y}_0(200\,\mathrm{keV}) = 12.924889\,\mathrm{cm}, \qquad \mathrm{Y}_0(108.412\,\mathrm{MeV}) = 309.47295\,\mathrm{cm}, \quad \mathrm{Y}_0(370.082556\,\mathrm{MeV}) = 608.30878\,\mathrm{cm}$$

The data file takes a reference $B\rho = 64.62444403717985$ kG cm (BORO quantity under OBJET), hence the relative $\mathrm{D} = \dfrac{\mathrm{B}\rho(\mathrm{E})}{\mathrm{BORO}}$ values:

$$\mathrm{D}(200\,\mathrm{keV}) = 1, \qquad \mathrm{D}(108.412\,\mathrm{MeV}) = 23.943951797, \quad \mathrm{D}(370.082556\,\mathrm{MeV}) = 47.06491129$$

A FIT could be used instead, or both! to cross-check (FIT is helpful anyway if field is inhomogeneous - not the case, here).

Input data file:

```
Cyclotron, classical. Synchronized spin kick in a uniform field
'OBJET'
64.62444403717985                    ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
3 1
12.924889  0. 0. 0. 0.   1.          'o'                                  ! Ggamma=1.793229 -> 0.200MeV;
309.47295  0. 0. 0. 0.  23.943951797 'i'                                  ! Ggamma=2 -> 108.411628MeV;
608.30878  0. 0. 0. 0.  47.064911290 'h'                                  ! Ggamma=2.5 -> 370.082556MeV.
1 1 1                       ! For any particle: set to 1 to enable ray-tracing, or to -9 to ignore.

'PARTICUL'                                             ! This is required for spin motion to be computed
PROTON                                          ! - by default zgoubi otherwise only requires rigidity.
'SPNTRK'                                                                   ! Request spin tracking.
4.1                                                      ! All initial spins taken parallel to Z axis.
0. 0. 1.

'SPNPRT'  PRINT             ! Log spin coordinates in zgoubi.res; PRINT adds log in zgoubi.SPNPRT.Out.

'INCLUDE'                                                          ! Grabs lines form a file, here, the
1          ! sequence of lines comprised between #S_60degSectorUnifB and #E_60degSectorUnifB MARKERS:
6* 60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]    ! a 60 degree sector, INCLUDEd 6 times.
'FAISCEAU'
'SPINR'
1                                                                                  ! Spin rotation,
0. 30.                                                        1 about the X-axis, by 10 degree here.

'FIT'                                                                    ! To be uncommented for use.
3                                                                                      ! 3 variables.
1 30 0 .4                                                                         ! Y0 of particle 1,
1 40 0 .4                                                                         ! Y0 of particle 2,
1 50 0 .4                                                                         ! Y0 of particle 3.
6  1e-15                   ! 6 constraints; penalty, all apply at end of sequence, which is here.
3.1 1 2 #End 0. 1. 0                      ! Partilce 1 Y0 equal to current Y, right before FIT,
3.1 1 3 #End 0. 1. 0                      ! partilce 1 T0 equal to current Y, right before FIT,
3.1 2 2 #End 0. 1. 0                      ! Partilce 2 Y0 equal to current Y, right before FIT,
3.1 2 3 #End 0. 1. 0                      ! partilce 2 T0 equal to current Y, right before FIT,
3.1 3 2 #End 0. 1. 0                      ! Partilce 3 Y0 equal to current Y, right before FIT,
3.1 3 3 #End 0. 1. 0                      ! partilce 3 T0 equal to current Y, right before FIT.

'FAISCEAU'                               ! Log current coordinates in zgoubi.res execution listing.
'FAISTORE'                     ! Create zgoubi.fai and log current coordinates therein (usually for
zgoubi.fai                                                      ! graphic or data post-treatmet).
1
'END'
```

The result of the FIT procedure can be found in zgoubi.res listing:

```
*******************************************************************************************************************************************
     24  Keyword, label(s) :  FIT                                                                                         IPASS= 1

     Pgm main. FITING=.T., FIT procedure launched.

                    FIT  variables  and  constraints  in  good  order,  FIT  will proceed.
 STATUS OF VARIABLES  (Iteration #     1 /    999 max.)
LMNT VAR PARAM  MINIMUM    INITIAL         FINAL         MAXIMUM     STEP        NAME   LBL1                 LBL2
   1   1    30    7.75        12.9       12.924889       18.1      3.447E-02  OBJET      -                    -
   1   2    40    186.        309.       309.47295       433.      0.825      OBJET      -                    -
   1   3    50    365.        608.       608.30878       852.       1.62      OBJET      -                    -
 STATUS OF CONSTRAINTS (Target penalty =   1.0000E-15)
TYPE  I   J LMNT#     DESIRED          WEIGHT          REACHED         KI2     NAME   LBL1                 LBL2      Nb param. [value]
  3   1   2    23    0.000000E+00    1.000E+00    4.160228E-11    8.59E-05 FAISCEAU   -                    -                  0
  3   1   3    23    0.000000E+00    1.000E+00    4.489777E-09    1.00E+00 FAISCEAU   -                    -                  0
  3   2   2    23    0.000000E+00    1.000E+00    1.250555E-12    7.76E-08 FAISCEAU   -                    -                  0
  3   2   3    23    0.000000E+00    1.000E+00    1.015507E-11    5.12E-06 FAISCEAU   -                    -                  0
  3   3   2    23    0.000000E+00    1.000E+00    3.410605E-13    5.77E-09 FAISCEAU   -                    -                  0
  3   3   3    23    0.000000E+00    1.000E+00    2.747802E-12    3.75E-07 FAISCEAU   -                    -                  0
 Fit reached penalty value   2.0160E-17
*******************************************************************************************************************************************
```

Current particle coordinates after the FIT can also be found in zgoubi.res listing (right hand side columns):

```
*******************************************************************************************************************************************
     26  Keyword, label(s) :  FAISCEAU                                                                                    IPASS= 1

0                                              TRACE DU FAISCEAU
                                           (follows element #     25)
                                                 3 TRAJECTOIRES

                                   OBJET                                                       FAISCEAU

          D       Y(cm)     T(mr)     Z(cm)     P(mr)       S(cm)      D-1      Y(cm)     T(mr)    Z(cm)    P(mr)      S(cm)


o   1   1.0000    12.925     0.000     0.000     0.000      0.0000   0.0000   12.925    0.000    0.000    0.000   8.120937E+01     1
               Time of flight (mus) :  0.13121675     mass (MeV/c2) :   938.272
i   1  23.9440   309.473     0.000     0.000     0.000      0.0000  22.9440  309.473   -0.000    0.000    0.000   1.944476E+03     2
               Time of flight (mus) :  0.14634703     mass (MeV/c2) :   938.272
h   1  47.0649   608.309     0.000     0.000     0.000      0.0000  46.0649  608.309   -0.000    0.000    0.000   3.822117E+03     3
               Time of flight (mus) :  0.18293379     mass (MeV/c2) :   938.272
*******************************************************************************************************************************************
```

2 to 4/ The 3 questions are answered using a single input data file.

SPINR is introduced, at the end of the sequence. REBELOTE is used for multiturn tracking. Note: A different possibility, instead of REBELOTE, could just be to INCLUDE the 60 deg dipole 6*N times, for N turns in the constant field.

Input data file:

```
Cyclotron, classical. Synchronized spin kick in a uniform field
'OBJET'
64.62444403717985                      ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
3 1
12.9248888074 0. 0. 0. 0.  1.     'm'                  ! Ggamma=1.793229,    0.200MeV
3.0947295453790e2 0. 0. 0. 0.  23.9439548880185 'm'   ! Ggamma=2,         108.411628MeV
6.08308775712857e2 0. 0. 0. 0. 47.0649136542578 'm'   ! Ggamma=2.5        370.082556MeV
1 1 1                           ! For any particle: set to 1 to enable ray-tracing, or -9 to ignore.

'PARTICUL'                                          ! This is required for spin motion to be computed,
PROTON                                          ! otherwise, by default zgoubi only requires rigidity.
'SPNTRK'                                                                   ! Request spin tracking.
4.1                                                      ! All initial spins taken parallel to Z axis.
0. 0. 1.

'SPNPRT'  PRINT

'INCLUDE'
1
6* ./60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]                     ! 360 degree sector.
'FAISCEAU'
'SPINR'
1                                                                                  ! Spin rotation,
0. 30.                                                           1 about the X-axis, by 30 degree here.

'REBELOTE'                                                                     ! Multiturn ray-tracing.
40 0.2  99
'SYSTEM'
3
gnuplot <./gnuplot_Zspnprt_spinOscillation.gnu
gnuplot < ./gnuplot_Zplt_spinTilt.gnu
gnuplot <./gnuplot_Zplt_spinTilt_3D.gnu
'END'
```

The 3 different types of figures at the bottom have been obtainde with the following gnuplot files:

- Plot spin component oscillations:

```
  # gnuplot_Zspnprt_spinOscillation.gnu
# set tit "Observation azimuth is at snake. \n If G.{/Symbol g}=half-int, spin precesses in (Y,Z) plane.
set xlabel "turn"; set ylabel "S_X,   S_Y,   S_Z"; set key b l
nbtrj=3 # number of trajectories tracked

do for [it=1:nbtrj] {
 unset label; set label sprintf("particle %3.5g",it) at 10, 0.8
plot [] [-1:1] \
  'zgoubi.SPNPRT.Out' every nbtrj::(it+2) u ($22):($13) w lp lw .3 pt 4 ps .8 lc rgb "red" ,\
  'zgoubi.SPNPRT.Out' every nbtrj::(it+2) u ($22):($14) w lp lw .3 pt 6 ps .8 lc rgb "blue" ,\
  'zgoubi.SPNPRT.Out' every nbtrj::(it+2) u ($22):($15) w lp lw .3 pt 8 ps .8 lc rgb "black"
pause .5
set terminal postscript eps blacktext color  enh
set output sprintf('gnuplot_Zspnprt_spinOsc_trj%i.eps',it)
replot
set terminal X11
unset output
}
```

• Plot spin motion in X-Y plane, lab. coordinates:

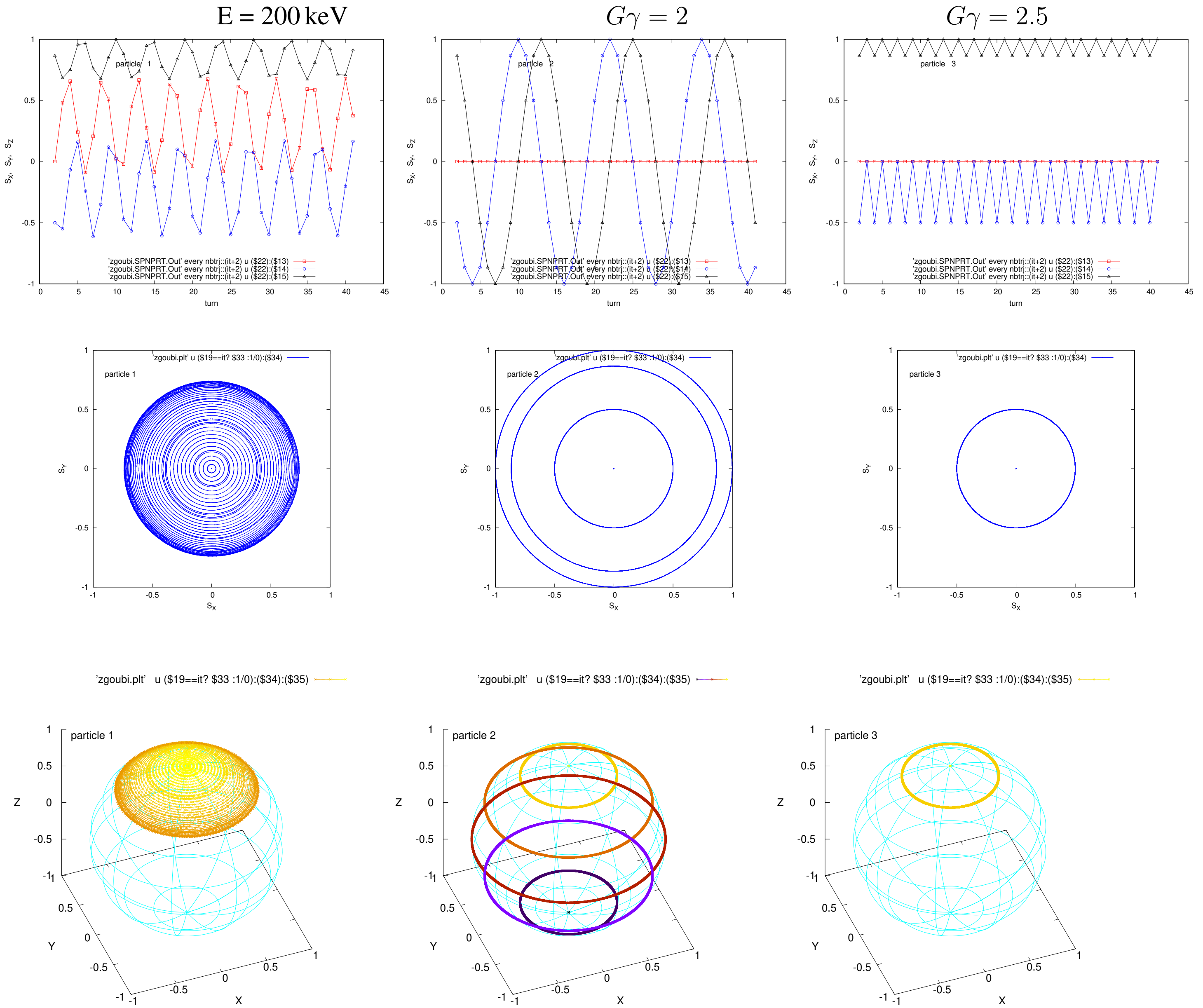
```
  # gnuplot_Zplt_spinTilt.gnu
# set tit "s_{/Symbol p} motion"
set xlabel "S_X"; set ylabel "S_Y"; set size ratio -1; set xrange [-1:1]; set yrange [-1:1]; set key t l
nbtrj=3 # number of trajectories tracked

do for [it=1:nbtrj] {
  unset label; set label sprintf("particle %i",it) at -.9, .8
  plot 'zgoubi.plt' u ($19==it? $33 :1/0):($34) w lp lw .3 ps .2 lc rgb "blue"
  pause .5
  set terminal postscript eps blacktext color  enh
  set output sprintf('gnuplot_Zplt_SX-SY_trj%i.eps',it)
  replot; set terminal X11; unset output
}
```

• Plot spin motion on a sphere:

```
  # gnuplot_Zplt_spinTilt_3D.gnu
set xlabel "X"; set ylabel "Y"; set zlabel "Z"; set xrange [-1:1]; set yrange [-1:1]; set zrange [-1:1]
set xyplane 0; set view equal xyz; set view 49, 339; unset colorbox
set urange [-pi/2:pi/2]; set vrange [0:2*pi]; set parametric; R = 1.   # radius of sphere
nbtrj=3 # number of trajectories tracked

do for [it=1:nbtrj] {
  unset label; set label sprintf("  particle %i",it) at -1, .9, 1.
  splot  R*cos(u)*cos(v),R*cos(u)*sin(v),R*sin(u) w l lw .2 lc rgb "cyan" notit ,\
  'zgoubi.plt'   u ($19==it? $33 :1/0):($34):($35) w lp lw .2 ps .4 lc palette
  pause .5
  set terminal postscript eps blacktext color  enh
  set output sprintf('gnuplot_Zplt_S3D_trj%i.eps',it)
  replot; set terminal X11; unset output
}
```




Top row: spin coordinates versus turn; middle row: projection in the median plane (the segment between two consecutive circles materializes the location of the X-kick by SPINR); bottom row: projection on a sphere. $G\gamma = 1.793229$: far from an integer, $\vec{S}$ remains within a reduced top angle cone. $G\gamma = 2$: the spin vector oscillates between up and down orientations, by 20 deg steps; it takes 180/20=9 orbits for the X-precession at SPINR to flip the spin; $G\gamma = 2.5$: the spin vector finds itself back in the (X,Z) at the location of SPINR, after one orbit and a half-integer number of precessions; it alternates between vertical and 20 deg from vertical, after each orbit around the cyclotron

• Spin tunes

Spin tunes are computed using “ ’SPNPRT’ MATRIX “. Tune is deduced from spin matrix trace, computation of the spin matrix requires the tracking of 3 rays with initial coordinates on the spin closed orbit and spins respectivele parallel to X, Y, Z axis.

OBJET[KOBJ=2] is used to create blocks of 3 rays with each block a different momentum. Spin matrix computation by “ ’SPNPRT’ MATRIX “ will take this into account and compute as many spin matrices.

The input data file is given below:

```
Cyclotron, classical. Synchronized spin kick in a uniform field
'OBJET'
64.62444403717985                    ! Reference Brho ("BORO" in the users' guide) -> 200keV proton.
2
9 3
12.924889  0. 0. 0. 0.   1.         'o'                              ! Ggamma=1.793229 -> 0.200MeV;
```

```
12.924889  0. 0. 0. 0.   1.          'o'                                  ! Ggamma=1.793229 -> 0.200MeV;
12.924889  0. 0. 0. 0.   1.          'o'                                  ! Ggamma=1.793229 -> 0.200MeV;
309.47295  0. 0. 0. 0.  23.943951797 'i'                                 ! Ggamma=2 ->    108.411628MeV;
309.47295  0. 0. 0. 0.  23.943951797 'i'                                 ! Ggamma=2 ->    108.411628MeV;
309.47295  0. 0. 0. 0.  23.943951797 'i'                                 ! Ggamma=2 ->    108.411628MeV;
608.30878  0. 0. 0. 0.  47.064911290 'h'                                 ! Ggamma=2.5 ->  370.082556MeV.
608.30878  0. 0. 0. 0.  47.064911290 'h'                                 ! Ggamma=2.5 ->  370.082556MeV.
608.30878  0. 0. 0. 0.  47.064911290 'h'                                 ! Ggamma=2.5 ->  370.082556MeV.
1 1 1 1 1 1 1 1 1             ! For any particle: set to 1 to enable ray-tracing, or to -9 to ignore.
! 309.47295  0. 0. 0. 0.  ***        'i'                                   ! Ggamma=2 -> 108.411628MeV;
! 608.30878  0. 0. 0. 0.  ***        'h'                                  ! Ggamma=2.5 -> 370.082556MeV.

'PARTICUL'                                          ! This is required for spin motion to be computed
PROTON                                         ! - by default zgoubi otherwise only requires rigidity.
'SPNTRK'                                                                  ! Request spin tracking.
4                                                     ! All initial spins taken parallel to Z axis.
1. 0. 0.
0. 1. 0.
0. 0. 1.
1. 0. 0.
0. 1. 0.
0. 0. 1.
1. 0. 0.
0. 1. 0.
0. 0. 1.

'SPNPRT'  PRINT             ! Log spin coordinates in zgoubi.res; PRINT adds log in zgoubi.SPNPRT.Out.

'INCLUDE'                                                           ! Grabs lines form a file, here, the
1           ! sequence of lines comprised between #S_60degSectorUnifB and #E_60degSectorUnifB MARKERS:
6* 60degSector.inc[#S_60degSectorUnifB:#E_60degSectorUnifB]    ! a 60 degree sector, INCLUDEd 6 times.
'FAISCEAU'
'SPINR'
1                                                                                  ! Spin rotation,
0. 30.                                                        1 about the X-axis, by 10 degree here.

! 'FIT'                                                                     ! To be uncommented for use.
! 3                                                                                     ! 3 variables.
! 1 30 0 .4
! 1 40 0 .4
! 1 50 0 .4
! 6  1e-15                      ! 6 constraints; penalty, all apply at end of sequence, which is here.
! 3.1 1 2 #End 0. 1. 0
! 3.1 1 3 #End 0. 1. 0
! 3.1 2 2 #End 0. 1. 0
! 3.1 2 3 #End 0. 1. 0
! 3.1 3 2 #End 0. 1. 0
! 3.1 3 3 #End 0. 1. 0
! 'END'

'FAISCEAU'                                    ! Log current coordinates in zgoubi.res execution listing.
'SPNPRT' MATRIX

'END'
```

## Resulting spin matrices and tunes:

This data are extracted from zgoubi.res

```
     26  Keyword, label(s) :  SPNPRT      MATRIX                                                               IPASS= 1

                         -- 3  GROUPS  OF  MOMENTA  FOLLOW   --

                    -----------------------------------------------------------------------------------
                    Momentum  group  #1 (D=   1.000000E+00) ; average  over 3 particles at this pass :

                Spin  components  of  each  of  the      3  particles,  and  rotation  angle :

                   INITIAL                                                  FINAL
           SX         SY         SZ         |S|          SX         SY         SZ         |S|        GAMMA   |Si,Sf|   (Z,Sf_yz) (Z,Sf)
                                                                                                            (deg.)     (deg.)    (deg.)
                                                                                                   (Sf_yz : projection of Sf on YZ plane)
 o  1  1.000000  0.000000  0.000000  1.000000     0.268269  0.834281  0.481672  1.000000      1.0002   74.439   63.435   61.205    1
 o  1  0.000000  1.000000  0.000000  1.000000    -0.963344  0.232327  0.134134  1.000000      1.0002   76.566   63.435   82.291    2
 o  1  0.000000  0.000000  1.000000  1.000000     0.000000 -0.500000  0.866025  1.000000      1.0002   30.000   49.107   30.000    3

                  Spin transfer matrix, momentum group # 1 :

          0.268269       -0.963344         0.00000
          0.834281        0.232327       -0.500000
          0.481672        0.134134        0.866025

     Determinant =       1.0000000000
     Trace =       1.3666212399;   spin precession acos((trace-1)/2) =      79.4373462271 deg
     Precession axis :  ( 0.3225, -0.2450,  0.9143)  ->  angle to (X,Y) plane, to X axis :    66.1073,   -37.2194 deg
     Spin precession/2pi (or Qs, fractional) :    2.2066E-01

                    -----------------------------------------------------------------------------------
                    Momentum  group  #2 (D=   2.394395E+01) ; average  over 3 particles at this pass :

                Spin  components  of  each  of  the      3  particles,  and  rotation  angle :

                   INITIAL                                                  FINAL
```

```
        SX         SY         SZ         |S|         SX         SY         SZ         |S|       GAMMA   |Si,Sf|   (Z,Sf_yz) (Z,Sf)
                                                                                                         (deg.)    (deg.)    (deg.)
                                                                                                 (Sf_yz : projection of Sf on YZ plane)
 i  1  1.000000  0.000000  0.000000  1.000000     1.000000  0.000000  0.000000  1.000000      1.1155    0.000   63.435   90.000    4
 i  1  0.000000  1.000000  0.000000  1.000000    -0.000000  0.866025  0.500000  1.000000      1.1155   30.000   63.435   60.000    5
 i  1  0.000000  0.000000  1.000000  1.000000     0.000000 -0.500000  0.866025  1.000000      1.1155   30.000   49.107   30.000    6

               Spin transfer matrix, momentum group # 2 :

          1.00000       -3.239976E-07     0.00000
         2.805902E-07    0.866025       -0.500000
         1.619988E-07    0.500000        0.866025

     Determinant =       1.0000000000
     Trace =       2.7320508076;   spin precession acos((trace-1)/2) =      30.0000000000 deg
     Precession axis :  ( 1.0000, -0.0000,  0.0000)  ->  angle to (X,Y) plane, to X axis :     0.0000,    -0.0000 deg
     Spin precession/2pi (or Qs, fractional) :    8.3333E-02

                 ------------------------------------------------------------------------------------
                 Momentum  group  #3 (D=   4.706491E+01) ; average  over 3 particles at this pass :

             Spin  components  of  each  of  the      3  particles,  and  rotation  angle :

                INITIAL                                                 FINAL
        SX         SY         SZ         |S|         SX         SY         SZ         |S|       GAMMA   |Si,Sf|   (Z,Sf_yz) (Z,Sf)
                                                                                                         (deg.)    (deg.)    (deg.)
                                                                                                 (Sf_yz : projection of Sf on YZ plane)
 h  1  1.000000  0.000000  0.000000  1.000000    -1.000000 -0.000000 -0.000000  1.000000      1.3944 -180.000  116.565   90.000    7
 h  1  0.000000  1.000000  0.000000  1.000000     0.000000 -0.866025 -0.500000  1.000000      1.3944 -150.000  116.565  120.000    8
 h  1  0.000000  0.000000  1.000000  1.000000     0.000000 -0.500000  0.866025  1.000000      1.3944   30.000   49.107   30.000    9

               Spin transfer matrix, momentum group # 3 :

         -1.00000        3.962777E-07     0.00000
        -3.431865E-07   -0.866025       -0.500000
        -1.981388E-07   -0.500000        0.866025

     Determinant =       1.0000000000
     Trace =      -1.0000000000;   spin precession acos((trace-1)/2) =     179.9999780994 deg
     Precession axis :  ( 0.0000,  0.2588, -0.9659)  ->  angle to (X,Y) plane, to X axis :   -75.0000,    90.0000 deg
     Spin precession/2pi (or Qs, fractional) :    5.0000E-01
```

5/ Spinor representation.

5.a/ The spin rotator causes a $s$-rotation of angle $\phi_s$, it is thus represented by the matrix (with $\vec{n}_s$ a unit vector along the $s$-axis)

$$T_{\text{spinR}} = e^{\frac{i}{2}(\vec{n}_s\cdot\vec{\sigma})\phi_s} = e^{\frac{i}{2}\sigma_s\phi_s} = I\cos\frac{\phi_s}{2} + i\sigma_s\sin\frac{\phi_s}{2} = \begin{pmatrix} \cos\frac{\phi_s}{2} & \sin\frac{\phi_s}{2} \\ -\sin\frac{\phi_s}{2} & \cos\frac{\phi_s}{2} \end{pmatrix}$$

which expectedly coincides with the $s$-axis spinor rotation matrix (“Spinor Methods” lecture, slide 10).

5.b/ The ring with $\phi_s$ torque is represented by the spinor matrix

$$T_{\text{ring}} = e^{\frac{i}{2}(\vec{n}_y\cdot\vec{\sigma})G\gamma\,2\pi}\; e^{\frac{i}{2}(\vec{n}_s\cdot\vec{\sigma})\phi_s} = (I\cos G\gamma\pi + i\sigma_y\sin G\gamma\pi)\,(I\cos\frac{\phi_s}{2} + i\sigma_s\sin\frac{\phi_s}{2})$$

so, with (slide 5) $-\sigma_y\sigma_s = i\sigma_x$

$$\boxed{T_{\text{ring}} = I\cos G\gamma\pi\ \cos\frac{\phi_s}{2} + i\sigma_x\sin G\gamma\pi\ \sin\frac{\phi_s}{2} + i\sigma_s\sin\frac{\phi_s}{2}\ \cos G\gamma\pi + i\sigma_y\ \sin G\gamma\pi\ \cos\frac{\phi_s}{2}}$$

Under explicit 2×2 matrix form: this is also simply the product of $y$-axis and $s$-axis spinor rotations, matrices in slide 10, so

$$T_{\text{ring}} = \begin{pmatrix} e^{iG\gamma\pi} & 0 \\ 0 & e^{-iG\gamma\pi} \end{pmatrix} \begin{pmatrix} \cos\frac{\phi_s}{2} & \sin\frac{\phi_s}{2} \\ -\sin\frac{\phi_s}{2} & \cos\frac{\phi_s}{2} \end{pmatrix}$$

$$\boxed{T_{\text{ring}} = \begin{pmatrix} e^{iG\gamma\pi}\cos\frac{\phi_s}{2} & e^{iG\gamma\pi}\sin\frac{\phi_s}{2} \\ -e^{-iG\gamma\pi}\sin\frac{\phi_s}{2} & e^{-iG\gamma\pi}\cos\frac{\phi_s}{2} \end{pmatrix}}$$

The spin tune satisfies: $\cos\pi\nu_{\rm sp} = \frac{1}{2}\mathrm{Tr}(T_{\rm ring}) = \cos G\gamma\pi\ \cos\frac{\phi_s}{2}$, so

$$\boxed{\mathrm{frac}(\nu_{\rm sp}) = \pm\frac{1}{\pi}\,\mathrm{acos}\big(\cos G\gamma\pi\ \cos\frac{\phi_s}{2}\big)}$$

Numerically:

$$\text{case E=200 keV},\ G\gamma = 1.793229,\ \phi_s = \frac{\pi}{6}\ \Rightarrow\ \mathrm{frac}(\nu_{\rm sp}) = 0.220656294278834$$

$$\text{case E=370.0825 MeV},\ G\gamma = 2,\ \phi_s = \frac{\pi}{6}\ \Rightarrow\ \mathrm{frac}(\nu_{\rm sp}) = 0.08333$$

$$\text{case E=370.0825 MeV},\ G\gamma = 2.5,\ \phi_s = \frac{\pi}{6}\ \Rightarrow\ \mathrm{frac}(\nu_{\rm sp}) = 0.5$$

all three examples in accord with tracking results, namely:

```
CASE G.gamma=1.793229:

  Spin transfer matrix, momentum group # 1 :

          0.268269       -0.963344         0.00000
          0.834281        0.232327       -0.500000
          0.481672        0.134134        0.866025

     Trace =       1.3666212399,    ;   spin precession acos((trace-1)/2) =      79.4373462271 deg
     Precession axis :  ( 0.3225, -0.2450,  0.9143)  ->  angle to (X,Y) plane,  angle to X axis :    66.1073,    70.5691  degree

>>>> Spin  tune  Qs (fractional) :     2.2066E-01
-----

CASE G.gamma=2

                Spin transfer matrix, momentum group # 2 :

           1.00000       -3.239976E-07     0.00000
          2.805902E-07    0.866025       -0.500000
          1.619988E-07    0.500000        0.866025

     Trace =       2.7320508076,    ;   spin precession acos((trace-1)/2) =      30.0000000000 deg
     Precession axis :  ( 1.0000, -0.0000,  0.0000)  ->  angle to (X,Y) plane,  angle to X axis :     0.0000,     0.0000  degree\

>>>> Spin  tune  Qs (fractional) :     8.3333E-02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CASE G.gamma=2.5:

                Spin transfer matrix, momentum group # 3 :

          -1.00000        3.962777E-07     0.00000
         -3.431865E-07   -0.866025       -0.500000
         -1.981388E-07   -0.500000        0.866025

     Trace =      -1.0000000000,    ;   spin precession acos((trace-1)/2) =     179.9999780994 deg
     Precession axis :  ( 0.0000,  0.2588, -0.9659)  ->  angle to (X,Y) plane,  angle to X axis :   -75.0000,   -90.0000  degree

>>>> Spin  tune  Qs (fractional) :     5.0000E-01
```

The periodic spin vector can be obtained by taking the $t_i$ components from the form (slide 7) $T_{\rm ring} = t_0 I + i\sigma_x t_x + i\sigma_y t_y + i\sigma_z t_z$, namely (slide 15)

$$\vec{n}_\pm = \begin{pmatrix} \pm\dfrac{t_{x,1-\rm turn}}{\sqrt{1-t^2_{0,1-\rm turn}}} \\ \pm\dfrac{t_{y,1-\rm turn}}{\sqrt{1-t^2_{0,1-\rm turn}}} \\ \pm\dfrac{t_{z,1-\rm turn}}{\sqrt{1-t^2_{0,1-\rm turn}}} \end{pmatrix}$$

Thus $T_{\rm ring}$ as obtained above yields

$$t_0 = \cos G\gamma\pi\ \cos\frac{\phi_s}{2} \quad\rightarrow\quad \sqrt{1-t^2_{0,1-\rm turn}} = \sin\pi\nu_{sp}$$

and

$$t_x = \sin G\gamma\pi \ \sin\frac{\phi_s}{2}, \ t_y = \sin\frac{\phi_s}{2} \ \cos G\gamma\pi, \ t_z = \sin G\gamma\pi \ \cos\frac{\phi_s}{2}$$

so that

$$\boxed{\vec{n}_\pm = \pm\begin{pmatrix} \sin G\gamma\pi \ \sin\frac{\phi_s}{2}/\sin\pi\nu_{\rm sp} \\ \cos G\gamma\pi \ \sin\frac{\phi_s}{2}/\sin\pi\nu_{\rm sp} \\ \sin G\gamma\pi \ \cos\frac{\phi_s}{2}/\sin\pi\nu_{\rm sp} \end{pmatrix}}.$$

Numerically (*** problem with sign of $n_y$, to be checked),

$$\text{case E=200 keV, ,} \quad \vec{n}_\pm = \pm\begin{pmatrix} 0.3225386 \\ -0.2449867 \\ -0.91430318 \end{pmatrix}$$

$$\text{case E=108.4116 MeV, ,} \quad \vec{n}_\pm = \pm\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{case E=370.0825 MeV, ,} \quad \vec{n}_\pm = \pm\begin{pmatrix} 0 \\ 0.258819 \\ 0.96592 \end{pmatrix}$$

in accord with tracking results, see above (*** problem with sign of $n_y$, to be checked).