# VUV and X-ray Free-Electron Lasers

# Transverse Focusing

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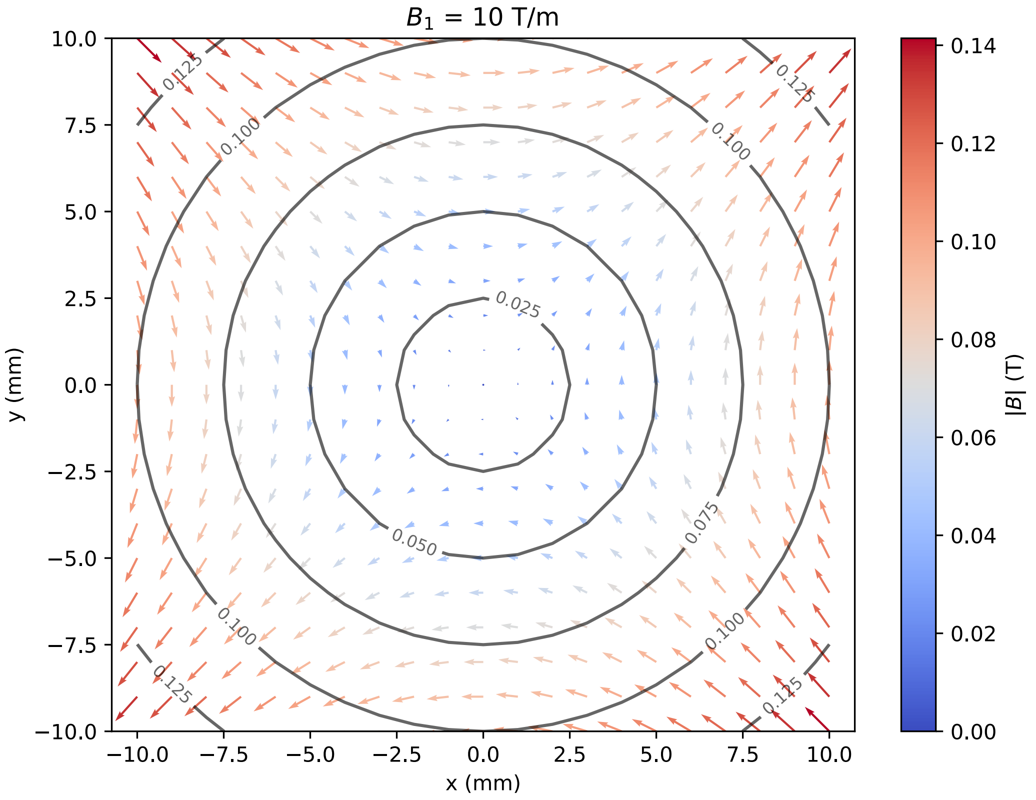
# Quadrupole Fields & Forces




An ideal upright quadrupole field is defined by

$$B_x(x,y) = B_1 y$$
$$B_y(x,y) = B_1 x$$

$$B_1 = \frac{dB_x}{dy}\Big|_{x,y=0}$$

The transverse force on charged particle moving along the *z* axis is then

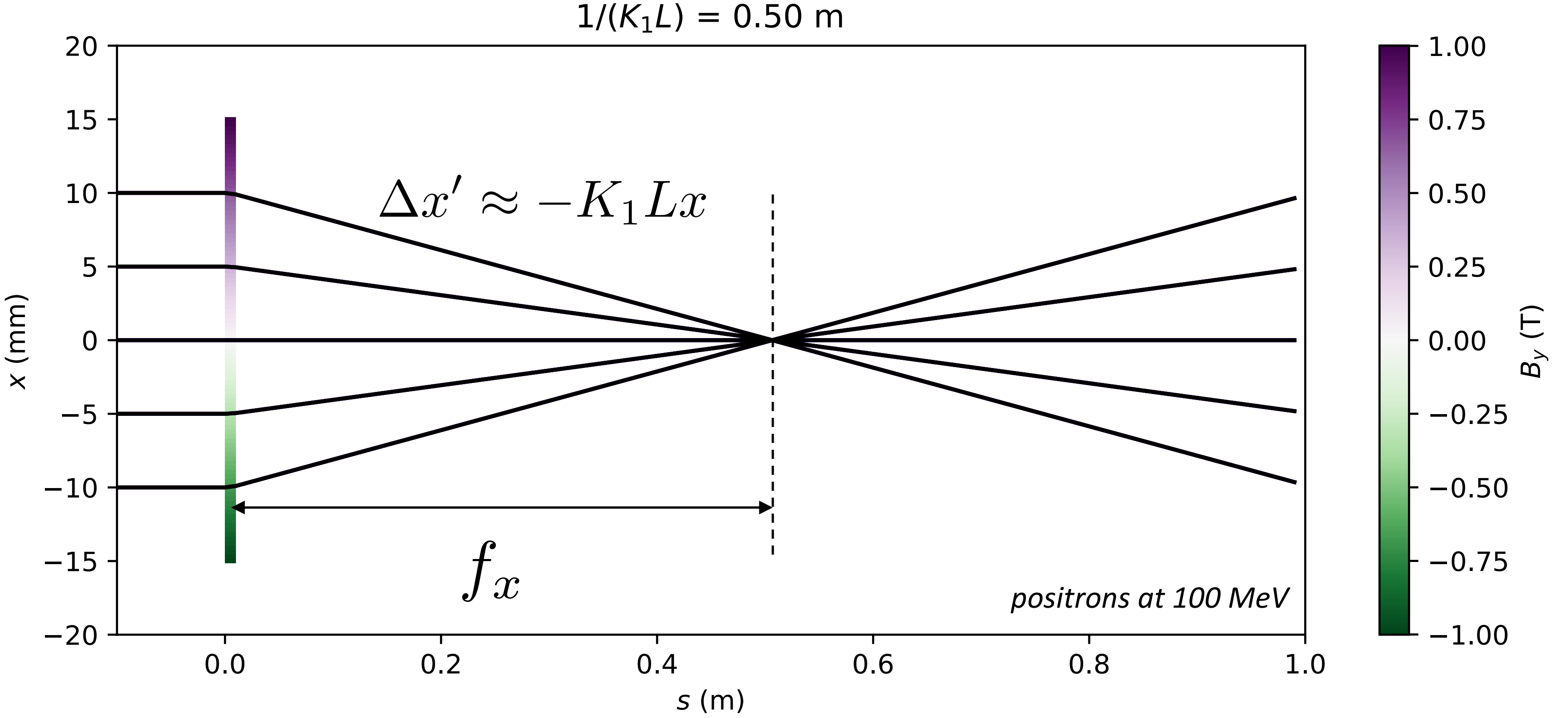
$$\frac{dx'}{ds} = -K_1 x$$
$$\frac{dy'}{ds} = +K_1 y$$

$$K_1 \equiv \frac{qB_1}{p_0}$$

*See Chapter 16: Electromagnetic Fields in the [Bmad manual]*

# Thin Quadrupole lens

For a thin quadrupole with effective length *L*, we can identify the focal lengths in the *x* and y planes




$$\frac{dx'}{ds} = -K_1 x$$

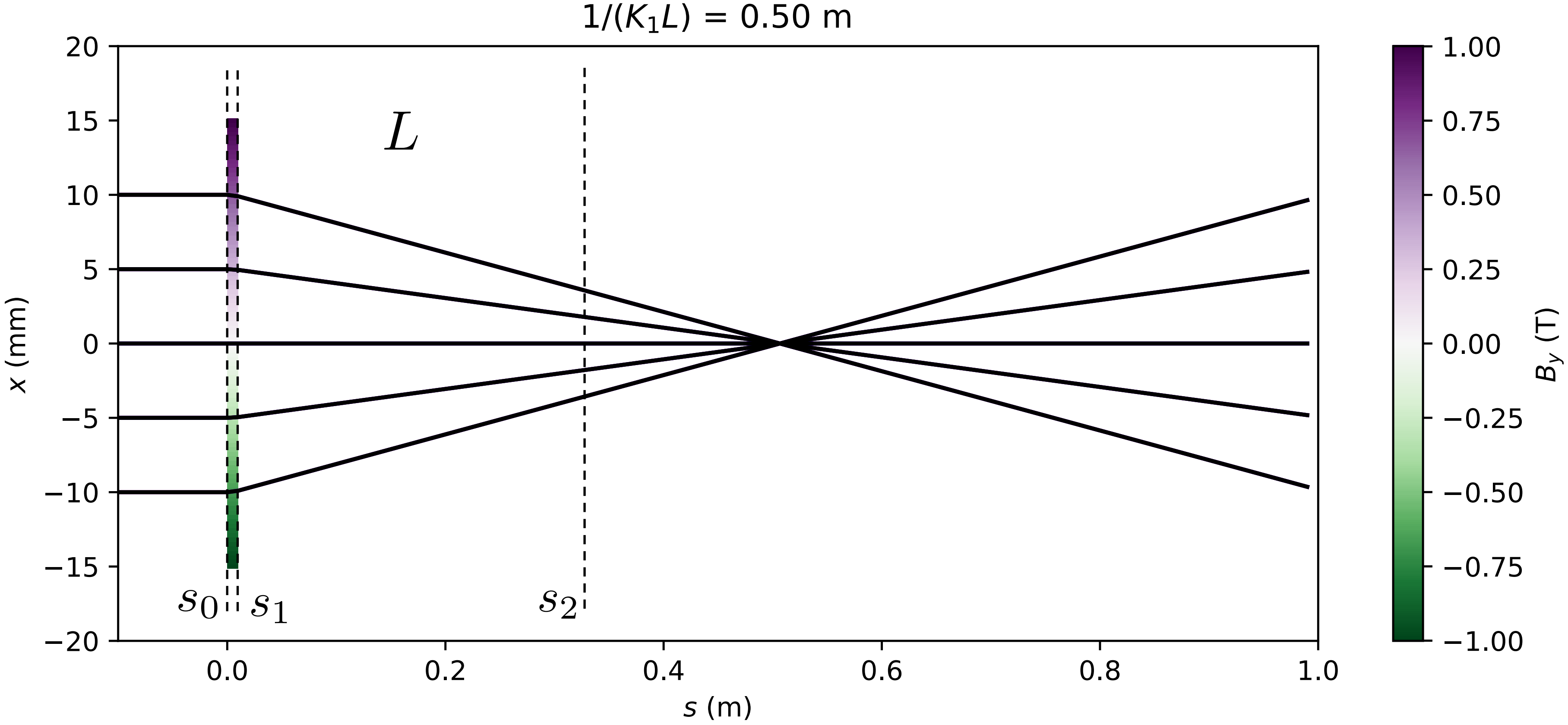
$$\frac{dy'}{ds} = +K_1 y$$

$$f_x = \frac{1}{K_1 L}$$

$$f_y = -f_x$$

# Thin Quad, Drift transfer matrices

Thin linear maps for quadrupole and drifts illustrate the building blocks of map analysis:




Thin quad map from 0 to 1:

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f_x & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Drift map from 1 to 2:

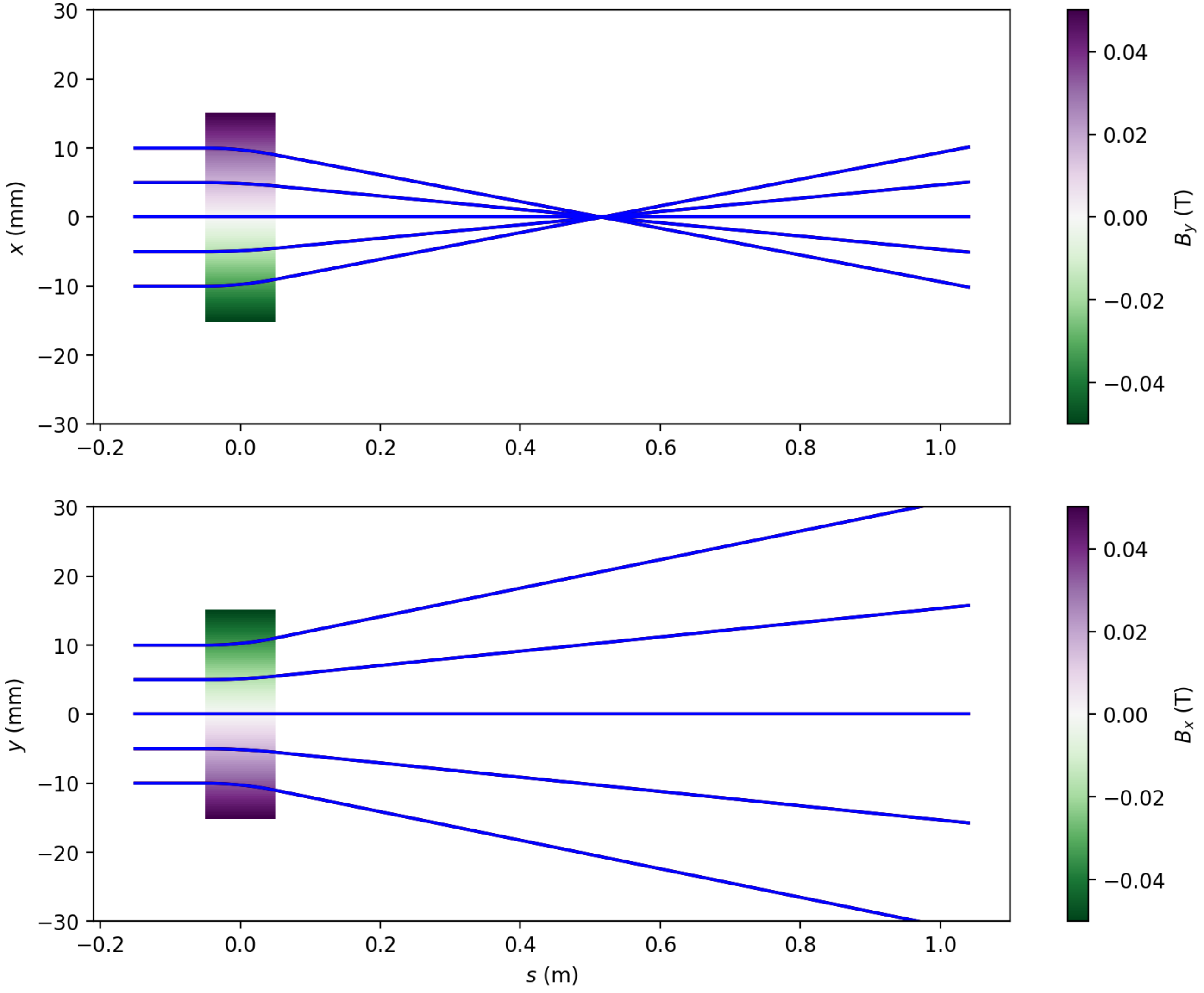
$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

Linear map from 0 to 2:

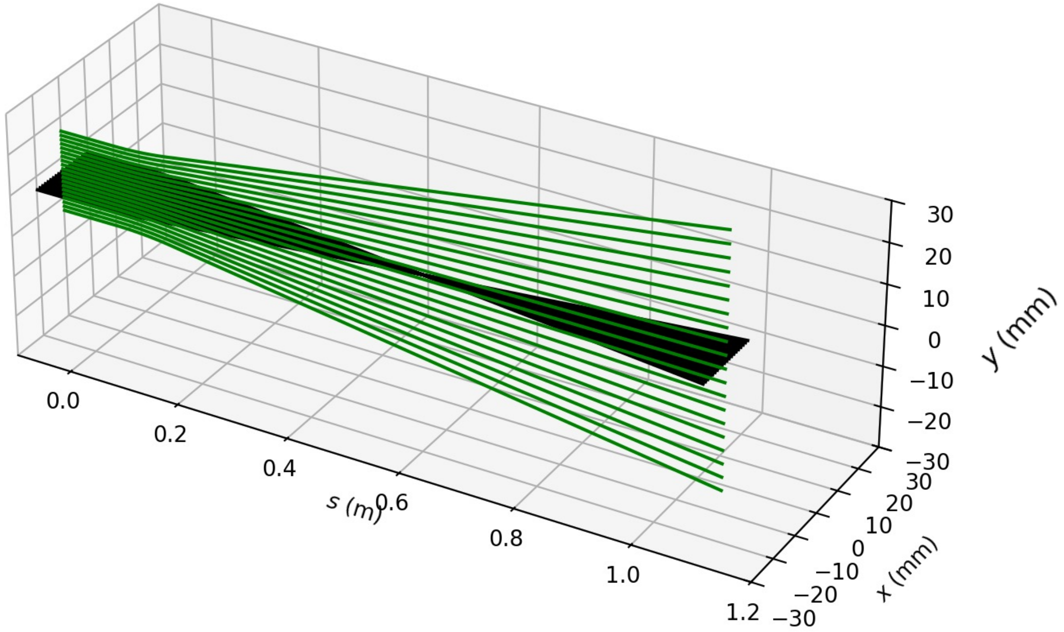
$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 - L/f_x & L \\ -1/f_x & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

*positrons at 100 MeV*

# Thick Quadrupole Focusing

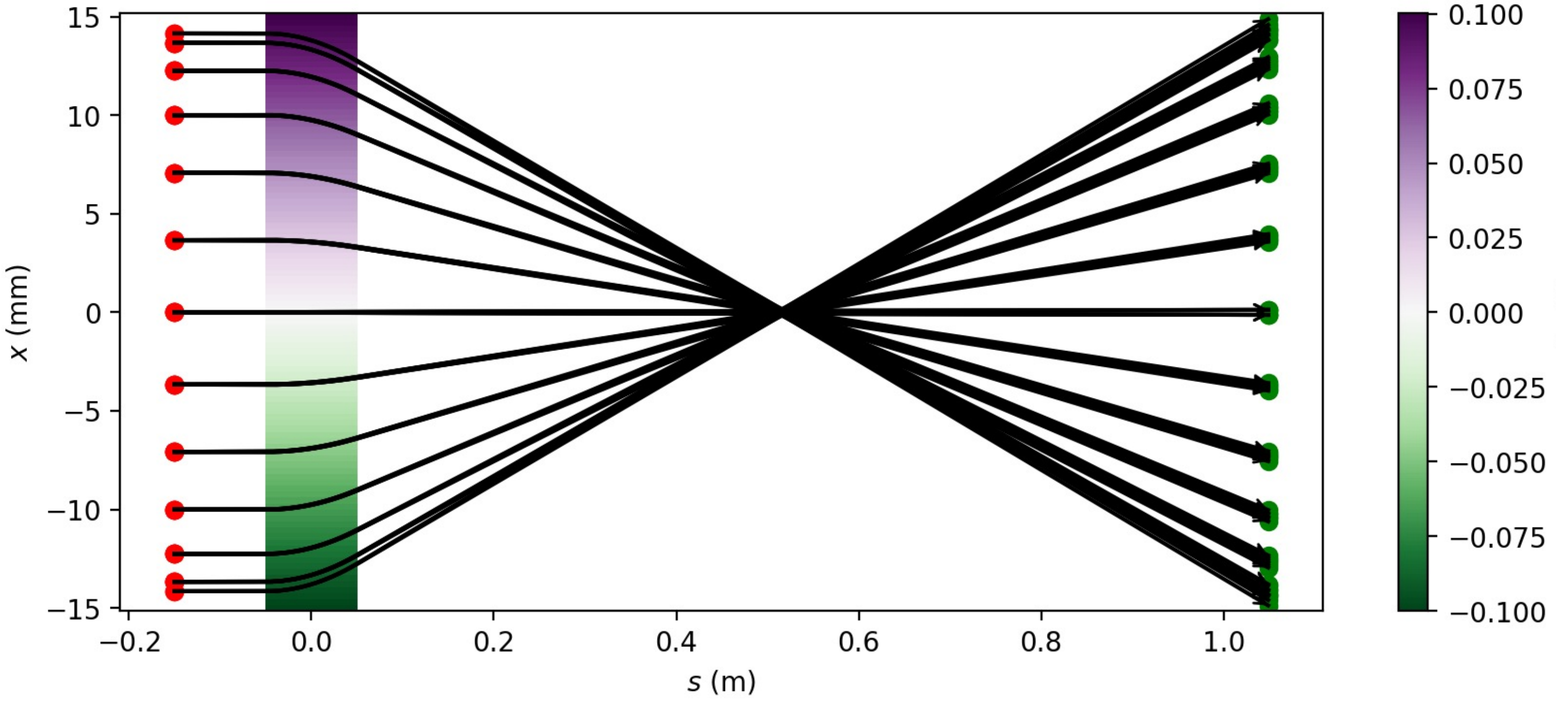


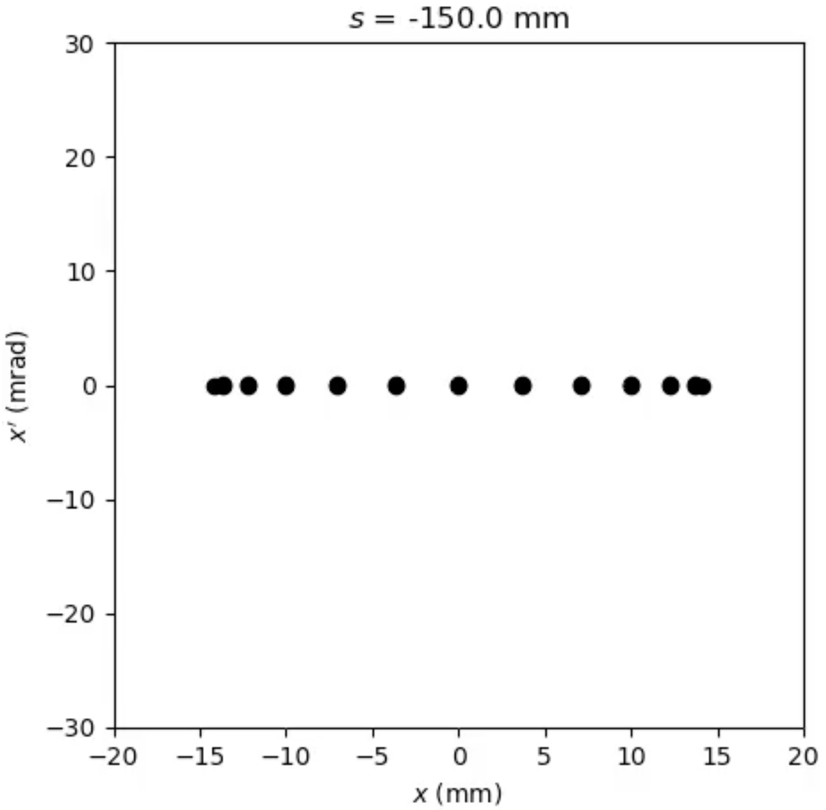

A quadrupole that focuses in one plane will always defocus in the other plane.



# 2D phase space

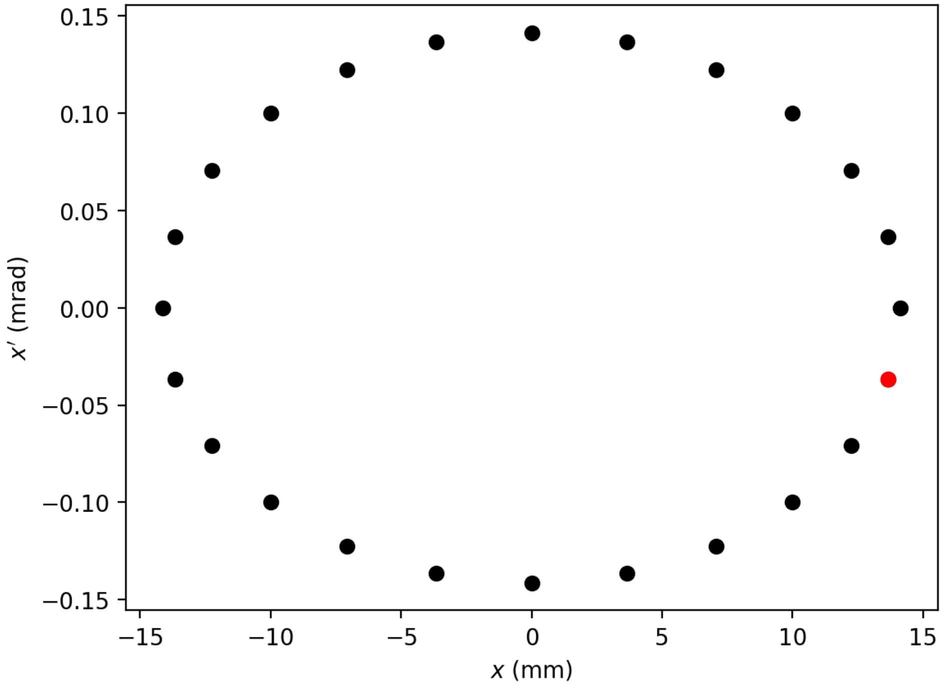
Here is the motion in 2D *(x, x')* phase space.



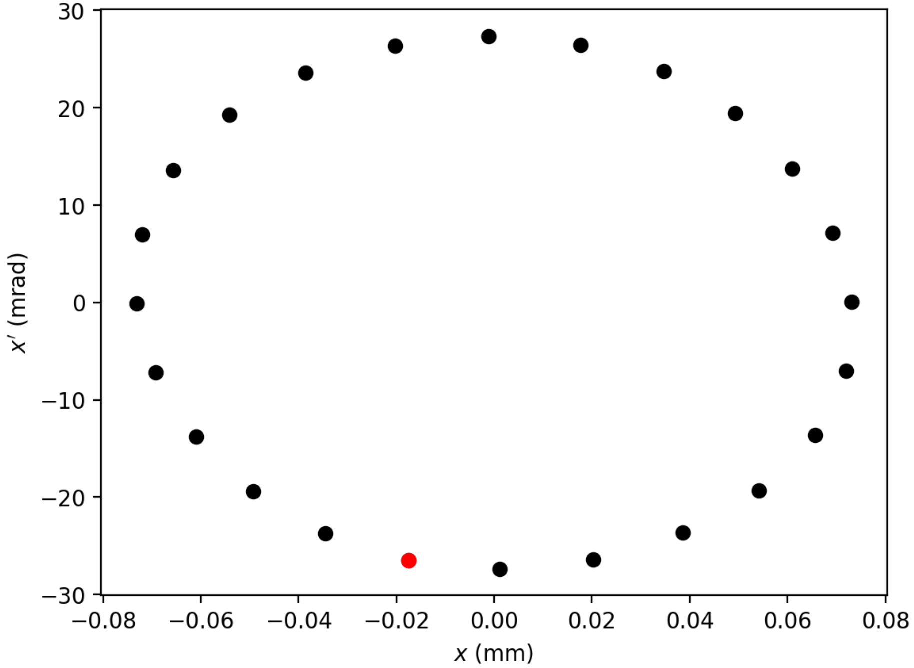



# 2D phase space

A closer look shows that the particles are initially started on an ellipse in phase space, and that the phase space area area enclosed by this ellipse is conserved. This is because a transfer matrix derived from a Hamiltonian is **symplectic**, which has determinant 1 (“area preserving”)




Entrance




Focus

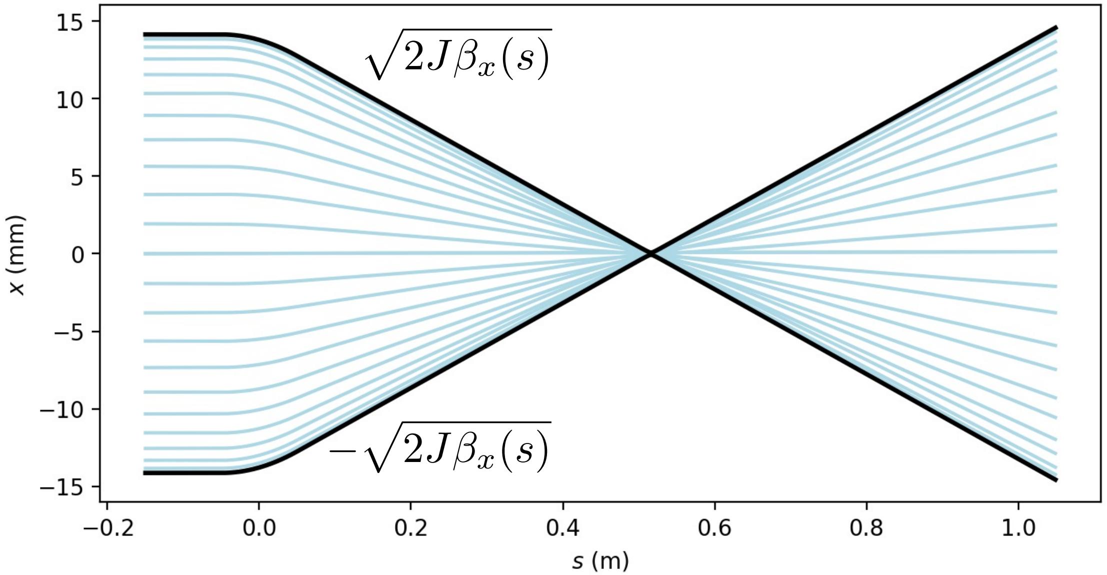
# Action-Angle coordinates

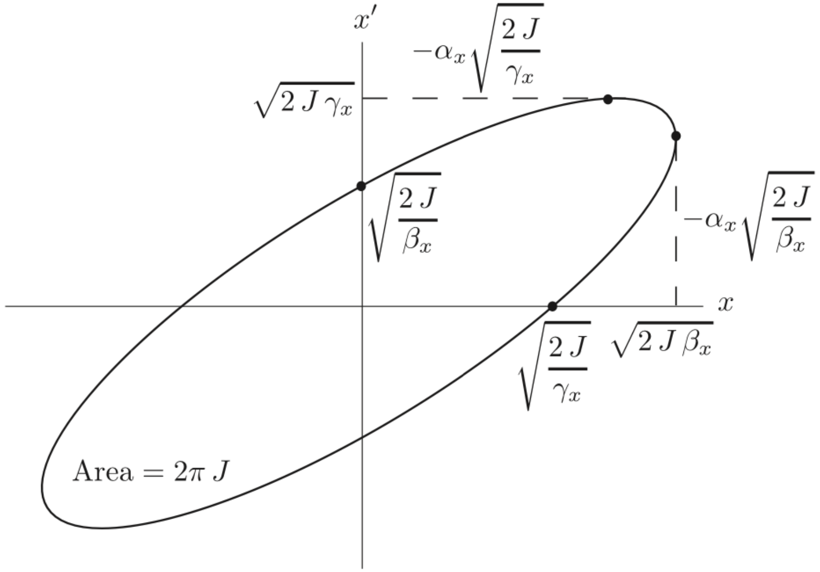
An alternative way to view particle motion in the transverse phase space is to use action-angle coordinates:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha_x}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix} \begin{pmatrix} \sin(\psi_x + \phi) \\ \cos(\psi_x + \phi) \end{pmatrix}$$

Amplitude: $J$
Phase: $\psi_x(s) + \phi$
Beta function: $\beta_x(s)$
Alpha function: $\alpha_x(s)$

Particles with the same amplitude will lie on an ellipse in transverse phase space:




$$\psi_x(s) \equiv \int_0^s \frac{1}{\beta_x(s)} \, \mathrm{d}s$$

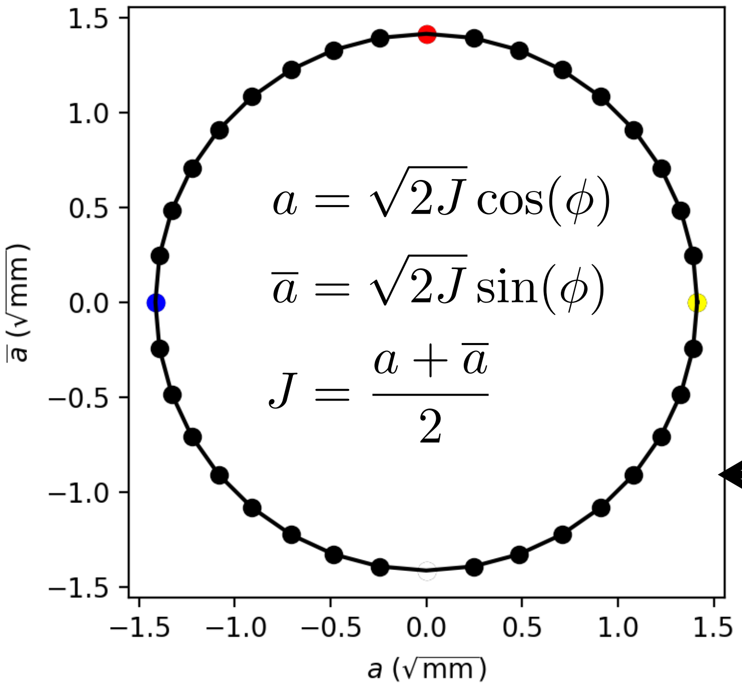
$$\alpha_x \equiv -\frac{1}{2}\beta'_x$$

$$\gamma_x = \frac{1 + \alpha_x^2}{\beta_x}$$

# Normalized Coordinates

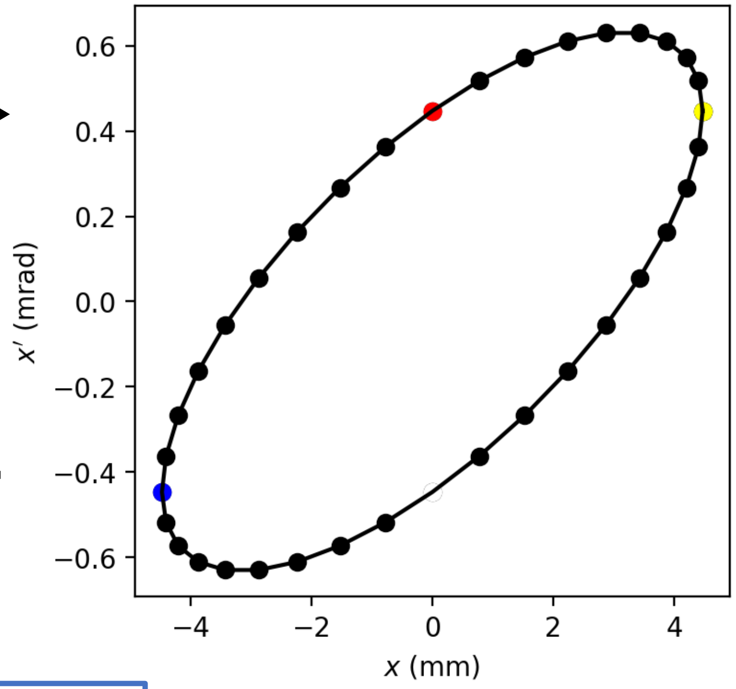
$$\begin{pmatrix} a \\ \overline{a} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} a \\ \overline{a} \end{pmatrix}$$




$$A = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

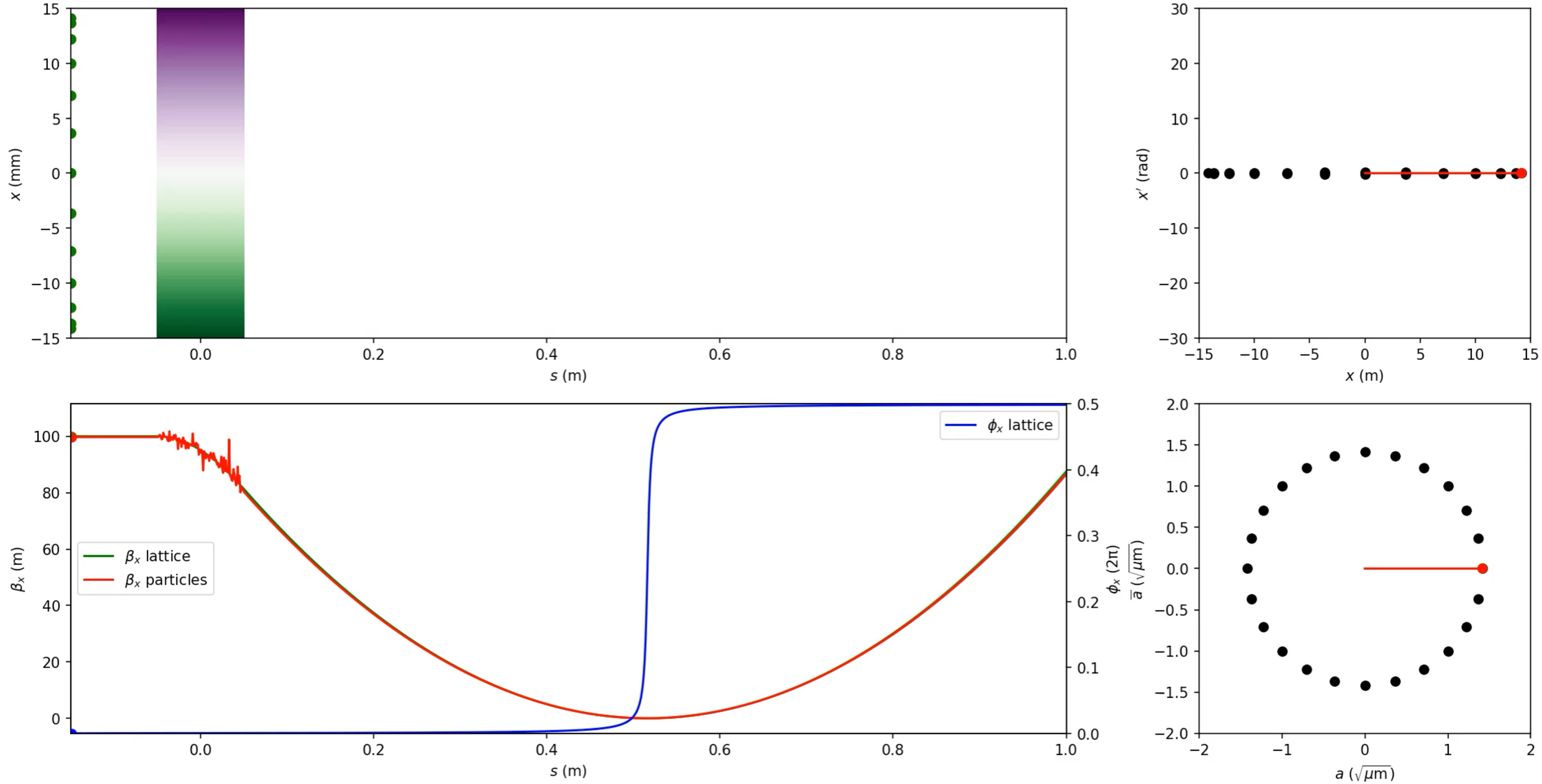
$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$




"Courant-Synder Invariant"

$$2J = \frac{(\alpha^2 + 1)}{\beta} x^2 + 2\alpha x x' + \beta x'^2$$

# Normalized Coordinates



# Twiss Propagation

It can be shown that the Twiss parameters at one location along *s* can be related to another by the transfer matrix *M* between these locations:

$$T_x(s_1) = M \cdot T_x(s_0) \cdot M^T$$

where the Twiss matrix is defined as:

$$T_x \equiv \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix}$$

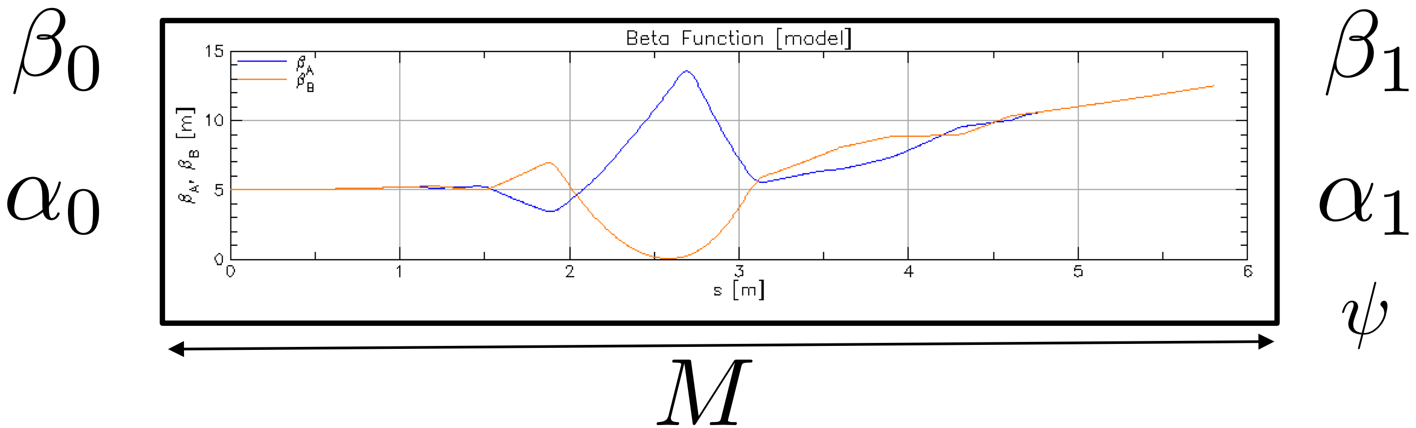
# Matrix from Twiss

A transfer matrix can also be written in terms of beginning and ending Twiss parameters, denoted by *0* and *1* respectively:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_1}{\beta_0}}\left(\cos\psi + \alpha_0 \sin\psi\right) & \sqrt{\beta_0\beta_1}\sin\psi \\ \frac{(\alpha_0 - \alpha_1)\cos\psi - (1+\alpha_0\alpha_1)\sin\psi}{\sqrt{\beta_0\beta_1}} & \sqrt{\frac{\beta_0}{\beta_1}}\left(\cos\psi - \alpha_1 \sin\psi\right) \end{pmatrix}$$

Note that this decomposition is not unique, because it depends on the initial Twiss to be defined.

However, with a periodic system, the decomposition is unique.



# Periodic Twiss from Matrix

The transfer matrix can also be used to define a periodic system. Setting beginning and ending Twiss to be the same in the previous slide gives:

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} \cos\psi + \alpha\sin\psi & \beta\sin\psi \\ -\frac{(1+\alpha^2)}{\beta}\sin\psi & \cos\psi - \alpha\sin\psi \end{pmatrix}$$

This is easily solved:

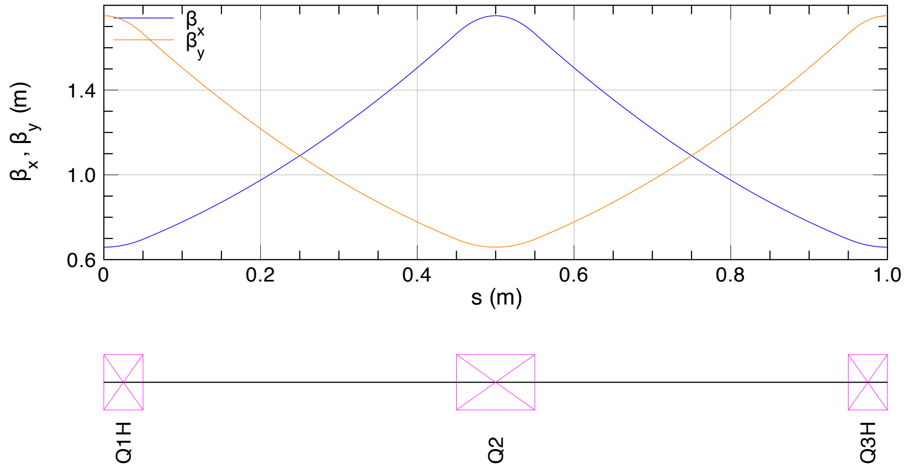
$$\cos\psi = \frac{1}{2}(M_{11} + M_{22})$$

$$\beta = \frac{M_{12}}{\sqrt{1-(M_{11}+M_{22})^2/4}}$$

$$\alpha = \frac{1}{2}\frac{M_{11} - M_{12}}{\sqrt{1-(M_{11}+M_{22})^2/4}}$$

There is a real periodic solution only if: $\frac{1}{2}\left|M_{11} + M_{22}\right| \leq 1$

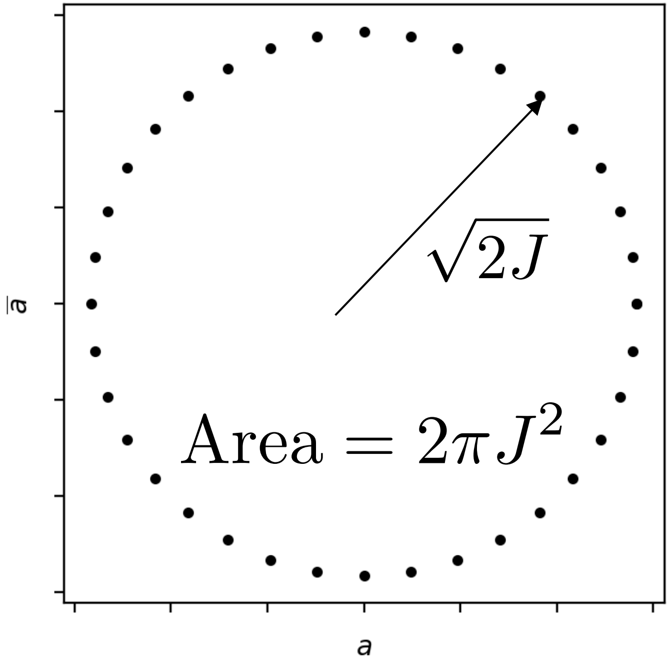
Example: FODO Cell




| | |
|---|---|
| 0.56682984 | 0.54276144 m |
| -1.25046455 | 0.56682984 m |

# Twiss from Particles

Twiss parameters can also be determined from particle distributions. To understand this, consider a distribution of particles with amplitude $J = \epsilon_x$ and uniform distribution in phases:




Transforming to physical coordinates:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \end{pmatrix}$$

and averaging over phases reveals:

$$\langle x^2 \rangle = \epsilon_x \beta_x$$

$$\langle x'^2 \rangle = \epsilon_x \gamma_x$$

$$\langle x, x' \rangle = -\epsilon_x \alpha_x$$

*geometric emittance:* $\epsilon_x = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x\, x' \rangle^2}$

$$\langle J \rangle = \epsilon_x$$

# Mismatch

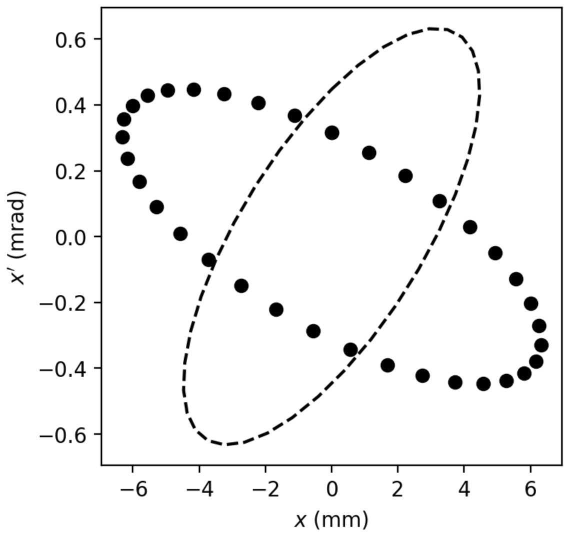
A beam is **mismatched** when its statistical Twiss parameters are not the same as the designed Twiss parameters. To quantify this, recall that the design Twiss parameters define the amplitude according to:

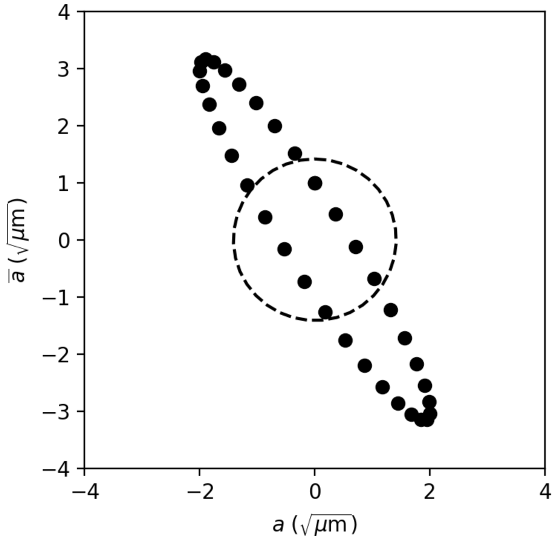
$$2J = \frac{(\alpha^2+1)}{\beta}x^2 + 2\alpha x x' + \beta x'^2$$

Labeling *0* for the design and *1* for the particles and averaging:

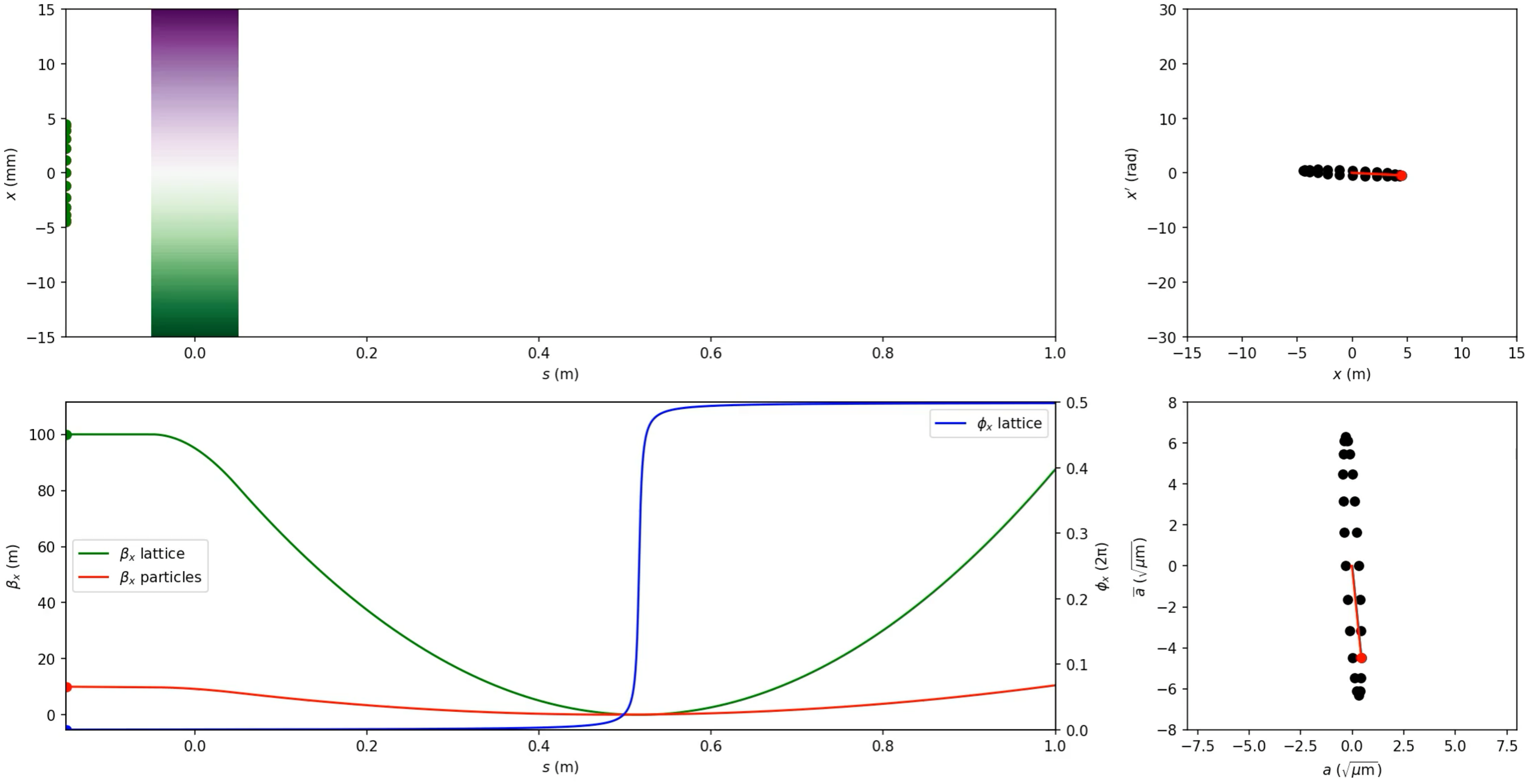
$$\langle 2J_1 \rangle = \frac{(1+\alpha_0^2)}{\beta_0} \underbrace{\langle x_1^2 \rangle}_{\beta_1 \epsilon_1} + 2\alpha_0 \underbrace{\langle x_1 x_1' \rangle}_{-\alpha_1 \epsilon_1} + \beta_0 \underbrace{\langle x_1'^2 \rangle}_{\frac{(1+\alpha_1^2)}{\beta_1}\epsilon_1}$$

$$\implies B_{\text{mag}} = \frac{\langle J_1 \rangle}{\epsilon_1} = \frac{1}{2}\left(\frac{\beta_1}{\beta_0} + \frac{\beta_0}{\beta_1} + \beta_0\beta_1\left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_0}{\beta_0}\right)^2\right)$$

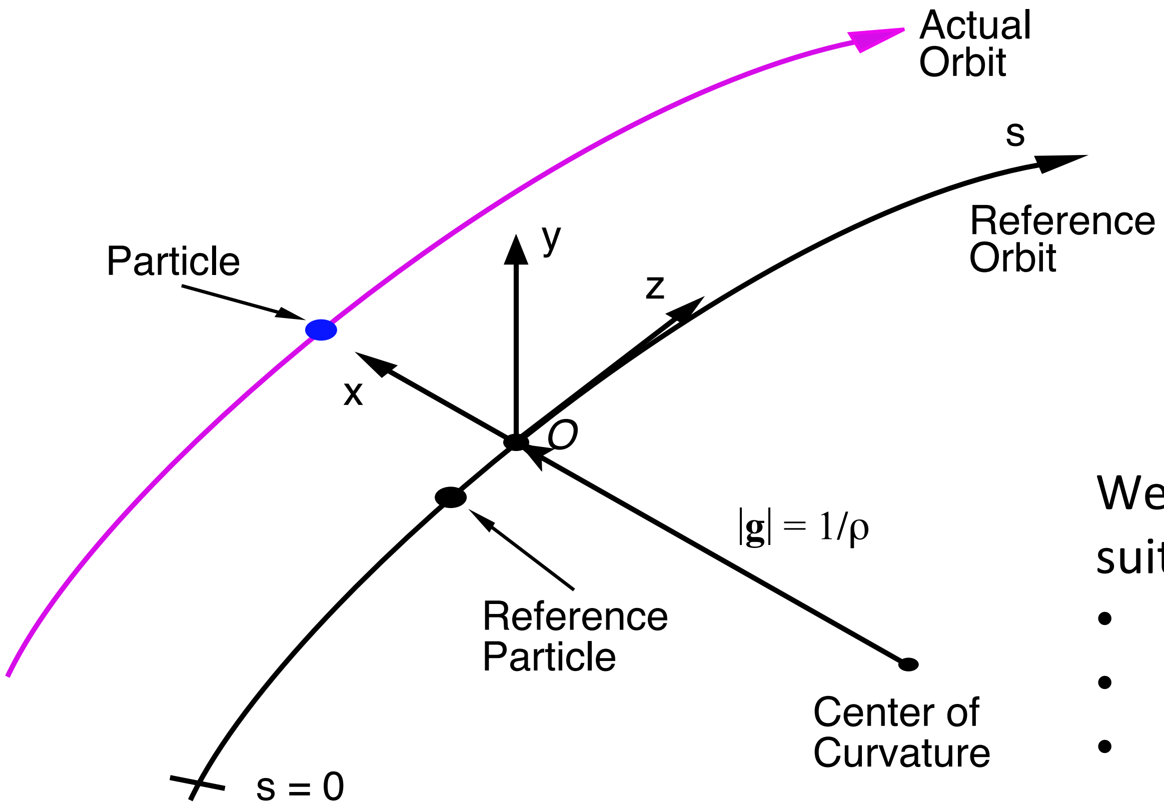


# Normalized Coordinates, Mismatched



# 6D Phase Space Coordinates




$$\underbrace{\begin{pmatrix} X \\ P_x \\ Y \\ P_y \\ Z \\ P_z \end{pmatrix}}_{\text{Lab frame}}(t) \rightarrow \underbrace{\begin{pmatrix} x & \\ p_x & \equiv P_x/p_0 \approx x' \\ y & \\ p_x & \equiv P_y/p_0 \approx y' \\ z & \equiv -\beta c(t-t_0) \\ p_z = \delta & \equiv P/p_0 - 1 \approx E/E_0 - 1 \end{pmatrix}}_{\text{local frame}}(s)$$

We will choose local phase space coordinates more suited for following a reference “particle”:

- The independent variable is *s,*
- *Z* is set to zero by construction in the local frame
- Often we write the relative momentum deviation:

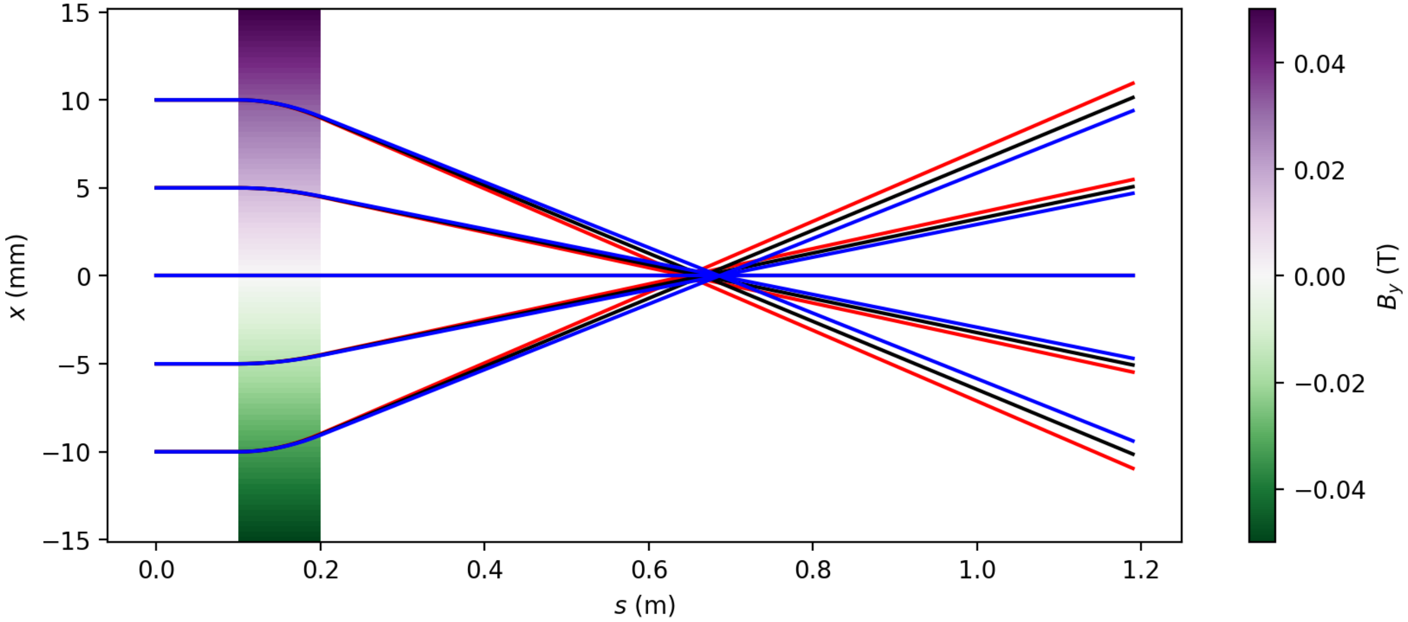
$$\delta \equiv p_z$$

Note that there are many conventions used. This is the convention used by *Bmad.*

# 6D Thick Quadrupole Map

In these coordinates, the body field of a quadrupole (neglecting the fringe fields) has a well-known analytic solution.

Here we see that the map varies with the momentum deviation (“chromatic”), red = -4%, blue = +4%:






The Hamiltonian for an upright quadrupole is

$$H = \frac{p_x^2 + p_y^2}{2(1+p_z)} + \frac{k_1}{2}(x^2 - y^2) \qquad (24.102)$$

This is simply solved

$$\begin{aligned} x_2 &= c_x\, x_1 + s_x\, \frac{p_{x1}}{1+p_{z1}} \\ p_{x2} &= \tau_x\, \omega^2\, (1+p_{z1})\, s_x\, x_1 + c_x\, p_{x1} \\ y_2 &= c_y\, y_1 + s_y\, \frac{p_{y1}}{1+p_{z1}} \\ p_{y2} &= \tau_y\, \omega^2\, (1+p_{z1})\, s_y\, y_1 + c_y\, p_{y1} \\ z_2 &= z_1 + m_{511}\, x_1^2 + m_{512}\, x_1\, p_{x1} + m_{522}\, p_{x1}^2 + m_{533}\, y_1^2 + m_{534}\, y_1\, p_{y1} + m_{544}\, p_{y1}^2 \\ p_{z2} &= p_{z1} \end{aligned} \qquad (24.103)$$

where

$$\omega \equiv \sqrt{\frac{|k_1|}{1+p_{z1}}} \qquad (24.104)$$

and

$$\begin{array}{lll@{\qquad}lll} & k_1 > 0 & k_1 < 0 & & k_1 > 0 & k_1 < 0 \\ c_x = & \cos(\omega L) & \cosh(\omega L) & c_y = & \cosh(\omega L) & \cos(\omega L) \\ s_x = & \frac{\sin(\omega L)}{\omega} & \frac{\sinh(\omega L)}{\omega} & s_y = & \frac{\sinh(\omega L)}{\omega} & \frac{\sin(\omega L)}{\omega} \\ \tau_x = & -1 & +1 & \tau_y = & +1 & -1 \end{array} \qquad (24.105)$$

with this

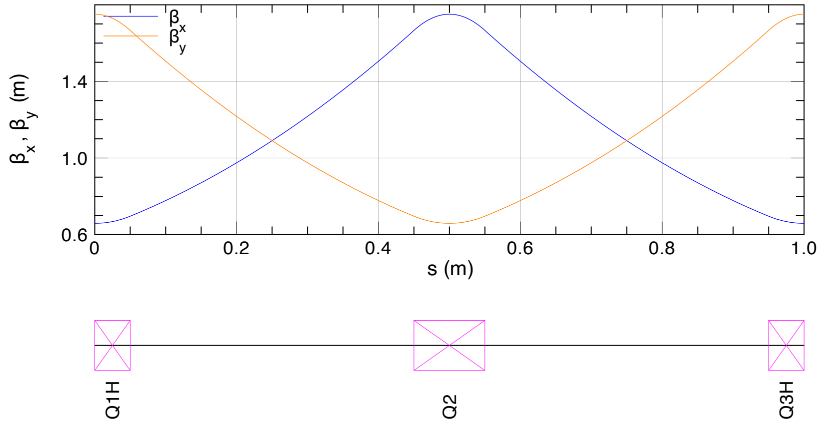
$$\begin{aligned} m_{511} &= \frac{\tau_x\, \omega^2}{4}(L - c_x\, s_x) & m_{533} &= \frac{\tau_y\, \omega^2}{4}(L - c_y\, s_y) \\ m_{512} &= \frac{-\tau_x\, \omega^2}{2\,(1+p_{z1})}\, s_x^2 & m_{534} &= \frac{-\tau_y\, \omega^2}{2\,(1+p_{z1})}\, s_y^2 \\ m_{522} &= \frac{-1}{4\,(1+p_{z1})^2}(L + c_x\, s_x) & m_{544} &= \frac{-1}{4\,(1+p_{z1})^2}(L + c_y\, s_y) \end{aligned} \qquad (24.106)$$

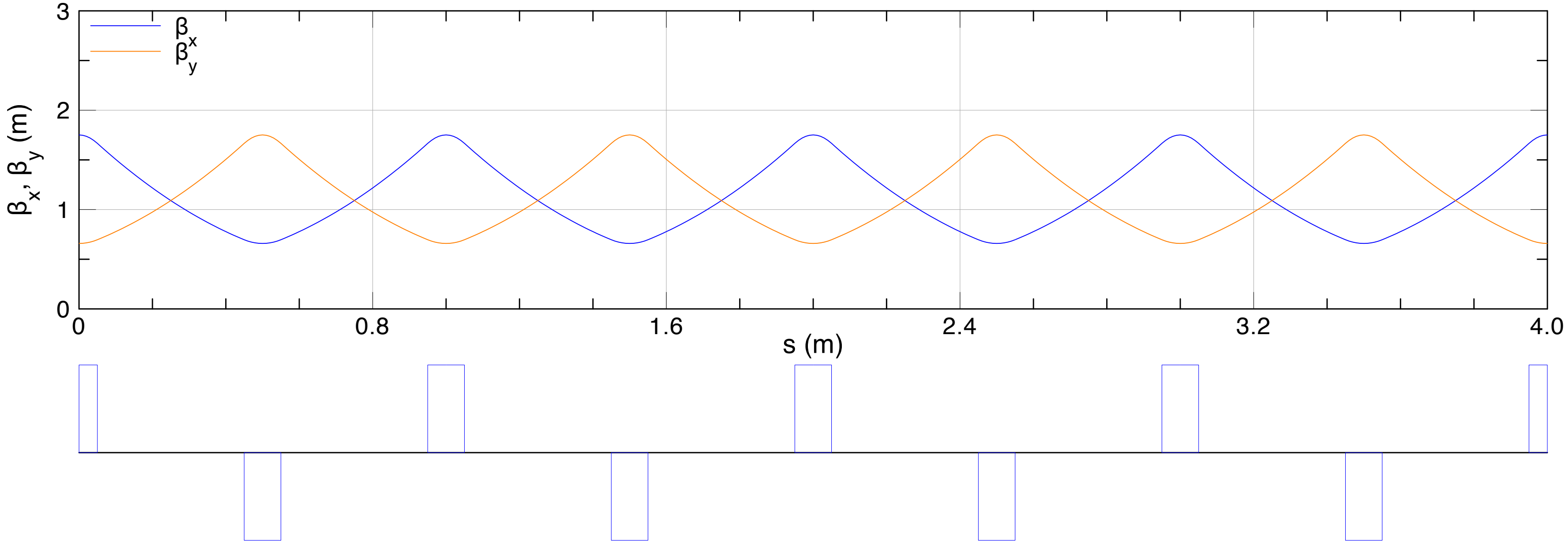
*From the Bmad manual*

# FODO Cell and Lattice

A system consisting of equally spaced quadrupole magnets with alternating field strengths is called a **FODO lattice** ("focusing, defocusing").

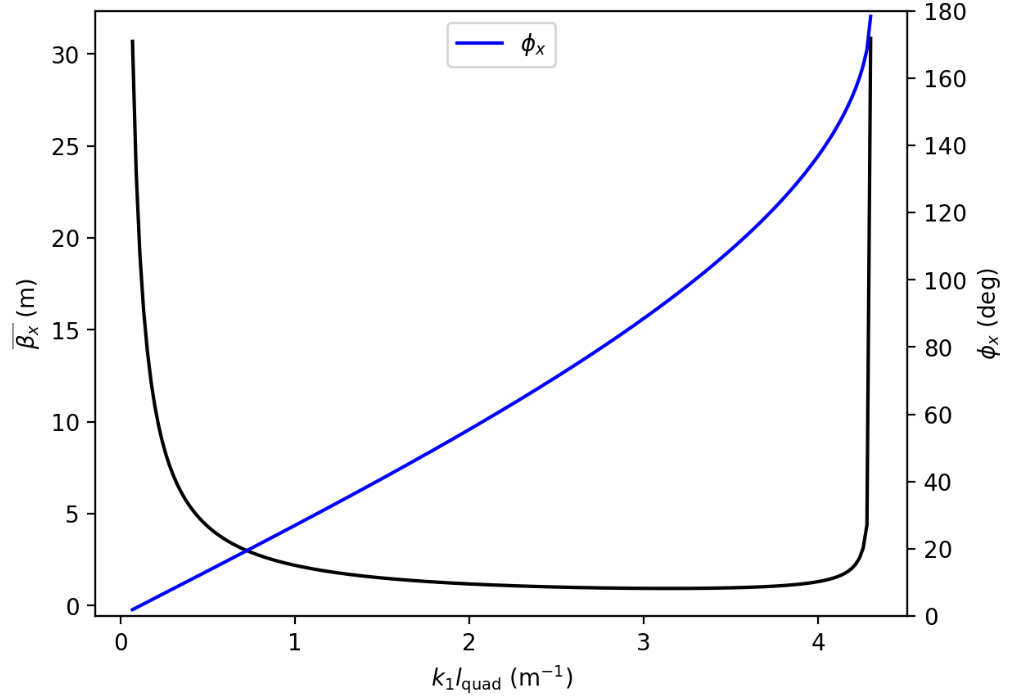
A system with a single period is called a **FODO cell**. Because of the periodicity, the cell could be defined to start anywhere, and repeated cells will yield periodic beta functions.



# FODO Stability

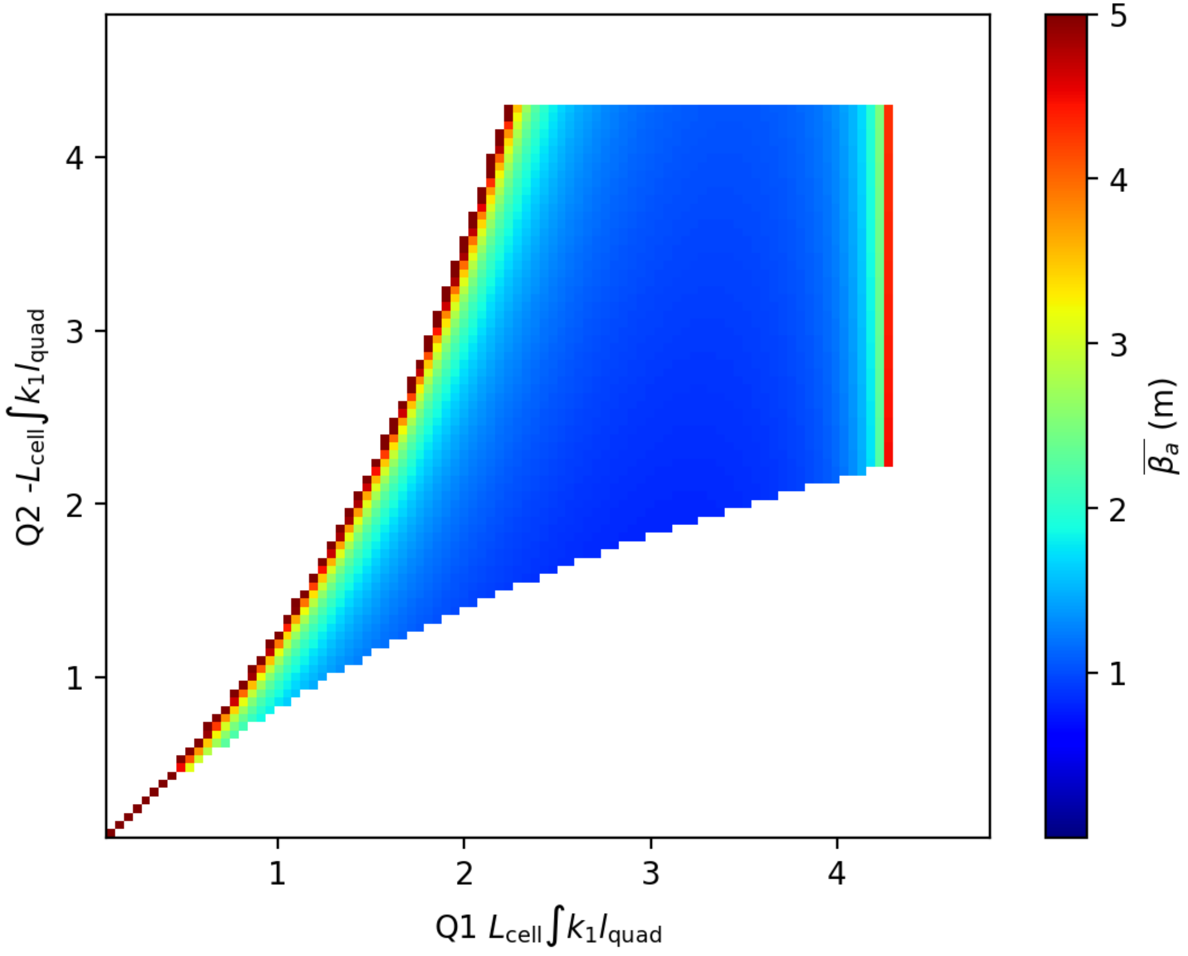
Scanning the focusing strength reveals average beta functions and stability for this **thick quad** system:



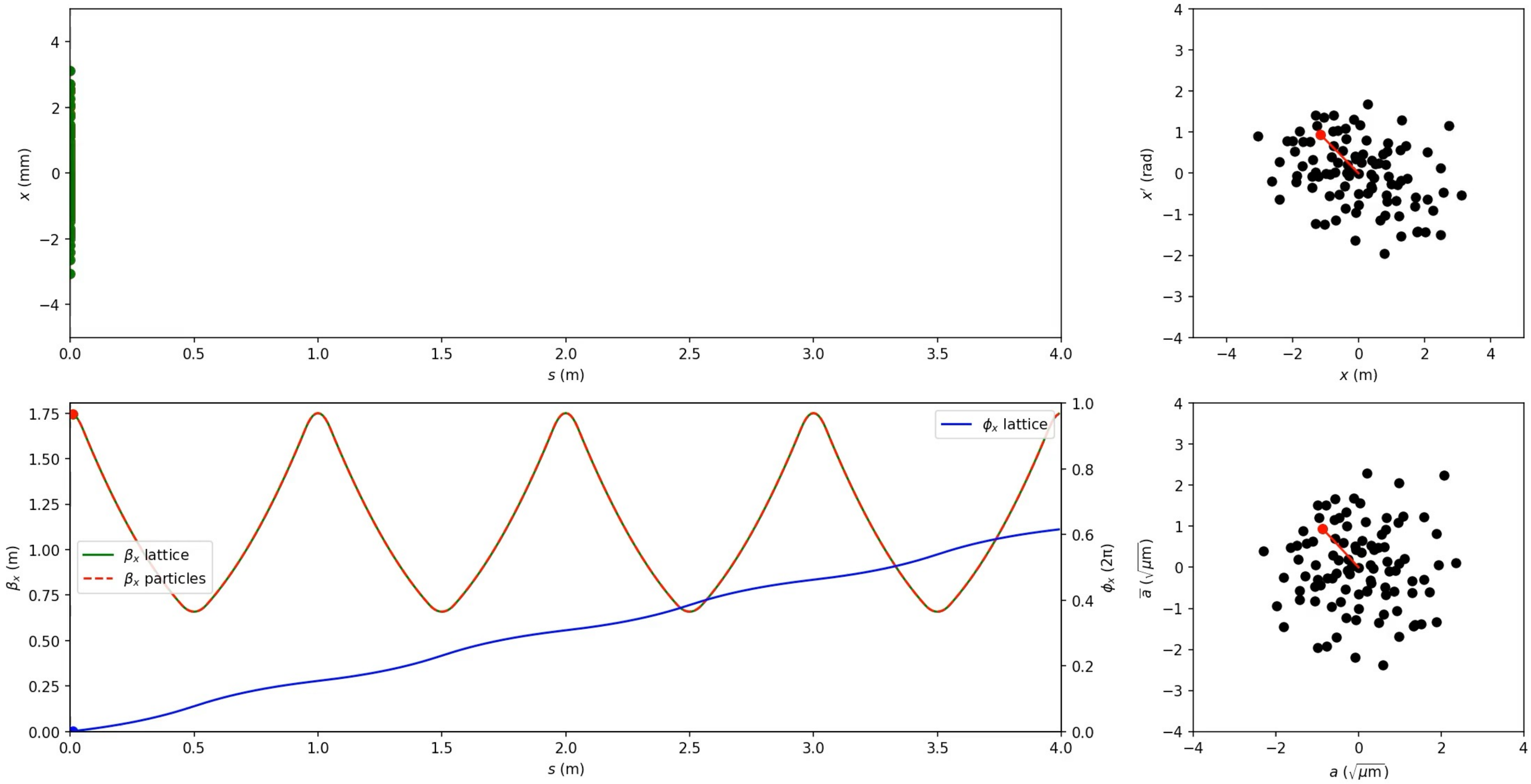

Generally, there are stable solutions when:

$$\int k_1(s)ds \lesssim 4/L_{\rm cell}$$

2D stability "necktie" plot



# FODO tracking



# References

This formalism was originally developed in
[*E. Courant & H. Snyder, Theory of the alternating-gradient synchrotron, Annals of Physics Volume 3, Issue 1 (1958)*]

It is now a sophisticated subject, and things become much more complicated when considering 4D and 6D coupled transport. See, for example:


- D. Sagan and D. Rubin, *Linear analysis of coupled Lattices*, PRAB 074001 (1999)
- A. Wolski, *Alternative approach to general coupled linear optics*, PRAB 024001 (2006)
- *P. Nishikawa, De Moivre's formula: are Sands'H-functions the same as Chao's, JINST 7 P07012 (2012)*


Practically we need to use numerical methods and computer simulations, many of which are described in


- D. Sagan, *Bmad Manual* (online) (continuously updated)


Also see these standard reference books:


- Forest, Etienne. *Beam Dynamics: A New Attitude and Framework*. Harwood Academic Publishers, 1998.
- Wiedemann, Helmut. *Particle Accelerator Physics*. 3rd ed., Springer, 2007.
- Wille, Klaus. *The Physics of Particle Accelerators: An Introduction*. Oxford: Oxford University Press, 2000.